ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2013, Article ID ama0105, 11 pages
ISSN 2307-7743
http://scienceasia.asia

ANOTHER EFFICIENT METHOD OF SOLVING WEIGHTED GOAL PROGRAMMING PROBLEM

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Abstract. Literature recorded that articles that used weighted goal programming in pre-1990 application papers is 25% whereas its use in the period 1990–2000 is 41%. The reasons for this increment are due to the greater flexibility by the weighted goal programming, and the aim of the decision makers to do more 'trade-off' analysis and direct comparison between goals. An efficient solution algorithm for solving weighted goal programming problems is presented. The algorithm incorporates rigid constraints in the model and utilized variant of simplex method. A step by step illustration of the procedure is provided. The computational method is efficient.

1. Introduction

In 1948, Dantzig [1] developed a systematic procedure for solving linear programming problems (LP) called simplex method which is efficient in practice and is a great improvement over earlier methods of solving LP such as Fourier-Motzkin elimination, even though it is exponential in complexity. Lp had received substantial international exposure and attention, and was hailed as one of the major developments of applied mathematics. Today, linear programming is the most widely employed of the methods used by those in such field as operations research and management sciences. However, as with many quantitative approaches to the modeling and solution of real problems, LP has its blemish, drawback and limitations. Its inability to directly and efficiently handle problems involving multiple objectives and goals: thus restricting the users of the technique to narrowing their problems to a single objective function. But in real life, decision situations frequently are characterized by multiple and conflicting objectives. For instance, Rifai [2] explained that, besides profit maximization, management has many other objectives to attain, including improved market share, maintaining full employment, providing quality ecological management, improved relationships with the public, production of a quality product, minimized pollution and the realization of a certain rate of return in investment. Many of these objectives may conflict, and so compete for scarce resources. LP is an inadequate technique for solving problems of this type. One way of handling problems with multiple objectives is to choose one of the goals as the supreme goal and to treat the others as constraints to ensure that some minimal 'satisficing' level of the other goals is achieved.

2010 Mathematics Subject Classification: 90C29.
Key words and phrases: weighted goal programming problem.

As a result of this drawback, a number of programming methods have evolved based on the Dantzig algorithm to solve programming problem of this situations. One of such class of problems is referred to as goal programming problems.

Goal programming (GP) can be considered as a popular branch of multi-objective optimization which itself is a part of multi-criteria decision analysis. This can also be considered as a generalization or extension of linear programming, in which more than one and in general conflicting objectives are to be optimized. GP technique was initially developed by Charnes and Cooper [3], Lee [4], Ignizio [5] and several others to handle multiple objective situations that cannot be addressed by linear programming. The essence of the technique is the achievement of the "best possible" solution, which comes as close as possible to meeting the goals.

However, Olson [6], described lack of an algorithm capable of reaching a compromised solution in a reasonable time as a major setback in applying GP.(see also Schniederjans and Kwaks [7] , Baykasoglu [8]. The most commonly used goal programming solution methods were introduced by Lee and Ignizio based on Dantzig's simplex method (see Authur and Ravindran [9], and Schniederjans and kwaks. Both methods require columns in the simplex tableau for positive and negative deviational variables. They also require separate objective function rows for each priority level, all of which add immensely to the computational time of the solution method. Tamiz and Jones [10] pointed out that a faster mathematical programming based solution method is given by Ignizio [11]. This has the same effect as imposing constraints on the unwanted deviational variables but reduces the problem in terms of number of variables and size of feasible region while keeping the number of constraints constant. Orumie and Ebong [12] developed an efficient method of solving a generalized linear goal programming problem.

However, weighted goal programming and Lexicographic goal programming are the most popular variants of the GP model as shown in Alp et al [13] and many others. The two methods do not generate the same solution and neither is one method superior to the other because, variant is designed to satisfy certain decision makers' preferences.
The purpose of this research is to present an efficient solution method for strictly linear weighted goal programming problems. In section two, the linear weighted goal programming problems will be formulated whereas the statement of problems and methodology will be presented in section three and four respectively. The new algorithm for solving weighted goal programming will be formulated in section five, followed by step-by-step illustration of a case problem in section six. Section seven and eight presents the conclusion and further research respectively.

