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K-RELATIVE FIX POINTS OF FUNCTIONS OF CLASS II

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Abstract. Introducing the idea of k-relative iteration we prove a fix point theorem for k-relative iterated functions of class II.

1. INTRODUCTION AND NOTATIONS

A single valued function $f(z)$ of the complex variable z is said to belong to (i) class I if $f(z)$ is entire transcendental and (ii) class II if it is regular in the complex plane punctured at $a, b\,(a \neq b)$ having an essential singularitiy at b, and a singularity at a and $f(z)$ omits the values a and b anywhere in the complex plane except possible at the point a.

For simplicity we take $a = 0$ and $b = \infty$.

The functions $f_n(z)$ of $f(z)$ are defined inductively by

$$f_0(z) = z \text{ and } f_{n+1}(z) = f(f_n(z)) \text{ for } n = 0, 1, 2, ... \qquad .$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and called a fix point of exact order n if α is a solution of $f_n(z) = z$ but not for $f_k(z) = z, k = 1, 2, ..., n-1$.

In this direction, Baker [1] proved the following theorem.

Theorem 1.1. . *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n, except for at most one value of n.*

Bhattacharyya [2] extended Theorem 1.1 for the functions of class II as follows.

Theorem 1.2. . *If $f(z)$ belongs to class II, then $f(z)$ has infinitely many fix points of exact order n, for every positive integer n.*

Keywords: Generalised fix points, exact order, class II functions.

After this in [5], Lahiri and Banerjee introducing the concept of relative iteration [defined below] proved a fixed point theorem.

Let $f(z)$ and $g(z)$ be functions of the complex variable z.

Let

$$\begin{aligned} f_1(z) &= f(z) \\ f_2(z) &= f(g(z)) = f(g_1(z)) \\ f_3(z) &= f(g(f(z))) = f(g(f_1(z))) \\ &\vdots \\ f_n(z) &= f(g(f(g...(f(z) \text{ or } g(z) \text{ according} \\ &\quad \text{as } n \text{ is odd or even})...))) \\ &= f(g_{n-1}(z)) = f(g(f_{n-2}(z))). \end{aligned}$$

Similarly

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f_2(z)) = g(f(g_1(z))) \\ &\vdots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are functions in class II, if $f(z)$ and $g(z)$ are so.

A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$, if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, k = 1, 2, 3, ..., n-1$.These points α are also called relative fix points.

Theorem 1.3. . *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many relative fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ is bounded.*

In this paper we first consider k (≥ 2) functions $f_1, f_2, ...f_k$ and form their k-relative iterations as follows:

$$\begin{aligned} F_1^1(z) &= f_1(z) \\ F_2^1(z) &= f_1(f_2(z)) = f_1\left(F_1^2(z)\right) \\ F_3^1(z) &= f_1(f_2(f_3(z))) = f_1\left(f_2\left(F_1^3(z)\right)\right) = f_1(F_2^2(z)) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
F_k^1(z) &= f_1(f_2(\ldots f_k(z))) = f_1\left(F_{k-1}^2(z)\right) = f_1\left(f_2\left(F_{k-2}^3(z)\right)\right) \\
&= \ldots = f_1\left(f_2 \ldots f_{k-1}\left(F_1^k(z)\right)\right) \\
&\vdots \\
F_n^1(z) &= f_1(f_2(f_3\ldots(f_1(z) \text{ or } f_2(z) \text{ or} \ldots \text{or } f_k(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or } \ldots \text{or } km)\ldots)) \\
&= f_1\left(F_{n-1}^2(z)\right) \\
&= f_1\left(f_2\left(F_{n-2}^3(z)\right)\right) \\
&= \ldots\ldots \\
&= f_1\left(f_2\left(\ldots\left(f_{k-1}\left(F_{n-(k-1)}^k(z)\right)\right)\right)\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
F_1^2(z) &= f_2(z) \\
F_2^2(z) &= f_2(f_3(z)) = f_2\left(F_1^3(z)\right) \\
F_3^2(z) &= f_2(f_3(f_4(z))) = f_2\left(f_3\left(F_1^4(z)\right)\right) = f_2(F_2^3(z)) \\
&\vdots \\
F_k^2(z) &= f_2(f_3(\ldots f_k(f_1(z)))) = f_2\left(F_{k-1}^3(z)\right) = \ldots \\
&= f_2\left(f_3\left(\ldots f_k\left(F_1^1(z)\right)\right)\right) \\
&\vdots \\
F_n^2(z) &= f_2(f_3(f_4\ldots(f_2(z) \text{ or } f_3(z) \text{ or } \ldots \text{or } f_k(z) \text{ or } f_1(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or } \ldots \text{or } (km-1) \text{ or } km)\ldots)) \\
&= f_2\left(F_{n-1}^3(z)\right) \\
&= f_2\left(f_3\left(F_{n-2}^4(z)\right)\right) \\
&= \ldots\ldots \\
&= f_2\left(f_3\left(\ldots\left(f_k\left(F_{n-(k-1)}^1(z)\right)\right)\right)\right)
\end{aligned}
$$

.

