

A NOTE ON VARIOUS TYPES OF CONES AND FIXED POINT RESULTS IN CONE METRIC SPACES

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ABSTRACT. Various types of cones in topological vector spaces are discussed. In particular, the usage of (non)-solid and (non)-normal cones in fixed point results is presented. A recent result about normable cones is shown to be wrong. Finally, a Geraghty-type fixed point result in spaces with cones which are either solid or normal is obtained.

1. INTRODUCTION

The usage of cones in vector spaces and their connection with order relations is a standard algebraic procedure. Ordered Banach and, later, topological vector spaces have been used since 1940's (see [18, 19] and [25, 27]).

The idea to use an ordered normed space instead of the set of real numbers as the codomain of a metric goes back to 1934 (Đ. Kurepa [20], see also [22]). Later, such spaces were called K -metric spaces and used to obtain several results, including fixed point ones (see a survey in [28]).

K -metric spaces (sometimes also called abstract metric spaces) were re-introduced in 2007 by L.G. Huang and X. Zhang [12] under the name of cone metric spaces. A novelty was the use of a relation \ll which could be defined under the supposition that the underlying cone had a nonempty interior (such cones are usually called solid). Thus, convergent sequences and completeness could be defined in a new way. Later, these definitions were extended for tvs-cone valued cone metric spaces in [5] (see also [8, 14]).

It should be noted that each cone metric space over a solid cone is metrizable, as was shown by various methods in [2, 3, 6, 8, 10, 15, 17]. However, not all fixed point results can be reduced in this way to their standard metric counterparts, so this line of investigation is still open (see, e.g., the recent paper [16]).

In [12], the underlying cones were supposed to be normal (see definition in the next section). It was shown in [24] that such assumption is usually not necessary. Hence, further, a lot of authors obtained (common) fixed point results for nonnormal solid cones (see a survey of these results until 2011 in [13]).

M. Asadi et al. [4] tried to prove that each cone in a Banach space can, by re-norming of the space, become normal with the normal constant equal to 1 (such cones are usually called monotone). We will show in Section 3 of this paper that their Theorem 2.1 contradicts some well known results and must be wrong.

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Solidness of the cone is an important assumption since without it interior points cannot be used to introduce convergence and no natural corresponding topology can be defined. Nevertheless, some results in the case when the cone is not solid were obtained by Kunze et al. in [21]. However, their approach is quite restrictive since rather strong assumptions (separability and reflexivity of the space) are used. Since there are a lot of examples of non-solid cones in spaces that do not satisfy these assumptions, a corresponding theory would be welcome.

In Section 4 of this paper we present a Geraghty-type fixed point result using cones which are either solid or normal. Thus, we generalize a result from [2], since we do not use dual bases of the cones.

2. PRELIMINARIES

Let E be a real Hausdorff topological vector space (*tvs* for short) with the zero vector θ . A proper nonempty and closed subset K of E is called a *cone* if $K + K \subset K$, $\lambda K \subset K$ for $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. If the cone K has a nonempty interior $\text{int } K$ then it is called *solid*.

Example 2.1. The easiest examples of solid cones are $\{x = (x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ and $\{x \in C[a, b] : x(t) \geq 0, a \leq t \leq b\}$.

On the other hand, similarly defined cones in many important spaces of functional analysis have empty interiors. These are, for example, c_0 , l^p ($p > 0$), L^p ($p > 0$) [1, 7, 27].

Each cone K induces a partial order \preceq on E by $x \preceq y \Leftrightarrow y - x \in K$. $x \prec y$ will stand for $(x \preceq y \text{ and } x \neq y)$. In the case when the cone K is solid, $x \ll y$ will stand for $y - x \in \text{int } K$. The pair (E, K) is an *ordered topological vector space*.

For a pair of elements x, y in E such that $x \preceq y$, put $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset A of E is said to be *order-convex* if $[x, y] \subset A$, whenever $x, y \in A$ and $x \preceq y$. Ordered topological vector space (E, K) is *order-convex* if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, the cone K is said to be *normal*.

