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SOLUTION OF THIRTEENTH ORDER BOUNDARY VALUE PROBLEMS BY DIFFERENTIAL TRANSFORMATION METHOD

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Abstract. The aim of this paper is to use the differential transformation method (DTM), a semi-numerical and semi-analytic technique for solving linear and nonlinear thirteenth order boundary value problems. The approximate solution of the problem is calculated in terms of a rapidly convergent series. Two numerical examples have been considered to illustrate the efficiency and implementation of the method and the results are compared with the method developed in [1].

1. Introduction

Higher order boundary value problems occure in the study of fluid dynamics, astrophysics, hydrodynamic, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences.
The theory of differential transformation was established by Zhou [10]. The author solved linear and non-linear IVPs arising in circuit analysis. Pukhov [4] also studied differential transformation method at the same time. The method constructs an analytical solution for differential equations in the form of a series. It is a semi-numerical and semi-analytic technique that formulizes the Taylor series in a totally different manner. It differs from the traditional higher order Taylor series method, which computationally, takes more time for higher orders BVPs. The DTM transforms the given BVP into a recurrence relation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. The method is useful to obtain both exact and approximate solutions of linear and non-linear BVPs. There is no need to discretization, linearization or perturbation, large computational work and round-off errors are avoided.
Siddiqi and Iftikhar used the variation of parameter method for solving the seventh-order boundary value problems in [7]. Siddiqi and Akram [5] used nonic spline and non-polynomial spline technique for the numerical solution of eighth-order linear special case boundary value problems. These methods have also been proven to be second order convergent. Recently, Akram and Rehman presented the numerical solution of eighth-order boundary value problems using the reproducing kernel space method [2]. Geng and Li [3] construct a reproducing kernel space and solve a class of linear tenth-order boundary value problems using reproducing

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kernel method. Siddiqi *et al.* [6] used the variational iteration technique for the solution of tenth order boundary value problem. Siddiqi and Akram [8] presented the numerical solutions of the tenth-order linear special case boundary value problems using eleventh degree spline. Siddiqi and Iftikhar [9] determined the approximate solutions of seventh, eighth, and tenth-order boundary value problems using homotopy analysis method (HAM). Adeosun *et al.* [1] presented the variational iteration method (VIM) to find the approximate solutions of linear and nonlinear thirteenth order boundary value problems
In the present paper, thirteenth-order boundary value problems are solved using DTM. The following thirteenth-order boundary value problems are considered

$$\left.\begin{array}{lcl} u^{(13)}(x) & = & f(x,u(x)), \quad a \le x \le b, \\ u^{(i)}(a) & = & A_i, \\ u^{(j)}(b) & = & B_j \end{array}\right\} \tag{1. 1}$$

where for $i = 0, 1, 2, ..., 6$ and $j = 0, 1, ..., 5$. A_i's and B_j's are finite real constants. Also $f(x, u(x))$ is a continuous function on $[a, b]$.

In the following section the differential transformation method is discussed.

2. Differential Transformation Method

The differential transformation of the kth derivative of a function $f(x)$, at $x = x_0$, is defined as

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2. 1}$$

and the inverse differential transformation of $F(k)$ is defined by

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2. 2}$$

In real applications, the function $f(x)$ can be expressed as a finite series and Eq. (2. 2) can be written as

$$f(x) = \sum_{k=0}^{N} F(k)(x - x_0)^k. \tag{2. 3}$$

Substituting $F(k)$ from Eq. (2. 1) into Eq. (2. 2), gives

$$f(x) = \sum_{k=0}^{\infty}(x - x_0)^k \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}, \tag{2. 4}$$

which is the Taylor series for $f(x)$ at $x = x_0$. From Eq. (2. 1) and Eq. (2. 2), the following theorems can be proved.
Theorem 1 If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Proof Using the definition (2. 1), yields

$$G(k) = \frac{1}{k!}\left[\frac{d^k g(x)}{dx^k}\right]_{x=0}, \tag{2. 5}$$

$$H(k) = \frac{1}{k!}\left[\frac{d^k h(x)}{dx^k}\right]_{x=0}, \tag{2. 6}$$

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}[g(x)+h(x)]\right]_{x=0}. \tag{2. 7}$$