.

.

And

$$
\begin{aligned}
F_1^k(z) &= f_k(z) \\
F_2^k(z) &= f_k(f_1(z)) = f_k\left(F_1^1(z)\right) \\
F_3^k(z) &= f_k(f_1(f_2(z))) = f_k\left(f_1\left(F_1^2(z)\right)\right) = f_k(F_2^1(z)) \\
&\vdots \\
F_k^k(z) &= f_k(f_1(...f_{k-2}(f_{k-1}(z)))) = ... \\
&= f_k\left(f_1\left(...f_{k-2}\left(F_1^{k-1}(z)\right)\right)\right) \\
&\vdots \\
F_n^k(z) &= f_k(f_1(f_2...(f_k(z) \text{ or } f_1(z) \text{ or } ...\text{or } f_{k-1}(z) \text{ according} \\
\text{as } n &= km-(k-1) \text{ or } km-(k-2) \text{ or...or } km)..).) \\
&= f_k\left(F_{n-1}^1(z)\right) \\
&= f_k\left(f_1\left(F_{n-2}^2(z)\right)\right) \\
&= .... \\
&= f_k\left(f_1\left(...\left(f_{k-2}\left(F_{n-(k-1)}^{k-1}(z)\right)\right)\right)\right).
\end{aligned}
$$

We introduce the following definition.

Definition 1.4. *.A point α is called a fix point of $f_1(z)$ of order n with respect to $f_2(z)$, $f_3(z), ..., f_k(z)$ if $F_n^1(\alpha)=\alpha$ and a fix point of $f_1(z)$ of exact order n with respect to $f_2(z)$, $f_3(z), ..., f_k(z)$ if $F_n^1(\alpha)=\alpha$ but $F_s^1(\alpha)\neq\alpha$, $s=1,2,...,n-1$. These points are called k-relative fix points.*

Example 1.5. *.Let $f_1(z)=z+1$, $f_2(z)=z+2$, $f_3(z)=z+3, ..., f_{k-1}(z)=z+(k-1)$ and $f_k(z)=2z+k$. Then $F_k^1(z)=2z+\frac{k(k+1)}{2}$. Then $z=-\frac{k(k+1)}{2}$ is fix point of $f_1(z)$ of exact order k.*

Let $f(z)$ be meromorphic in $0<r_0\leq|z|<\infty$.

From the first fundamental theorem

$$m(r,a,f)+N(r,a,f)=T(r,f)+O(\log r), \tag{1}$$

where $0<r_0\leq|z|<\infty$.

Suppose that $f(z)$ is non-constant and let $a_{1,}a_{2,...,}a_q; q\geq 2$ be distinct finite complex numbers with $\delta>0$ and $|a_\mu-a_\upsilon|\geq\delta$ for $1\leq\mu<\upsilon\leq q$. Then

$$m(r,f)+\sum_{\upsilon=1}^{q}m(r,a_\upsilon,f)\leq 2T(r,f)-N_1(r)+S(r), \tag{2}$$

where

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r,f) - N(r,f'),$$

and

$$S(r) = m\left(r, \frac{f'}{f}\right) + \sum_{\upsilon=1}^{q} m\left(r, \frac{f'}{f-a_\upsilon}\right) + O(\log r).$$

Adding $N(r,f) + \sum_{\upsilon=1}^{q} N(r, a_\upsilon, f)$ to both sides of (2) and using (1), we have

$$(q-1)\,T(r,f) \leq \overline{N}(r,f) + \sum_{\upsilon=1}^{q} \overline{N}(r,a_\upsilon,f) + S_1(r), \tag{3}$$

where $S_1(r) = O(\log T(r,f))$ and $\overline{N}$ corresponds to distinct roots.

Further if f_n has an essential singularity at ∞,we have $\frac{\log r}{T(r,f_n)} \to 0$ as $r \to \infty$ [3].

2. LEMMA

To prove the main result we need the following lemma.

Lemma 2.1. . *If n is any positive integer and $f_1, f_2, ..., f_k$ are k (≥ 2) functions in class II, then for any $r_0 > 0$ and a suitable positive integer M_1, we have*

$$\frac{T\left(r, F_{n+p}^1\right)}{T\left(r, F_n^1\right)} > M_1 \text{ or } \frac{T\left(r, F_{n+p}^2\right)}{T\left(r, F_n^1\right)} > M_1 \text{ or ..... or } \frac{T\left(r, F_{n+p}^k\right)}{T\left(r, F_n^1\right)} > M_1$$

according as $p = km$ or $km-1$ or...or $km-(k-1)\,, m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Proof. Let $p = km - l,\ l \in \{0, 1, 2, ..., k-1\}$.