If E is a normed space, the last condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number k such that $x, y \in E$ and $0 \preceq x \preceq y$ implies that $\|x\| \leq k\|y\|$. The minimal such constant k is called the normal constant of K . Obviously, the normal constant is always greater or equal to 1. In the case when $k = 1$, i.e., when $0 \preceq x \preceq y$ implies that $\|x\| \leq \|y\|$, the cone is called *monotone*.

The next lemma contains a result on cones in ordered Banach spaces that is rather old (1940, see [18]). It is interesting that most authors (working with normal cones after 2007) do not use this result, which can be applied to reduce a lot of results to the setting of ordinary metric spaces.

Lemma 2.2. [18] *The following conditions are equivalent for a cone K in the Banach space $(E, \|\cdot\|)$:*

- (1) K is normal;
 - (2) for arbitrary sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in E ,
- (2.1) $(\forall n) x_n \preceq y_n \preceq z_n$ and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x$ imply $\lim_{n \rightarrow \infty} y_n = x$;
- (3) there exists a norm $\|\cdot\|_1$ on E , equivalent to the given norm $\|\cdot\|$, such that the cone K is monotone w.r.t. $\|\cdot\|_1$.

Proof of this important lemma can be found, e.g., in [1, 7, 26, 27].

Example 2.3. [26] Let $E = C_{\mathbb{R}}^1[0, 1]$, with $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$, $K = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. This cone is solid but non-normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n \rightarrow \infty} y_n = \theta$, but $\|x_n\| = \max_{t \in [0, 1]} \left| \frac{t^n}{n} \right| + \max_{t \in [0, 1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence $\{x_n\}$ does not converge to zero. It follows that K is a non-normal cone.

Now consider the set $E = C_{\mathbb{R}}^1[0, 1]$ endowed with the strongest locally convex topology t^* . Then K is also t^* -solid (it has the nonempty t^* -interior), but not t^* -normal. Indeed, if it were normal then, according to Theorem 2.4 (see further), the space (E, t^*) would be normed, which is impossible since an infinite-dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [25]).

Note the following properties of bounded sets in an ordered *tvs* E .

If the cone K is solid, then each topologically bounded subset of (E, K) is also order-bounded, i.e., it is contained in a set of the form $[-c, c]$ for some $c \in \text{int } K$. On the other hand, if the cone K is normal, then each order-bounded subset of (E, K) is topologically bounded. Hence, if the cone is both solid and normal these two properties of subsets of E coincide. Moreover, a proof of the following assertion can be found, e.g., in [26].

Theorem 2.4. *If the underlying cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space.*

Following [5, 8, 12, 14], we give the following

Definition 2.5. Let \mathcal{X} be a nonempty set and (E, K) be an ordered *tvs*. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow E$ is called a *tvs-cone metric* and (\mathcal{X}, d) is called a *tvs-cone metric space* if the following conditions hold:

- (c1) $\theta \preceq d(x, y)$ for all $x, y \in \mathcal{X}$ and $d(x, y) = \theta$ if and only if $x = y$;
- (c2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
- (c3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Further, we distinguish two possibilities.

Case 1. Suppose that the cone K in E is solid. We will call a sequence $\{u_n\}$ in K a *c-sequence* [9] if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are *c-sequences* in K and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a *c-sequence*. We will say that a sequence $\{x_n\}$ in (\mathcal{X}, d) *c-converges* to $x \in \mathcal{X}$ (denoted as $x_n \xrightarrow{c} x, n \rightarrow \infty$) if $\{d(x_n, x)\}$ is a *c-sequence* in K . Similarly, Cauchy sequences and completeness of the space are defined, see [12, 13, 14] and references therein.

Case 2. The cone K is arbitrary (i.e., possibly non-solid). We will call a sequence $\{u_n\}$ in K a θ -*sequence* if $u_n \rightarrow \theta$ as $n \rightarrow \infty$ (in the sense of the topology of E). We will say that a sequence $\{x_n\}$ in (\mathcal{X}, d) *d-converges* to $x \in \mathcal{X}$ (denoted as $x_n \xrightarrow{d} x, n \rightarrow \infty$) if $\{d(x_n, x)\}$ is a θ -sequence in K .