Using Eqs. (2. 5) - (2. 7), yield

$$F(k) = G(k) + H(k). \tag{2. 8}$$

Similarly, it can be proved that

$$F(k) = G(k) - H(k). \tag{2. 9}$$

Theorem 2 If $f(x) = cg(x)$, then $F(k) = cG(k)$, where c is a constant.
Proof Using the definition (2. 5),

$$G(k) = \frac{1}{k!}\left[\frac{d^k g(x)}{dx^k}\right]_{x=0}, \tag{2. 10}$$

$$\begin{aligned} F(k) &= \frac{1}{k!}\left[\frac{d^k}{dx^k}[cg(x)]\right]_{x=0} \\ &= c\frac{1}{k!}\left[\frac{d^k g(x)}{dx^k}\right]_{x=0}. \end{aligned} \tag{2. 11}$$

Using Eqs. (2. 10) and (2. 11), give

$$F(k) = cG(k). \tag{2. 12}$$

Theorem 3 If $f(x) = \frac{d^m g(x)}{dx^m}$, then $F(k) = (k+m)G(k+m)$.
Proof By definition (2. 5),

$$\begin{aligned} F(k) &= \frac{1}{k!}\left[\frac{d^k}{dx^k}\left[\frac{d^m g(x)}{dx^m}\right]\right]_{x=0} \\ &= \frac{(k+1)(k+2)\cdots(k+m)}{(k+m)!}\left[\frac{d^{k+m} g(x)}{dx^{k+m}}\right]_{x=0}, \end{aligned} \tag{2. 13}$$

then

$$F(k) = \frac{(k+m)!}{k!}G(k+m). \tag{2. 14}$$

Theorem 4 If $f(x) = g(x)h(x)$, then $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.
Proof By definition (2. 5),

$$\begin{aligned} F(0) &= \frac{1}{0!}\left[g(x)h(x)\right]_{x=0} = G(0)H(0), && (2.\ 15)\\ F(1) &= \frac{1}{1!}\left[\frac{d}{dx}[g(x)h(x)]\right]_{x=0}\\ &= \left[\frac{dg(x)}{dx}h(x) + g(x)\frac{dh(x)}{dx}\right]_{x=0}\\ &= G(1)H(0) + G(0)H(1), && (2.\ 16)\\ F(2) &= \frac{1}{2!}\left[\frac{d^2}{dx^2}[g(x)h(x)]\right]_{x=0}\\ &= G(2)H(0) + G(1)H(1) + G(0)H(2). && (2.\ 17)\end{aligned}$$

In general, it can be written as

$$F(k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} G(k_1)H(k_2-k_1)L(k-k_2). \tag{2. 18}$$

Theorem 5 If $f(x) = e^{\lambda x}$, then $F(k) = \frac{\lambda^k}{k!}$.
Proof Using definition (2. 5), gives

$$\begin{aligned} F(k) &= \frac{1}{k!}\left[\frac{d^k}{dx^k}e^{\lambda x}\right]_{x=0}\\ &= \frac{\lambda^k}{k!}. && (2.\ 19)\end{aligned}$$

Theorem 6 (i) If $f(x) = Sin(\alpha x + \beta)$, then $F(k) = \frac{\alpha^k}{k!}Sin\left(\frac{k\pi}{2} + \beta\right)$.
(ii) If $f(x) = Cos(\alpha x + \beta)$, then $F(k) = \frac{\alpha^k}{k!}Cos\left(\frac{k\pi}{2} + \beta\right)$.