Then we consider the equation $F_{n+p}^i(z) = a, i = 1, 2, ..., k$ according as $l = 0, 1, 2, ..., k-1$, where $a \neq 0, \infty$.

We now show that $F_{n+p}^k(z) = F_p^k(F_n^1(z))$ where $p = km - (k-1)$.

Since

$$\begin{aligned} F_{n+p}^k(z) &= f_k(F_{n+p-1}^1(z)) \\ &= f_k(f_1(F_{n+p-2}^2(z))) \\ &= f_k(f_1(f_2(F_{n+p-3}^3(z)))) \end{aligned}$$

$$
\begin{aligned}
&= ...\\
&= f_k(f_1(f_2...(f_k(F^1_{n+p-(k+1)}(z)))))\\
&= \underbrace{f_k(f_1(f_2(...f_k}_{\substack{\text{no. of functions}\\ 2k-(k-1)}}(F^1_{n+p-\{2k-(k-1)\}}(z)))))\\
&= ...\\
&= \underbrace{f_k(f_1(f_2(...f_k(f_1(f_2(...f_k}_{\substack{\text{no. of functions}\\ 3k-(k-1)}}(F^1_{n+p-\{3k-(k-1)\}}(z))))))))\\
&= ...\\
&= \underbrace{f_k(f_1(f_2(...f_k}_{\substack{\text{no. of functions}\\ km-(k-1)}}(F^1_{n+p-\{km-(k-1)\}}(z)))))\\
&= F^k_p\left(F^1_n(z)\right).
\end{aligned}
$$

Therefore $F^i_p\left(F^1_n(z)\right) = a$. This is equivalent to $F^i_p\left(u^l_j\right) = a$ where $F^1_n(z) = u^l_j, (j = 1, 2, 3, ...)$.

Because F^i_p is transcendental, $F^i_p\left(u^l_j\right) = a$ has infinitely many roots for every complex number a with two expections $a = 0, \infty$.

We have from (1),

$$
\begin{aligned}
T\left(r, F^i_{n+p}\right) &= m\left(r, a, F^i_{n+p}\right) + N\left(r, a, F^i_{n+p}\right) + O(\log r)\\
&\geq \overline{N}\left(r, a, F^i_{n+p}\right) + O(\log r)\\
&\geq \sum_{j=1}^{M} \overline{N}\left(r, u^l_j, F^1_n\right)
\end{aligned}
$$

for a fixed $M\ (> 4)$.

From (3), taking $a_v = u^l_j$, $f = F^1_n$ and $q = M$, we obtain

$$
\sum_{j=1}^{M} \overline{N}\left(r, u^l_j, F^1_n\right) \geq (M-1)\, T\left(r, F^1_n\right) - \overline{N}\left(r, F^1_n\right) - S_1(r)
$$

Since for large r,

$$
S_1(r) \leq T\left(r, F^1_n\right),
$$

so

$$\sum_{j=1}^{M}\overline{N}\left(r,u_j^l,F_n^1\right) \geq (M-3)\,T\left(r,F_n^1\right)$$

$$(4) \qquad > (M-4)\,T\left(r,F_n^1\right).$$

Therefore

$$T\left(r,F_{n+p}^i\right) > M_1 T\left(r,F_n^1\right), \text{ where } M_1 = M-4,$$

outside a set of r intervals of total finite length. □

3. THEOREM

Our main result is the following theorem.

Theorem 3.1. . *If $f_1,\ f_2,...,\ f_k$ belong to class II, then $f_1(z)$ has an infinity of k-relative fix points of exact order n $(>k)$ for every positive integer n, provided $\frac{T\left(r,F_n^i\right)}{T\left(r,F_n^1\right)}$, $i=2,3,...,k$ are bounded.*

Proof. Here we consider the function

$$g(z) = \frac{F_n^1(z)}{z}\ ,\ r_0 < |z| < \infty\ .$$

Then

$$T(r,g) = T\left(r,F_n^1\right) + O(\log r)\,. \tag{5}$$

Now we assume that $f(z)$ has only finite number of k-relative fix points of exact order n. Taking $q=2$, $a_1=0$, $a_2=1$, we have from (3)

$$T(r,g) \leq \overline{N}(r,\infty,g) + \overline{N}(r,0,g) + \overline{N}(r,1,g) + S_1(r,g)\,, \tag{6}$$

where $S_1(r,g) = O(\log T(r,g))$ outside a set of r intervals of finite length [4].