2 Weighted Goal Programming (Non-Pre-emptive)

If the decision maker is more interested in direct comparisons of the objectives, then weighted or non pre-emptive goal programming (WGP) should be used. This variant allows for direct trade-offs between all the deviational variables to be penalized by placing them in a weighted, normalised single achievement function. I.e all the unwanted deviations are multiplied by weights, reflecting their relative importance, and added together as a single sum to form the achievement function.(Weights are attached to each of the objectives to measure the relative importance of deviations from their target). I.e It is assumed that all the goals are of equal important with only slight differences that can be measured by assigning weights to these goals. WGP handles several objectives simultaneously by establishing a specific numeric goal for each of the objectives and then find a solution that comes close to each of these goals. The overall purpose is to minimize the weighted sum of deviations of these objective functions from their respective target values. It is important to realize the fact that deviational variables measured in different units cannot be added because of the principles of incommensurability. Hence each unwanted deviation is multiplied by a normalization constant to allow direct comparison. Popular choices for normalization constants are the goal target value of the corresponding objective (hence turning all deviations into percentages) or the range of the corresponding objective (between the best and the worst possible values, hence mapping all deviations onto a zero-one range).

The algebraic formulation of a WGP as presented in Min and Storbeck [14] is given as:

$$\min \quad z = \sum_{i}^{m} (w_i^- d_i^- + w_i^+ d_i^+) \tag{1}$$

s.t

$$\sum_{j}^{n} a_{ij} x_{ij} + d_i^- - d_i^+ = b_i \qquad (i = 1,2,\ldots, m), \tag{2}$$

$$x_{ij}, d_i^-, d_i^+ \geq 0, w_i > 0 \tag{3}$$

$$(i = 1,2,\ldots, m : \quad j = 1,2,3\ldots, n) \tag{4}$$

where w_i^-, w_i^+ are non-negative constants assigned to negative and positive deviational variable d_i^- *and* d_i^+ (≥ 0) respectively. Equation (1) represents the achievement function which measures the unwanted weighted sum of all the goal deviational variables to be penalized. Equation (2) is the objective goal whose deviation from the target value b_i is to be minimized in the achievement function. Tamiz et al.[15] reported that articles that used weighed goal programming in pre-1990 application papers is 25%. Tamiz and Jones [16] reported that its use in the period 1990–2000 41%. The reasons for this increment are due to the greater flexibility by the weighted goal programming, and the aim of the decision makers to do more 'trade-off' analysis and direct comparison between goals.

3. Statement of problems

According to Min and Storbeck[14], formulation of a WGP is given in equations (1)–(4). The above statement (model) omitted the fact that a goal programming model may include a rigid constraints like $\sum_{j}^{n} a_{ij} x_j \leq b_i$. In order to incorporate the rigid constraints in the model, the researcher decided to augument the above rigid constraints to equations (1)–(4) and to develop new algorithm for linear weighted goal programming as will be shown in section five.

4 Methodology/Initialization (Initial Table formulation)

The new goal programming algorithm is formulated into initial tableau in the same format with that of Orumie and Ebong . The different is that in the new algorithm, the deviational variables that appeared in the achievement functions with their weight attached to them, together with slacks from the rigid constraints if exists forms the basis and the column of the slacks variables from the rigid constraints added to the non basic. The table 4.1 below represents the initial table. The decision variables x_j, the slack variables from the rigid constraints, and the deviational variable that appeared in the achievement function are placed in row 1, column2. Last Column constitute the right-hand-values, b_i. The coefficient of the decision variables a_{ij}, the slack variables (s) from the rigid constraints, and the deviational variable that appeared in the achievement function ($c_{ij}^{(v)}$) which constitute the coefficient matrix G $=[(g_{ij})]$, where $g_{ij}=\{a_{ij}, s, c_{ij}^{(v)}\}$. ∋ $g_{i.}=(a_{i1}, a_{i2}, \ . \ . \ . \ a_{in}, s_i, c_{i1}, c_{i1}, \ . \ . \ .c_{it},)$ be the ith row of G and $g_{.j}$ the jth column of G are placed in column 2. The weihted factors w_f for the deviational variable, including slacks variables are presented in column 1 which forms the basis.