Proof (i) Using definition (2. 5), gives

$$\begin{aligned} F(k) &= \frac{1}{k!}\left[\frac{d^k}{dx^k}Sin(\alpha x + \beta)\right]_{x=0}\\ &= \frac{\alpha^k}{k!}Sin\left(\frac{k\pi}{2} + \beta\right). && (2.\ 20)\end{aligned}$$

(ii) Using definition (2. 5), gives

$$\begin{aligned} F(k) &= \frac{1}{k!}\left[\frac{d^k}{dx^k}Cos(\alpha x + \beta)\right]_{x=0}\\ &= \frac{\alpha^k}{k!}Cos\left(\frac{k\pi}{2} + \beta\right). && (2.\ 21)\end{aligned}$$

Theorem 7 If $f(x) = g(x)h(x)l(x)$, then
$F(k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} G(k_1)H(k_2-k_1)L(k-k_2)$.

Proof By definition (2. 5),

$$\begin{aligned}
F(0) &= \frac{1}{0!}\left[g(x)h(x)l(x)\right]_{x=0} = G(0)H(0)L(0), && (2.\ 22)\\
F(1) &= \frac{1}{1!}\left[\frac{d}{dx}[g(x)h(x)l(x)]\right]_{x=0}\\
&= \left[\frac{dg(x)}{dx}h(x)l(x) + g(x)\frac{dh(x)}{dx}l(x) + g(x)h(x)\frac{dl(x)}{dx}\right]_{x=0}\\
&= G(1)H(0)L(0) + G(0)H(1)L(0) + G(0)H(0)L(1), && (2.\ 23)\\
F(2) &= \frac{1}{2!}\left[\frac{d^2}{dx^2}[g(x)h(x)l(x)]\right]_{x=0}\\
&= G(2)H(0)L(0) + G(1)H(1)L(0) + G(1)H(0)L(1)\\
&\quad + G(1)H(1)L(0) + G(0)H(2)L(0) + G(0)H(1)L(1)\\
&\quad + G(1)H(0)L(1) + G(0)H(1)L(1) + G(0)H(0)L(2). && (2.\ 24)
\end{aligned}$$

In general, it can be written as

$$F(k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} G(k_1)H(k_2-k_1)L(k-k_2). \tag{2. 25}$$

2.1. **Analysis of the Method.** The following nth order BVP is considered

$$u^{(n)} = f(x,u,u',u'',\cdots,u^{(n-1)}), \quad a \le x \le b, \tag{2. 26}$$

with the boundary conditions

$$B\left(u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \cdots, \frac{d^{(n-1)}u}{dx^{(n-1)}}\right). \tag{2. 27}$$

The differential transformation of the problem (2. 26) is

$$U(k+n) = \frac{F(k)}{(k+n)!}, \tag{2. 28}$$

where $F(k)$ is the differential transformation of $f(x,u,u',u'',\cdots,u^{(n-1)})$.
The transformed boundary conditions can be written as

$$U(k) = A, U(m) = \sum_{k=0}^{N}\prod_{i=1}^{m-1}(k-i)U(k) = B_m,\ (m < n), \tag{2. 29}$$

where m is the order of the derivative in the boundary conditions and $A,\ B_m$ are real constants.
Using Eqs. (2. 28) and (2. 29), the values of $U(i),\ i = 1,2,3,\cdots$ can be determined and then using the inverse differential transformation, the following approximate solution upto $O(x^{N+1})$ can be determined

$$\widetilde{u} = \sum_{k=0}^{N} x^k U(k). \tag{2. 30}$$

To illustrate the applicability and effectiveness of the method, two numerical examples are considered in the following section.

