K-RELATIVE FIX POINTS OF FUNCTIONS OF CLASS II

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ABSTRACT. Introducing the idea of k-relative iteration we prove a fix point theorem for k-relative iterated functions of class II.

1. INTRODUCTION AND NOTATIONS

A single valued function \( f(z) \) of the complex variable \( z \) is said to belong to (i) class I if \( f(z) \) is entire transcendental and (ii) class II if it is regular in the complex plane punctured at \( a, b (a \neq b) \) having an essential singularity at \( b \), and a singularity at \( a \) and \( f(z) \) omits the values \( a \) and \( b \) anywhere in the complex plane except possible at the point \( a \).

For simplicity we take \( a = 0 \) and \( b = \infty \).

The functions \( f_n(z) \) of \( f(z) \) are defined inductively by

\[
f_0(z) = z \text{ and } f_{n+1}(z) = f(f_n(z)) \text{ for } n = 0, 1, 2, \ldots.
\]

A point \( \alpha \) is called a fix point of \( f(z) \) of order \( n \) if \( \alpha \) is a solution of \( f_n(z) = z \) and called a fix point of exact order \( n \) if \( \alpha \) is a solution of \( f_n(z) = z \) but not for \( f_k(z) = z, k = 1, 2, \ldots, n-1 \).

In this direction, Baker [1] proved the following theorem.

Theorem 1.1. If \( f(z) \) belongs to class I, then \( f(z) \) has fix points of exact order \( n \), except for at most one value of \( n \).

Bhattacharyya [2] extended Theorem 1.1 for the functions of class II as follows.

Theorem 1.2. If \( f(z) \) belongs to class II, then \( f(z) \) has infinitely many fix points of exact order \( n \), for every positive integer \( n \).

Keywords: Generalised fix points, exact order, class II functions.
After this in [5], Lahiri and Banerjee introducing the concept of relative iteration [ defined below ] proved a fixed point theorem.

Let \( f(z) \) and \( g(z) \) be functions of the complex variable \( z \).

Let

\[
\begin{align*}
f_1(z) & = f(z) \\
f_2(z) & = f(g(z)) = f(g_1(z)) \\
f_3(z) & = f(g(f(z))) = f(g(f_1(z))) \\
\vdots \\
f_n(z) & = f(g(f(...(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even}))) \\
& = f(g_{n-1}(z)) = f(g(f_{n-2}(z))).
\end{align*}
\]

Similarly

\[
\begin{align*}
g_1(z) & = g(z) \\
g_2(z) & = g(f(z)) = g(f_1(z)) \\
g_3(z) & = g(f_2(z)) = g(f(g_1(z))) \\
\vdots \\
g_n(z) & = g(f_{n-1}(z)) = g(g(f_{n-2}(z))).
\end{align*}
\]

Clearly all \( f_n(z) \) and \( g_n(z) \) are functions in class \( II \), if \( f(z) \) and \( g(z) \) are so.

A point \( \alpha \) is called a fix point of \( f(z) \) of order \( n \) with respect to \( g(z) \), if \( f_n(\alpha) = \alpha \) and a fix point of exact order \( n \) if \( f_n(\alpha) = \alpha \) but \( f_k(\alpha) \neq \alpha, k = 1, 2, 3, ..., n-1 \). These points \( \alpha \) are also called relative fix points.

**Theorem 1.3.** If \( f(z) \) and \( g(z) \) belong to class \( II \), then \( f(z) \) has infinitely many relative fix points of exact order \( n \) for every positive integer \( n \), provided \( \frac{T(r,g_n)}{T(r,f_n)} \) is bounded.

In this paper we first consider \( k \) \((\geq 2)\) functions \( f_1, f_2, ..., f_k \) and form their \( k \)-relative iterations as follows:

\[
\begin{align*}
F_1^1(z) & = f_1(z) \\
F_2^1(z) & = f_1(f_2(z)) = f_1(F_1^2(z)) \\
F_3^1(z) & = f_1(f_2(f_3(z))) = f_1(f_2(F_1^3(z))) = f_1(F_2^2(z)).
\end{align*}
\]
\[ F_k^1(z) = f_1(f_2(...f_k(z))) = f_1(F_{k-1}^2(z)) = f_1(f_2(F_{k-2}^3(z))) = ... = f_1(f_2...f_{k-1}(F_1^k(z))) \]

\[ F_n^1(z) = f_1(f_2(f_3...(f_1(z) \text{ or } f_2(z) \text{ or } ... \text{ or } f_k(z) \text{ according as } n = km - (k-1) \text{ or } km - (k-2) \text{ or } ... \text{ or } km)...) = f_1(F_{n-1}^2(z)) = f_1(f_2(F_{n-2}^3(z))) = ... = f_1(f_2(... (f_{k-1}(F_{n-(k-1)}^k(z))))). \]