We have

$$\overline{N}(r,0,g) = \int_{r_0}^{r}\frac{\overline{n}(t,0,g)}{t}dt,$$

where $\overline{n}(t,0,g)$ is the number of roots of $g(z)=0$ in $r_0<|z|\leq t$, each multiple root taken once at a time. The distinct roots of $g(z)=0$ in $r_0<|z|\leq t$ are the roots of $F_n^1(z)=0$ in $r_0<|z|\leq t$. Since $F_n^1(z)$ has a singularity at $z=0$, an essential singularity at $z=\infty$ with $F_n^1(z)\neq 0,\infty$, so $\overline{n}(t,0,g)=0$. So $\overline{N}(r,0,g)=0$. Similarly $\overline{N}(r,\infty,g)=0$. So

$$T(r,g) \leq \overline{N}(r,1,g) + S_1(r,g)$$

Again $F_n^1(z) = z$ when $g(z) = 1$.
So

$$\begin{aligned}\overline{N}(r,1,g) &= \overline{N}\left(r,0,F_n^1 - z\right) \\ &\leq \sum_{j=1}^{n-1} \overline{N}\left(r,0,F_j^1 - z\right) + O(\log r),\end{aligned}$$

here $O(\log r)$ arises due to the assumption that $f_1(z)$ has only a finite number of fix points of exact order n.

We have using (6),

$$\begin{aligned}T(r,g) &\leq \sum_{j=1}^{n-1} \overline{N}\left(r,0,F_j^1 - z\right) + O(\log r) + O(\log T(r,g)) \\ &\leq \sum_{j=1}^{n-1} T\left(r,F_j^1\right) + O(\log T(r,g)) + O(\log r) \\ &= T\left(r,F_n^1\right)[\{\frac{T\left(r,F_{j_1}^1\right)}{T\left(r,F_n^1\right)} + \frac{T\left(r,F_{j_{k+1}}^1\right)}{T\left(r,F_n^1\right)} + ... + \frac{T\left(r,F_{j_{kp_1-(k-1)}}^1\right)}{T\left(r,F_n^1\right)}\} \\ &\quad +\{\frac{T\left(r,F_{j_2}^2\right)}{T\left(r,F_n^2\right)} + \frac{T\left(r,F_{j_{k+2}}^2\right)}{T\left(r,F_n^2\right)} + ... + \frac{T\left(r,F_{j_{kp_2-(k-2)}}^2\right)}{T\left(r,F_n^2\right)}\}\frac{T\left(r,F_n^2\right)}{T\left(r,F_n^1\right)} \\ &\quad +............ \\ &\quad +............ \\ &\quad +\{\frac{T\left(r,F_{j_k}^1\right)}{T\left(r,F_n^k\right)} + \frac{T\left(r,F_{j_{2k}}^1\right)}{T\left(r,F_n^k\right)} + ... + \frac{T\left(r,F_{j_{kp_k}}^1\right)}{T\left(r,F_n^k\right)}\}\frac{T\left(r,F_n^k\right)}{T\left(r,F_n^1\right)} \\ &\quad +\frac{O\left(\log\left\{T\left(r,F_n^1\right)\left(1+\frac{O(\log r)}{T(r,F_n^1)}\right)\right\}\right)}{T\left(r,F_n^1\right)} + \frac{O(\log r)}{T\left(r,F_n^1\right)}], \quad \text{where}\end{aligned}$$

$j_1,\ j_{k+1}, ...,\ j_{kp_1-(k-1)};\ j_2,\ j_{k+2}, ...,\ j_{kp_2-(k-2)}; ...;\ j_k, j_{2k}, ...,\ j_{kp_k}$ are strictly less than n and are of the form $kp_1-(k-1), kp_2-(k-2), ..., kp_k, (p_1,p_2,...,p_k \in \mathbb{N})$

$$\begin{aligned}&< T\left(r,F_n^1\right)[\frac{n-1}{2kn} + \frac{n-1}{2kn} + ... + \frac{n-1}{2kn} \\ &\quad +\frac{O\left(\log\left\{T\left(r,F_n^1\right)\left(1+\frac{O(\log r)}{T(r,F_n^1)}\right)\right\}\right)}{T\left(r,F_n^1\right)} + \frac{O(\log r)}{T\left(r,F_n^1\right)}], \text{ for all large } r, \text{ by}\end{aligned}$$

$$\text{Lemma 2.1 and since } \frac{T\left(r,F_n^i\right)}{T\left(r,F_n^1\right)},\ i=2,3,..,k \text{ are bounded}$$
$$< \quad \frac{1}{2}T\left(r,F_n^1\right).$$

Therefore, $T\left(r,g\right)<T\left(r,F_n^1\right)$, for all large r. This contradicts (5).
Therefore $f_1\left(z\right)$ has infinity many k-relative fix points of exact order n.
This completes the proof. □

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