3. Numerical Examples

Example 3.1 The linear thirteenth order BVP, is considered as

$$u^{(13)}(x) = Cosx - Sinx, \qquad 0 \le x \le 1, \tag{3. 1}$$

subject to the boundary conditions

$$\left.\begin{array}{rcll} u(0) & = & 1, & u(1) = Cos(1) + Sin(1), \\ u^{(1)}(0) & = & 1, & u^{(1)}(1) = Cos(1) - Sin(1), \\ u^{(2)}(0) & = & -1, & u^{(2)}(1) = -Sin(1) - Cos(1), \\ u^{(3)}(0) & = & -1, & u^{(3)}(1) = -Cos(1) + Sin(1), \\ u^{(4)}(0) & = & 1, & u^{(4)}(1) = Cos(1) + Sin(1), \\ u^{(5)}(0) & = & 1, & u^{(5)}(1) = Cos(1) - Sin(1). \\ u^{(6)}(0) & = & -1, & \end{array}\right\} \tag{3. 2}$$

The exact solution of the problem is $u(x) = Cosx + Sinx$.
Applying the theorems 1, 3 and 6 the differential transformation of the problem (3. 1) can be determined as

$$U(k+13) = \frac{k!}{(k+13)!}\left[\frac{1}{k!}Cos\left(\frac{\pi k}{2}\right) - \frac{1}{k!}Sin\left(\frac{\pi k}{2}\right)\right]. \tag{3. 3}$$

Using the Eq. (2. 1), the boundary conditions (3. 2) at $x_0 = 0$, can be transformed as

$$\left.\begin{array}{rcl} U(0) & = & 1, U(1) = 1, U(2) = \frac{-1}{2!}, \\ U(3) & = & \frac{-1}{3!}, U(4) = \frac{1}{4!}, U(5) = \frac{1}{5!}, U(6) = \frac{-1}{6!}, \end{array}\right\} \tag{3. 4}$$

$$\left.\begin{array}{l} \sum_{k=0}^{n} U(k) = Cos(1) + Sin(1), \\ \sum_{k=0}^{n} kU(k) = Cos(1) - Sin(1), \\ \sum_{k=0}^{n} k(k-1)U(k) = -Sin(1) - Cos(1), \\ \sum_{k=0}^{n} k(k-1)(k-2)U(k) = -Cos(1) + Sin(1), \\ \sum_{k=0}^{n} k(k-1)(k-2)(k-3)U(k) = Cos(1) + Sin(1), \\ \sum_{k=0}^{n} k(k-1)(k-2)(k-3)(k-4)U(k) = Cos(1) - Sin(1), \end{array}\right\} \tag{3. 5}$$

where n is a sufficiently large integer. Using the recurrence relation (3. 3), the transformed boundary conditions (3. 4) and the inverse differential transformation (2. 3), the following series solution for $n = 13$, can be written

$$\begin{aligned} u(x) &= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} + Ax^7 + Bx^8 + Cx^9 \\ &\quad + Dx^{10} + Ex^{11} + Fx^{12} + \frac{x^{13}}{6227020800} + O(x^{14}), \end{aligned} \tag{3. 6}$$

where, according to the definition (2. 1)

$$\left.\begin{array}{l} A = \frac{u^{(7)}(0)}{7!} = U(7),\ B = \frac{u^{(8)}(0)}{8!} = U(8),\ C = \frac{u^{(9)}(0)}{9!} = U(9), \\ D = \frac{u^{(10)}(0)}{10!} = U(10),\ E = \frac{u^{(11)}(0)}{11!} = U(11),\ F = \frac{u^{(12)}(0)}{12!} = U(12). \end{array}\right\} \tag{3. 7}$$

From the transformed boundary conditions (3. 5), the following system of linear equations in terms of A, B, C, D, E and F can be determined

$$\frac{8605396801}{6227020800}+A+B+C+D+E+F=Cos[1]+Sin[1],$$
$$-\frac{143700479}{479001600}+7A+8B+9C+10D+11E+12F=Cos[1]-Sin[1],$$
$$-\frac{54885599}{39916800}+42A+56B+72C+90D+110E+132F=-Cos[1]-Sin[1],$$
$$\frac{1209601}{3628800}+210A+336B+504C1+720D1+990F+1320G=-Cos[1]+Sin[1],$$
$$\frac{544321}{362880}+840A+1680B+3024C1+5040D1+7920F+11880G=Cos[1]+Sin[1],$$
$$\frac{1}{40320}+2520A+6720B+15120C1+30240D1+55440F+95040G=Cos[1]-Sin[1].$$