Similarly

\[ F_1^2(z) = f_2(z) \]
\[ F_2^2(z) = f_2(f_3(z)) = f_2(F_1^3(z)) \]
\[ F_3^2(z) = f_2(f_3(f_4(z))) = f_2(f_3(F_1^4(z))) = f_2(F_2^3(z)) \]

\[ F_k^2(z) = f_2(f_3(...f_k(f_1(z)))) = f_2(F_{k-1}^3(z)) = ... = f_2(f_3(...f_k(F_1^1(z)))) \]

\[ F_n^2(z) = f_2(f_3(f_4...(f_2(z) \text{ or } f_3(z) \text{ or } ... \text{ or } f_k(z) \text{ or } f_1(z) \text{ according as } n = km - (k-1) \text{ or } km - (k-2) \text{ or } ... \text{ or } (km-1) \text{ or } km)...) = f_2(F_{n-1}^3(z)) = f_2(f_3(F_{n-2}^4(z))) = ... = f_2(f_3(...(f_k(F_{n-(k-1)}^1(z))))). \]

And
\[
F_k^1(z) = f_k(z) \\
F_k^2(z) = f_k(f_1(z)) = f_k(F_1^1(z)) \\
F_k^3(z) = f_k(f_1(f_2(z))) = f_k(f_1(F_1^2(z))) = f_k(F_2^1(z)) \\
\vdots \\
F_k^n(z) = f_k(f_1(f_2\ldots f_{k-2}(f_{k-1}(z)))) = \ldots \\
= f_k(f_1(\ldots f_{k-2}(F_{1}^{k-1}(z)))) \\
\vdots \\
= f_k\left(f_1(\ldots (f_{k-2}\left(F_{n-(k-1)}^{k-1}\left(z\right)\right))\right)).
\]

We introduce the following definition.

**Definition 1.4.** A point \( \alpha \) is called a fix point of \( f_1(z) \) of order \( n \) with respect to \( f_2(z), f_3(z), \ldots, f_k(z) \) if \( F_n^1(\alpha) = \alpha \) and a fix point of \( f_1(z) \) of exact order \( n \) with respect to \( f_2(z), f_3(z), \ldots, f_k(z) \) if \( F_n^1(\alpha) = \alpha \) but \( F_s^1(\alpha) \neq \alpha \), \( s = 1, 2, \ldots, n-1 \). These points are called \( k \)-relative fix points.

**Example 1.5.** Let \( f_1(z) = z + 1, f_2(z) = z + 2, f_3(z) = z + 3, \ldots, f_{k-1}(z) = z + (k - 1) \) and \( f_k(z) = 2z + k \). Then \( F_k^1(z) = 2z + \frac{k(k+1)}{2} \). Then \( z = -\frac{k(k+1)}{2} \) is fix point of \( f_1(z) \) of exact order \( k \).

Let \( f(z) \) be meromorphic in \( 0 < r_0 \leq |z| < \infty \).

From the first fundamental theorem

\[
m(r, a, f) + N(r, a, f) = T(r, f) + O(\log r),
\]

where \( 0 < r_0 \leq |z| < \infty \).

Suppose that \( f(z) \) is non-constant and let \( a_1, a_2, \ldots, a_q; q \geq 2 \) be distinct finite complex numbers with \( \delta > 0 \) and \( |a_\mu - a_\nu| \geq \delta \) for \( 1 \leq \mu < \nu \leq q \). Then

\[
m(r, f) + \sum_{\nu=1}^{q} m(r, a_\nu, f) \leq 2T(r, f) - N_1(r) + S(r),
\]
where

\[ N_1(r) = N \left( r, \frac{1}{f'} \right) + 2N(r, f) - N(r, f'), \]

and

\[ S(r) = m \left( r, \frac{f'}{f} \right) + \sum_{v=1}^{q} m \left( r, \frac{f'}{f - a_v} \right) + O(\log r). \]

Adding \( N(r, f) + \sum_{v=1}^{q} N(r, a_v, f) \) to both sides of (2) and using (1), we have

\[ (q - 1)T(r, f) \leq N(r, f) + \sum_{v=1}^{q} N(r, a_v, f) + S_1(r), \]

where \( S_1(r) = O(\log T(r, f)) \) and \( N \) corresponds to distinct roots.