It starts by not including the deviational variables column that did not appear in the basis from the tableau (i.e while searching for the optimal solution), but others can be augmented when necessary since positive deviational variables columns coefficient is the same as negative of the negative deviational column coefficient as represented in table (4.1) below. This is because $d_i^- = -d_i^+$. The new algorithm starts with the highest weighted row constraints.

Let $d_i^v = d_i^+$ or d_i^- $\in$ z ∋ $v=\begin{cases} + & if \quad v \quad is \quad positive \quad sign \\ - & if \quad v \quad is \quad positive \quad sign \end{cases}$

Let $c_i^{(v)}$ be the coefficient of $d_i^v \in$ z in the basis, then initially, the coefficient of $d_i^{(v)}$ is given by

$$c_i^{(v)} = \begin{cases} 1 & \text{if } v \text{ is negative sign} \\ -1 & \text{if } v \text{ is negative sign} \end{cases}$$

Here $d_i^{(v)}$ are the deviational variables in the objective function, and t is the number of deviational variables in the objective function. Let g_{hj} be the coefficient of weighted row(s) corresponding to w_k with initial table below.

(Table 4.1) Innitial Table of the new algorithm

Variable in basis with $w_k.C_B$	X_1 X_2 ... X_n S $d_1^{(v)}$ $d_2^{(v)}$ . . .$d_t^{(v)}$	Solution value b_i (rhs)
	a_{11} a_{12}. ... a_{1n} s_1 $c_{11}^{(v)}$ $c_{11}^{(v)}$. . . $c_{1t}^{(v)}$	b_1
	a_{21} a_{22} ... a_{2n} s_2 $c_{21}^{(v)}$ $c_{22}^{(v)}$. . .$c_{2t}^{(v)}$	b_2
	. . . .	. . .
	a_{m1} a_{m2} ... a_{mn} s_m $c_{m1}^{(v)}$ $c_{m2}^{(v)}$. . . $c_{mt}^{(v)}$	b_m

5 The New Algorithm For Weighted Goal Programming (WGP)

Consider the weighted goal programming where weights are assigned to the goal constraints, (deviational variables in the achievement function) .The WGPP formulation for n variables, m goal constraints, t deviational variables and F weight factors that that includes the rigid constraints is defined below:

$$\min \quad z = \sum_{f}^{F} (w_f^- d_{i_t}^- + w_f^+ d_{i_t}^+) \qquad \text{for } f \subset i_t \subset \{1,2, \ . \ . \ ., m\} \tag{5}$$

such that

$$\sum_{i}^{m} a_{ij} x_j + d_i^- - d_i^+ = b_i \tag{6}$$

$$\sum_{j}^{n} a_{ij} x_j \le b_i \tag{7}$$

Where

$$w_f \ge 0, \quad x_j, d_i^+, d_i^- \ge 0 \, , \; x_{ij}, d_i^-, d_i^+ \ge 0 \tag{8}$$

for $(i = 1,2,\ldots, m : \quad j = 1,2,3\ldots, n)$ (3)

where w_f= f^{th} weighted factor f= 1,2, . . .F

$(w_f^- d_{i_f}^+ + w_f^+ d_{i_f}^+)$ are set of the weighted sum of the deviational variables in z.

e.g

Min $(4d_2^+ + 3d_1^- + 2(d_3^- + d_4^+))$

s.t.

$$x_1 + 3x_2 \le 120$$
$$2x_1 + x2 + 2x_3 \le 80$$
$$10x_1 + 15x_2 + d_1^- - d_1^+ = 100$$
$$x_1 + 2x_2 + d_2^- - d_2^+ = 50$$
$$2x_1 + x_2 + x_4 + d_3^- - d_3^+ = 100$$
$$x_1 + d_4^- - d_4^+ = 80$$