The solution of the above system, yields
$A=-0.00019841261645169112,\quad B=0.000024801110561331 96,$
$C=2.7568717766803008\times10^{-6},\ D=-2.769904021536794\times10^{-7},$
$E=-2.4115063057288555\times10^{-8},\ F=1.8105803100768588\times10^{-9}.$
The series solution can, thus, be written as

$$\begin{aligned} u(x) &= 1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}-\frac{x^6}{720}-0.000198413x^7+0.0000248011x^8\\ &\quad +2.75687\times10^{-6}x^9-2.7699\times10^{-7}x^{10}-2.41151\times10^{-8}x^{11}+1.81058\times10^{-9}x^{12}\\ &\quad +\frac{x^{13}}{6227020800}+O(x^{14}). \end{aligned} \tag{3. 8}$$

The comparison of the approximate solution of Example 3.1 obtained with the help of DTM and the approximate solution using VIM [1] is given in Table 1. From the numerical results, it is clear that the DTM is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained. The results are also expressed graphically in Figure 1. The solid line represents the curve corresponding to the exact solution whereas the dotted line corresponds to the approximate solution. In Figure 2, absolute errors are plotted.

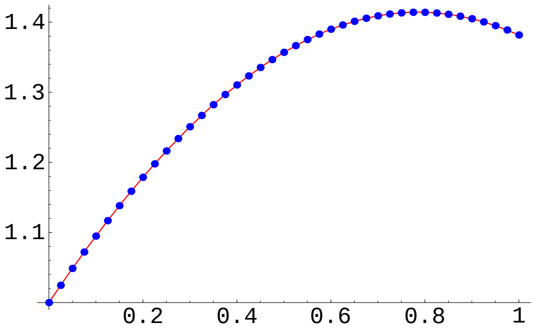

FIGURE 1. Comparison of the approximate solution with the exact solution for problem 3.1. Dotted line: approximate solution, solid line: the exact solution.

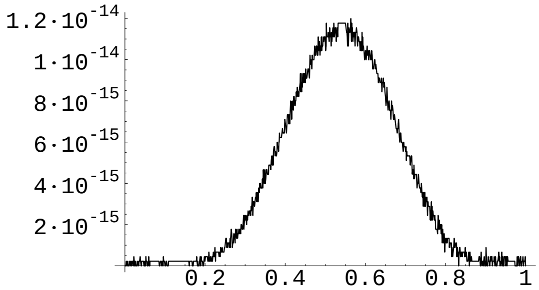

FIGURE 2. Absolute errors for problem 3.1.

Example 3.2 The thirteenth order non-linear BVP is considered as

$$u^{(13)}(x) = e^{-x}u^2(x), \quad 0 \le x \le 1, \tag{3. 9}$$

subject to the boundary conditions

$$\left.\begin{array}{rcll} u(0) &=& 1, & u(1) = e, \\ u^{(1)}(0) &=& 1, & u^{(1)}(1) = e, \\ u^{(2)}(0) &=& 1, & u^{(2)}(1) = e, \\ u^{(3)}(0) &=& 1, & u^{(3)}(1) = e, \\ u^{(4)}(0) &=& 1, & u^{(4)}(1) = e, \\ u^{(5)}(0) &=& 1, & u^{(5)}(1) = e. \\ u^{(6)}(0) &=& 1, & \end{array}\right\} \tag{3. 10}$$

The exact solution of the problem is $v(x) = e^x$.
Applying the theorems (1-7), the differential transformation of the problem (3. 9) can be determined as

$$U(k+13) = -\frac{k!}{(k+13)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}U(k-k_2)U(k_2-k_1)}{k_1!}. \tag{3. 11}$$