Further if \( f_n \) has an essential singularity at \( \infty \), we have \( \log r \rightarrow 0 \) as \( r \rightarrow \infty \) [3].

2. LEMMA

To prove the main result we need the following lemma.

Lemma 2.1. If \( n \) is any positive integer and \( f_1, f_2, ..., f_k \) are \( k (\geq 2) \) functions in class II, then for any \( r_0 > 0 \) and a suitable positive integer \( M_1 \), we have

\[ \frac{T(r, F_{n+p}^i)}{T(r, F_n^{a_i})} > M_1 \quad \text{or} \quad \frac{T(r, F_{n+p}^{a_i})}{T(r, F_n^i)} > M_1 \quad \text{or} \quad \frac{T(r, F_{n+p}^k)}{T(r, F_n^k)} > M_1 \]

according as \( p = km \) or \( km - 1 \) or...or \( km - (k - 1) \), \( m \in \mathbb{N} \) for all large \( r \), except a set of \( r \) intervals of total finite length.

Proof. Let \( p = km - l, \ l \in \{0, 1, 2, ..., k - 1\} \).

Then we consider the equation \( F_{n+p}^i(z) = a, \ i = 1, 2, ..., k \) according as \( l = 0, 1, 2, ..., k - 1 \), where \( a \neq 0, \ \infty \).

We now show that \( F_{n+p}^k(z) = F_p^k(F_n^i(z)) \) where \( p = km - (k - 1) \).

Since

\[ F_{n+p}^k(z) = f_k(F_{n+p-1}^i(z)) = f_k(f_1(F_{n+p-2}^i(z))) = f_k(f_1(f_2(F_{n+p-3}^i(z)))) \]
\[ f_k(f_1(f_2(\ldots f_k(F_{n+p-1}(z))))) \]
no. of functions
2k - (k - 1)
= ...
= f_k(f_1(f_2(\ldots f_k(F_{n+p-1}(2k-(k-1))(z))))))
no. of functions
3k - (k - 1)
= ...
= f_k(f_1(f_2(\ldots f_k(F_{n+p-1}(3k-(k-1))(z))))))
no. of functions
km - (k - 1)
= F_p^k(F_n^1(z)).

Therefore \( F_p^i(F_n^1(z)) = a \). This is equivalent to \( F_p^i(u_j^l) = a \) where \( F_n^1(z) = u_j^l, (j = 1, 2, 3, \ldots) \).

Because \( F_p^i \) is transcendental, \( F_p^i(u_j^l) = a \) has infinitely many roots for every complex number \( a \) with two exceptions \( a = 0, \infty \).

We have from (1),

\[ T(r, F_{n+p}^i) = m(r, a, F_{n+p}^i) + N(r, a, F_{n+p}^i) + O(\log r) \]
\[ \geq \overline{N}(r, a, F_{n+p}^i) + O(\log r) \]
\[ \geq \sum_{j=1}^{M} \overline{N}(r, u_j^l, F_n^1) \]

for a fixed \( M (> 4) \).

From (3), taking \( a_v = u_j^l, f = F_n^1 \) and \( q = M \), we obtain

\[ \sum_{j=1}^{M} \overline{N}(r, u_j^l, F_n^1) \geq (M - 1) T(r, F_n^1) - \overline{N}(r, F_n^1) - S_1(r) \]

Since for large \( r \),

\[ S_1(r) \leq T(r, F_n^1), \]

so
\[ \sum_{j=1}^{M} \mathcal{N}(r, u'_j, F^1_n) \geq (M - 3) T(r, F^1_n) \]

(4)

\[ > (M - 4) T(r, F^1_n) \ . \]

Therefore

\[ T(r, F^i_{n+p}) > M_1 T(r, F^1_n) , \] where \( M_1 = M - 4 \), outside a set of \( r \) intervals of total finite length. \( \square \)

3. THEOREM

Our main result is the following theorem.

**Theorem 3.1.** If \( f_1, f_2, ..., f_k \) belong to class II, then \( f_1(z) \) has an infinity of \( k \)-relative fix points of exact order \( n \) (\( > k \)) for every positive integer \( n \), provided \( T(r, F^i_n) \) are bounded.