In the case of a solid cone K , it is easy to prove that if $\{u_n\}$ is a θ -sequence in K then it is also a *c-sequence* in K . Hence, each *d-convergent* sequence in (\mathcal{X}, d) is *c-convergent*. The converse does not hold in general, as Lemma 2.2 and Example 2.3 show. If the cone K is both solid and normal, then these two types of convergence coincide.

3. RESULTS FOR (NON)-NORMAL CONES

In the case when the underlying cone K of a (tvs-valued) cone metric space (\mathcal{X}, d) is solid and normal, most of the fixed point problems can be reduced to their standard metric counterparts. It is enough to apply Theorem 2.4 and Lemma 2.2 and, after possible re-norming of the codomain space, use the metric $D(x, y) = \|d(x, y)\|$ (see details in [13]).

In an attempt to extend this procedure to arbitrary cones, M. Asadi et al. presented the following result ([4, Theorem 2.1]).

Let $(E, \|\cdot\|)$ be a real Banach space with a positive cone K . There exists a norm on E such that K is a normal cone with constant $k = 1$, with respect to this norm.

Moreover, they stated in Remark 2.1 that these two norms are equivalent.

However, this result cannot be true (an error appears in the last step of the proof, when proving that K is monotone in the new norm). Indeed, taking into account Lemma 2.2, this would imply that all cones in Banach spaces are normal, which is obviously not true (see Example 2.3).

Hence, fixed point theory in (tvs-valued) cone metric spaces over non-normal cones remains a possible area of investigations.

4. RESULTS FOR (NON)-SOLID CONES

We first present the following variant of [21, Proposition 1].

Proposition 4.1. *Let K be a solid cone in a tvs E and $\{u_n\}$ be a sequence in K . Then the following conditions are equivalent:*

- (1) $\{u_n\}$ is a c -sequence.
- (2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.
- (3) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \preceq c$ for $n \geq n_0$.

Proof. (1) \implies (2) and (2) \implies (3) are obvious. Let (3) hold and let $c \gg \theta$ be arbitrary. Then $\frac{c}{2} \gg \theta$ and choosing n_0 such that $n \geq n_0$ implies $u_n \preceq \frac{c}{2} \ll c$ we get that $u_n \ll c$ for $n \geq n_0$. \square

As a consequence, the usage of relation \ll can be avoided in the formulations and proofs concerning c -convergence.

If the underlying cone K of a cone metric space is normal (solid or not), then using the metric $D : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ given by $D(x, y) = \|d(x, y)\|$ (after possible renorming, see Lemma 2.2), all the results from [12] (and other papers that use normal cones) can be reduced to their standard metric counterparts (see, e.g., [13]).

In particular, the cone metric d is in this case a continuous function in two variables (which may not be the case if the cone is not normal). Indeed, the following lemma can be proved without using interior points of the cone.

Lemma 4.2. [12, Lemma 5] *Let (\mathcal{X}, d) be a cone metric space over a normal cone K . If $\{x_n\}$ and $\{y_n\}$ are sequences in \mathcal{X} such that $x_n \xrightarrow{d} x$ and $y_n \xrightarrow{d} y$, then $d(x_n, y_n) \rightarrow d(x, y)$ in E .*

Proof. From

$$d(x_n, y_n) \preceq d(x_n, x) + d(x, y) + d(y, y_n)$$

and

$$d(x, y) \preceq d(x, x_n) + d(x_n, y_n) + d(y_n, y),$$

it follows that

$$-(d(x_n, x) + d(y_n, y)) \preceq d(x_n, y_n) - d(x, y) \preceq d(x_n, x) + d(y_n, y).$$

It follows from the Sandwich Theorem (which holds in the case of a normal cone, see Lemma 2.2) that $d(x_n, y_n) \rightarrow d(x, y)$ as $n \rightarrow \infty$. \square

The following Geraghty-type fixed point result is a generalization of [2, Theorem 2.2] since it does not use any conditions concerning dual cones.

Theorem 4.3. *Let (\mathcal{X}, d) be a complete tvs-cone metric space over a cone K which is solid or normal. Let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping satisfying*

$$d(Tx, Ty) \preceq \alpha(d(x, y))d(x, y),$$

for all $x, y \in \mathcal{X}$, where $\alpha : K \rightarrow [0, 1)$ has the property that $\alpha(u_n) \rightarrow 1$ implies $u_n \rightarrow \theta$ in E as $n \rightarrow \infty$. Then T has a unique fixed point x^* and $T^n x \rightarrow x^*$ in \mathcal{X} as $n \rightarrow \infty$ for each $x \in \mathcal{X}$.