Using Eq. (2. 1), the boundary conditions (3. 10) at $x_0 = 0$, can be transformed as

$$\left.\begin{array}{rcl} U(0) &=& 1, U(1) = 1, U(2) = \frac{1}{2!}, \\ U(3) &=& \frac{1}{3!}, U(4) = \frac{1}{4!}, U(5) = \frac{1}{5!}, U(6) = \frac{1}{6!}, \end{array}\right\} \tag{3. 12}$$

$$\left.\begin{array}{l} \sum_{k=0}^{n} U(k) = e, \\ \sum_{k=0}^{n} kU(k) = e, \\ \sum_{k=0}^{n} k(k-1)U(k) = e, \\ \sum_{k=0}^{n} k(k-1)(k-2)U(k) = e, \\ \sum_{k=0}^{n} k(k-1)(k-2)(k-3)U(k) = e, \\ \sum_{k=0}^{n} k(k-1)(k-2)(k-3)(k-4)U(k) = e, \end{array}\right\} \tag{3. 13}$$

where n is a sufficiently large integer. Using the recurrence relation (3. 11), the transformed boundary conditions (3. 12) and the inverse differential transformation (2. 3), the following series solution upto $O(x^{14})$ can be determined

$$\begin{aligned} u(x) &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+Ax^7+Bx^8+Cx^9 \\ &\quad +Dx^{10}+Ex^{11}+Fx^{12}+\frac{x^{13}}{6227020800}+O(x^{14}), \end{aligned} \tag{3. 14}$$

where, according to the definition (2. 1)

$$\left.\begin{array}{l} A=\frac{u^{(7)}(0)}{7!}=U(7),\ B=\frac{u^{(8)}(0)}{8!}=U(8),\ C=\frac{u^{(9)}(0)}{9!}=U(9), \\ D=\frac{u^{(10)}(0)}{10!}=U(10),\ E=\frac{u^{(11)}(0)}{11!}=U(11),\ F=\frac{u^{(12)}(0)}{12!}=U(12). \end{array}\right\} \tag{3. 15}$$

From the transformed boundary conditions (3. 13), the following system of linear equations in terms of A, B and C can be determined

$$\left.\begin{array}{l} \frac{16925388481}{6227020800}+A+B+C+D+E+F=e, \\ \frac{1301287681}{479001600}+7A+8B+9C+10D+11E+12F=e, \\ \frac{108108001}{39916800}+42A+56B+72C+90D+110E+132F=e, \\ \frac{9676801}{3628800}+210A+336B+504C1+720D1+990F+1320G=e, \\ \frac{907201}{362880}+840A+1680B+3024C1+5040D1+7920F+11880G=e, \\ \frac{0641}{40320}+2520A+6720B+15120C1+30240D1+55440F+95040G=e. \end{array}\right\} \tag{3. 16}$$

The solution of the above system, yields
$A = 0.00019841261021669805$, $B = 0.000024802098855594142$,
$C = 2.75451301223356 \times 10^{-6}$, $D = 2.7708206549274656 \times 10^{-7}$,
$E = 2.4060411409761992 \times 10^{-8}$, $F = 2.3783375635762833 \times 10^{-9}$.
The series solution can, thus, be written as

$$\begin{aligned} u(x) &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+0.000198413x^7+0.0000248021x^8 \\ &\quad +2.75451\times10^{-6}x^9+2.77082\times10^{-7}x^{10}+2.40604\times10^{-8}x^{11} \\ &\quad +2.37834\times10^{-9}x^{12}+\frac{x^{13}}{6227020800}+O(x^{14}). \end{aligned} \tag{3. 17}$$

The approximate solutions of Example 3.2 obtained with the help of DTM are compared with the results of the VIM [1] in Table 2. From the numerical results, it is clear that the DTM is more efficient and accurate. By increasing the order of approximation more accuracy can be obtained. The graphical comparison of exact and approximate solutions is shown in Figure 3. The solid line represents the curve corresponding to the exact solution whereas the dotted line corresponds to the approximate solution. In Figure 4, absolute errors are plotted.