$x_1, x_2, x_3, x_4, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \ge 0$

f=1,2,3, j=1,…4, i=1,2,. . .6, t=1,2,3,4

The algorithm (WGP):

Let $w_f^v = w_f^-$ or w_f^+ be the f^{th} weight assigned to d_i^v,then,

Step 1. **Initialization:**

Set $f \leftarrow 1$

Step 2. **Feasibility**:

If $b_i = 0$ for all i=1,2,. . ., m, go to Step 8. {solution optimal}

Set $b_i \leftarrow |b_i|$ for all I =1,2,. . ., m, {ensure feasibility}

Step 3. **Optimality test:**

If $g_{hj} \le 0$ for all j $\neq$ *pivot column* , $h \in i_f$ go to Step 7.

{all coefficients of weighted row, h ,non positive ,so $w_f^v : v = +$ or $-$ is satisfied}

Step 4. **Entering variable:**

Entering variable is the variable with highest positive coefficient in the row $g_{h.}, h \in i_f$ for the $w_f^v(d_{i_t}^- + d_{i_t}^+)$ rows of the objective functions which does not violate the satisfied weighted deviational variable. If $g_{h.}, h \in \{1,2,...F\}$ is the highest coefficient, but has been previously satisfied or has more weight attached to it than the leaving variable under consideration then consider the next higher value on the same row, otherwise go to step 7.

(The weight attached to the entering variable should be placed alongside with it into the basis).

In case of ties $\{g_{hj_1}, \ldots, g_{hj_\theta}\}$, then the entering variable is the variable for which $\min_h \left\{ {b_i}/{g_{hj_\rho}} : g_{hj_\rho} > 0 \right\}$ is maximum. $j_\rho : \rho = 1, \ldots, \theta$ {ties in the weighted rows}.

Step 5. **Leaving variable**:

If y_0 is the column corresponding to the entering variable in step 4, then the leaving variable is the basic variable with minimum $\left\{ {b_i}/{g_{.y_0}} : g_{.y_0} > 0, i = 1, 2, \ldots m \right\}$. $\{g_{.y_0}$ is the pivot column$\}$.

In case of ties, the variable with the smallest right hand side leaves the basis.

Step 6. **Interchange basic variable with non basic**:

Perform Gauss Jordan row operations to update the table.If w_f^v is still in the basis (C_B), go to Step 3.

Step 7. **Increment the process:**

Set $f \leftarrow f + 1$.If $f \leq F$, go to Step 3. Satisfied weighted variable will not reenter for the lesser one to leave, instead the next highest coefficient variable enters.

Step 8. **Solution is optimal:**

i.) The coefficient of the weighted rows are all negative or zero

ii.) The right hand sides of the weighted rows are all zero

iii.) The weighted rule is satisfied.

The optimal solution is the value of $w_f^v(d_i^- + d_i^+)$ in the objective function as appeared in the last iteration.

Note: Just as in the method of artificial variables, a variable of higher or equal weight that has been satisfied should not be allowed to re-enter the table. In this case the next higher coefficient of g_{hj} will be considered.

6 Step by step illustration of the algorithm on weighted goal programming problems. (The algorithm WGP)

(i) $\min \ z = (2d_1^- + d_2^+)$

s.t $\quad 4x_1 + 8x_2 + d_1^+ - d_1^+ = 45$

$\quad 8x_1 + 24x_2 + d_2^- - d_2^+ = 100$

$x_1 + 2x_2 \leq 10$

$x_1 \leq 9$

$x_i \geq 0, d_i^+, d_i^- \geq 0,\ i = 1,2,\ \ j = 1,2,\ F = 2,$

Source: Cohon, [17]

The G matrix of the above problem is $\begin{bmatrix} 4 & 8 & 1 & 0 & 0 & 0 \\ 8 & 24 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ with the b_i $= \begin{bmatrix} 45 \\ 100 \\ 10 \\ 9 \end{bmatrix}$.