**Proof.** Here we consider the function

\[ g(z) = \frac{F^1_n(z)}{z} , \quad r_0 < |z| < \infty . \]

Then

(5)

\[ T(r, g) = T(r, F^1_n) + O(\log r) . \]

Now we assume that \( f(z) \) has only finite number of \( k \)-relative fix points of exact order \( n \). Taking \( q = 2, a_1 = 0, a_2 = 1 \), we have from (3)

(6)

\[ T(r, g) \leq \mathcal{N}(r, \infty, g) + \mathcal{N}(r, 0, g) + \mathcal{N}(r, 1, g) + S_1(r, g) , \]

where \( S_1(r, g) = O(\log T(r, g)) \) outside a set of \( r \) intervals of finite length [4].

We have

\[ \mathcal{N}(r, 0, g) = \int_{r_0}^{r} \frac{\pi(t, 0, g)}{t} dt, \]

where \( \pi(t, 0, g) \) is the number of roots of \( g(z) = 0 \) in \( r_0 < |z| \leq t \), each multiple root taken once at a time. The distinct roots of \( g(z) = 0 \) in \( r_0 < |z| \leq t \) are the roots of \( F^1_n(z) = 0 \) in \( r_0 < |z| \leq t \). Since \( F^1_n(z) \) has a singularity at \( z = 0 \), an essential singularity at \( z = \infty \) with \( F^1_n(z) \neq 0, \infty \), so \( \pi(t, 0, g) = 0 \). So \( \mathcal{N}(r, 0, g) = 0 \). Similarly \( \mathcal{N}(r, \infty, g) = 0 \). So

\[ T(r, g) \leq \mathcal{N}(r, 1, g) + S_1(r, g) \]
Again \( F_n^1(z) = z \) when \( g(z) = 1 \).

So

\[
N(r, 1, g) = N(r, 0, F_n^1 - z)
\]

\[
\leq \sum_{j=1}^{n-1} N(r, 0, F_j^1 - z) + O(\log r),
\]

here \( O(\log r) \) arises due to the assumption that \( f_1(z) \) has only a finite number of fix points of exact order \( n \).

We have using (6),

\[
T(r, g) \leq \sum_{j=1}^{n-1} N(r, 0, F_j^1 - z) + O(\log r) + O(T(r, g))
\]

\[
\leq \sum_{j=1}^{n-1} T(r, F_j^1) + O(\log T(r, g)) + O(\log r)
\]

\[
= T(r, F_n^1) \left[ \left\{ \frac{T(r, F_{j_1}^1)}{T(r, F_n^1)} + \frac{T(r, F_{j_{k+1}}^1)}{T(r, F_n^1)} + \ldots + \frac{T(r, F_{j_{k-1}}^1)}{T(r, F_n^1)} \right\} T(r, F_n^1) + \frac{T(r, F_{j_2}^2)}{T(r, F_n^1)} + \frac{T(r, F_{j_{k+2}}^2)}{T(r, F_n^1)} + \ldots + \frac{T(r, F_{j_{k-1}}^2)}{T(r, F_n^1)} \right] T(r, F_n^1)
\]

\[
+ \ldots
\]

\[
+ O \left( \log \left\{ T(r, F_n^1) \left( 1 + \frac{O(\log r)}{T(r, F_n^1)} \right) \right\} \right) + O(\log r), \quad \text{where}
\]

\( j_1, j_{k+1}, \ldots, j_{kp_1-(k-1)}; j_2, j_{k+2}, \ldots, j_{kp_2-(k-2)}; \ldots; j_k, j_{2k}, \ldots, j_{kp_k} \)

are strictly less than \( n \) and are of the form \( kp_1 - (k - 1), kp_2 - (k - 2), \ldots, kp_k \), \( p_1, p_2, \ldots, p_k \in \mathbb{N} \)

\[
< T(r, F_n^1) \left[ \frac{n-1}{2kn} + \frac{n-1}{2kn} + \ldots + \frac{n-1}{2kn} \right] + \frac{O(\log r)}{T(r, F_n^1)} \right] \right\} \right) + O(\log r), \quad \text{for all large } r, \text{ by}
\]
Lemma 2.1 and since $\frac{T(r, F_i^n)}{T(r, F_{1n}^1)}$, $i = 2, 3, ..., k$ are bounded

$$< \frac{1}{2} T(r, F_{1n}^1).$$

Therefore, $T(r, g) < T(r, F_{1n}^1)$, for all large $r$. This contradicts (5).

Therefore $f_1(z)$ has infinity many k-relative fix points of exact order $n$.

This completes the proof. $\square$

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