Proof. Suppose that the cone K is solid. Let $e \in \text{int } K$ be arbitrary and let q_e be the corresponding Minkowski functional of $[-e, e]$. Then $d_q = q_e \circ d$ is a (real-valued) metric on \mathcal{X} (see [15, Theorem 3.1]). Moreover,

$$d_q(Tx, Ty) \leq \alpha(d(x, y))d_q(x, y)$$

holds for all $x, y \in \mathcal{X}$. Let $x \in \mathcal{X}$ be arbitrary and denote $x_n = T^n x$, $n \in \mathbb{N}$. Supposing that $x_{n+1} \neq x_n$ for each n , we have that

$$d_q(x_{n+1}, x_n) = d_q(Tx_n, Tx_{n-1}) \leq \alpha(d(x_n, x_{n-1}))d_q(x_n, x_{n-1}) < d_q(x_n, x_{n-1}),$$

i.e., $\{d_q(x_{n+1}, x_n)\}$ is a strictly decreasing sequence in \mathbb{R}^+ . It follows that $d_q(x_{n+1}, x_n) \rightarrow d^* \geq 0$ as $n \rightarrow \infty$. Suppose that $d^* > 0$. Then

$$0 < \frac{d_q(x_{n+1}, x_n)}{d_q(x_n, x_{n-1})} \leq \alpha(d(x_n, x_{n-1})) < 1,$$

implying that $\alpha(d(x_n, x_{n-1})) \rightarrow 1$ as $n \rightarrow \infty$. It follows that $d(x_n, x_{n-1}) \rightarrow \theta$ in E as $n \rightarrow \infty$. As q_e is a continuous semi-norm, we get that $q_e(d(x_n, x_{n-1})) \rightarrow q_e(\theta) = 0$, i.e., $d_q(x_n, x_{n-1}) \rightarrow 0$ as $n \rightarrow \infty$, a contradiction.

Suppose now that $\{x_n\}$ is not a Cauchy sequence in the metric space (\mathcal{X}, d_q) . Using [23, Lemma 2.1], we obtain that there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, such that $d_q(x_{2n_k+1}, x_{2m_k}) \rightarrow \varepsilon$ and $d_q(x_{2n_k}, x_{2m_k-1}) \rightarrow \varepsilon$ as $k \rightarrow \infty$. Then

$$d_q(x_{2n_k+1}, x_{2m_k}) \leq \alpha(d(x_{2n_k}, x_{2m_k-1}))d_q(x_{2n_k}, x_{2m_k-1}),$$

implying that

$$0 < \frac{d_q(x_{2n_k+1}, x_{2m_k})}{d_q(x_{2n_k}, x_{2m_k-1})} \leq \alpha(d(x_{2n_k}, x_{2m_k-1})) < 1,$$

and $\alpha(d(x_{2n_k}, x_{2m_k-1})) \rightarrow 1$. Hence, $d(x_{2n_k}, x_{2m_k-1}) \rightarrow \theta$ as $n \rightarrow \infty$. Again, as before, it follows that $d_q(x_{2n_k}, x_{2m_k-1}) \rightarrow q_e(\theta) = 0$ as $n \rightarrow \infty$, a contradiction.

It follows that $\{x_n\}$ is a d_q -Cauchy sequence, and hence also a d -Cauchy sequence in \mathcal{X} . Hence, there exists $x^* \in \mathcal{X}$ such that $x_n \xrightarrow{d} x^*$ as $n \rightarrow \infty$. The rest of the proof is standard.

If the cone K is normal, then the space can be renormed such that the normal constant of the cone becomes equal to 1 (see Lemma 2.2). Further, a similar procedure as before can be done, using the metric $D(x, y) = \|d(x, y)\|$. \square

As corollaries, we obtain [2, Theorem 2.2] (without using dual bases of the cones) and the classical Geraghty result from 1979 [11].

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