Table (6.1) below is the initial table of the problem (i). The first column is the column of the deviational variables to be minimized with weight attached to them and the slacks from the hard constraints (basic variables). s_1 and s_2 are the slacks variables attached to the third and forth constraints respectively (dummy variables) .

Table (6.1) Initial table of problem i

	X_1	X_2	d_1^-	d_2^+	S_1	S_2	RHS
$w_1\, d_1^-$	4	8	1	0	0	0	45
$w_2\, d_2^+$	8	24	0	-1	0	0	100
S_1	1	2	0	0	1	0	10
S_2	1	0	0	0	0	1	9

x_2 enter

d_2^+ leaves

Step1. Set $f = 1$.

Step 2. $\exists\ b_i \neq o\ \ for\ \ i = 1,\ldots,\ m$.{ feasible.}

Step 3. $\exists\ g_{1j} > 0\ \ for\ \ for\ \ some\ \ j$. {Solution not optimal}

Step4. Max$\{g_{1j}\}$ = max{4,8,1,0,0,0}= 8 at g_{12} . So,x_2 enters.

Step5. Min$\left\{\frac{b_h}{g_{i2}} : g_{i2} > 0\right\}$ = min $\{\frac{45}{8}, \frac{100}{24}, 5\} = \frac{100}{24}\ at\ \frac{b_2}{g_{22}}$. d_2^+ leaves the basis.

Step6. Perform the normal Gauss Jordan row operations to update the new table (see table 6.2) and check if w_1 is still in the basis (C_B) to test for optimality.

Table (6.2) 1st Iteration for problem i

	x_1	x_2	d_1^-	d_2^+	S1	S2	RHS
W1 d_1^-	4/3	0	1	1/3	0	0	35/3
x_2	1/3	1	0	-1/24	0	0	25/6
S1	1/3	0	0	1/12	1	0	5/3
S2	1	0	0	0	0	1	9

x_1 enter

S_1 leaves

In table (3.5.2), w_1 is not satisfied, so we go back to step 3.

Step3. $\exists\, g_{1j} > 0$ *for for some* j. {Solution not optimal}

Step4. $\exists \max\{ g_{1j}\} = \text{Max}\{\frac{4}{3}, 0, 1, 1/3, 0, 0\} = 4/3$ *at* g_{11} .x_1 enters.

Step5. $\text{Min}\left\{\frac{b_h}{g_{i1}} : g_{i1} > 0\right\} = \min\ \{\frac{35}{4}, \frac{25}{2}, 5, 9\} = 5$ *at* $\frac{b_3}{g_{31}}$. s_1 leaves.

Step6. Perform the normal gauss Jordan row operations to update the tableau (see table 6.3) and check if w_1 is still in the basis (C_B) to test for optimality.

Table (6.3) 2nd Iteration for problem i

	x_1	x_2	d_1^-	d_2^+	S1	S2	RHS
W1 d_1^-	0	0	1	0	-4	0	5
x_2	0	1	0	-1/8	-1	0	5/2
x_1	1	0	0	¼	3	0	5
S2	0	0	0	-1/4	-3	1	4

In this table, w_1 is not satisfied, so we go back to step 3.

Step3. $g_{1j} \le 0$ *for all* j . {Solution optimal}

Step4. $\exists \max\{ g_{1j}\} = \text{Max}\{0, 0, 1, 0, -4, 0\} = 1$ at d_1^+ but recall that higher weighted factor will not re-enter the basis for lower one to leave. So analyses stop.

Step8. Thus, table (6.3) is therefore an optimal tableau with z=10, x_1=5, x_2=2.5, d_1^-=5 and s_2=4.

7 Conclusion

The new method is applied to various problems of different variables sizes, goals and constraints. The proposed method is efficient in reaching optimality and its formulation represents a better model than the commonly used.

8 Area for Further Research

Further research should be carried out on the modification of the new algorithm to solve other types of GPP.

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