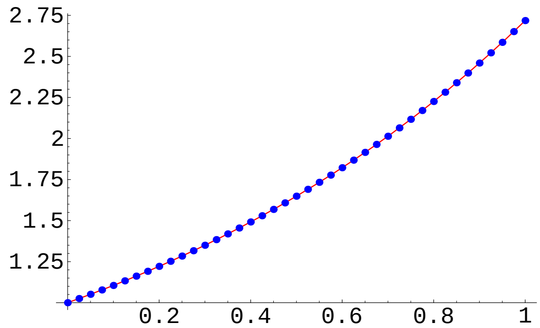

FIGURE 3. Comparison of the approximate solution with the exact solution for problem 3.2. Dotted line: approximate solution, solid line: the exact solution.

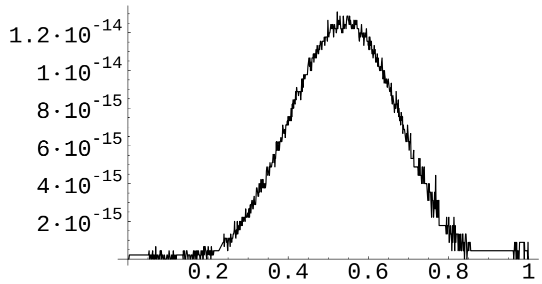

FIGURE 4. Absolute errors for problem 3.2.

Conclusion In this paper, the differential transformation method has been applied to obtain the numerical solution of linear and nonlinear thirteenth order boundary value problems. The present method has been applied in a direct way without using linearization, discretization, or perturbation. By increasing the order of approximation more accuracy can be obtained. Comparison of the numerical results with the existing technique [1] shows that the present method is more accurate.

TABLE 1. Comparison of numerical results for Example 3.1

x	Exact solution	Approximate series solution	Absolute Error present method	Absolute Error [1]
0.0	1.0000	1.0000	0.0000	0.0000
0.1	1.09484	1.09484	2.22045E-16	3.88578E-15
0.2	1.17874	1.17874	0.0000	1.46216E-13
0.3	1.25086	1.25086	2.22045E-15	8.80518E-13
0.4	1.31048	1.31048	6.66134E-15	2.35822E-12
0.5	1.35701	1.35701	1.11022E-14	3.8014E-12
0.6	1.38998	1.38998	1.04361E-14	5.14766E-11
0.7	1.40906	1.40906	5.32907E-15	1.56224E-11
0.8	1.41406	1.41406	8.88178E-16	8.99409E-11
0.9	1.40494	1.40494	0.0000	4.70031E-10
1.0	1.38177	1.38177	0.0000	2.06386E-09

TABLE 2. Comparison of numerical results for Example 3.2

x	Exact solution	Approximate series solution	Absolute Error present method	Absolute Error [1]
0.0	1.0000	1.0000	0.0000	0.0000
0.1	1.10517	1.10517	4.44089E-16	4.17444E-14
0.2	1.2214	1.2214	4.44089E-16	2.64144E-12
0.3	1.34986	1.34986	2.44249E-15	2.99314E-11
0.4	1.49182	1.49182	7.32747E-15	1.67101E-10
0.5	1.64872	1.64872	1.22125E-14	6.30955E-10
0.6	1.82212	1.82212	1.11022E-14	1.84757E-09
0.7	2.01375	2.01375	5.77316E-15	4.47866E-09
0.8	2.22554	2.22554	1.77636E-15	9.21592E-09
0.9	2.4596	2.4596	8.88178E-16	1.58906E-08
1.0	2.71828	2.71828	0.0000	2.09057E-08

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