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THE DEFICIENT QUARTIC SPLINE INTERPOLATION

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Abstract: In the present paper, we have studied the existence, uniqueness and convergence properties of deficient quartic spline interpolation.

1. INTRODUCTION

Quartic and higher degree splines are still popular for best approximation (see de'Boor [1]). In the study of lower degree piecewise spline functions there is a disadvantage, that we have corners at joint of two lower degree pieces and therefore to achieve a prescribed accuracy more data than higher methods are needed. Rana and Dubey [9] generalized the result of Garry and Howell [7] and obtained best error bounds for quartic spline interpolation. Dubey and Shukla [4] have obtained error bound for quartic spline interpolation. For various aspects of cubic, quartic and spline of degree six reference may be given to Meir and Sharma [8], Hall and Meyer [6], Gemling-Meyling [5], Dubey [3]. In this paper, we have investigated the existence uniqeueness and convergence properties of deficient quartic spline interpolation which match the given function at values at mesh points, interior points with appropriate boundary conditions.

2. EXISTENCE AND UNIQUENESS

Consider a mesh P of $[0,1]$ given by $0 = x < x_0 < \dots\dots\dots < x_n = 1$ such that $x_{i+1} - x_i = h_i$ i=0,1,.....n-1 and $h = \max h_i$. Let $S(4,P)$ denote the set of real algebraic polynomials of degree not greater than 4 i.e.

$$S(4,P) = \{s_i, s_i \in C^2[0,1], \text{i} = 1,2.....\text{n-1}\}.$$

where s_i is the restriction of s over $[x_i, x_{i+1}]$. Let S* (4,P) denotes the class of all deficient quartic splines S(4,P) which satisfy the boundary conditions,

$$s''(x_0) = f''(x_0),$$

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$$s''(x_n) = f''(x_n).\tag{2.1}$$

We shall prove the following.

THEOREM 2.1 : Let f', f'' exist. Then there exist a unique deficient quartic spline in S* (4, P) which satisfies the interpolatory conditions,

$$s(x_i) = f(x_i) \qquad i = 0, 1, 2, \ldots\ldots n \tag{2.2}$$

$$s(\alpha_i) = f(\alpha_i) \qquad i = 0, 1, \ldots., n-1 \tag{2.3}$$

where $\alpha_i = x_i + \theta h_i$ if $0 < \theta < 3 - \sqrt{6}$ and $< h_i >$

is non increasing.

PROOF : Consider a quartic polynomial $P(z)$ on $[0,1]$. We can easily verify that

$$P(z) = P(0)\,q_1(z) + P(1)\,q_2(z) + P(\theta)\,q_3(z)$$

$$+ P''(0)\,q_4(z) + P''(1)\,q_5(z), \tag{2.4}$$

where

$$q_1(z) = 1 + [\{(-6\theta^4 - 6 + 12\theta^3)\,z - 12(\theta - 1)z^3 + 6(\theta - 1)z^4\,]/6A,$$

$$q_2(z) = [(6\theta^4 - 12\theta^3)z + 12\theta z^3 - 6\theta z^4\,]/6A\,,$$

$$q_3(z) = [6z - 12z^3 + 6z^4\,]/6A\,,$$

$$q_4(z) = [(2\theta^4 + 5\theta^3 - 3\theta^2)\,z + 3(\theta^4 + \theta - 2\theta^3)z^2$$

$$+ (-\theta^4 - 5\theta + 6\theta^2)^3 z + (\theta^3 - 3\theta^2 + 2\theta)\,z^4\,]/6A\,,$$

$$q_5(z) = [(-\theta^4 + \theta^3)z + (\theta^4 - \theta)\,z^3 - (\theta^3 - \theta)\,z^4\,]/6A\,,$$

with $A = (\theta^4 + \theta - 2\theta^3)$

Now writing $t = \dfrac{(x - x_i)}{h_i}$, (2.4) may be expressed in term of restriction s_i of s as follows :-

$$s_i(x) = f(x_i)\,q_1(t) + f(x_{i+1})\,q_2(t) + f(\alpha_i)\,q_3(t) + q_4(t)\,s''_{i-1}(x_i) + q_5(t)\,s_i(x) \tag{2.5}$$

which clearly satisfies the condition (2.1) - (2.3) and $s_i(x)$ is a quartic in $[x_i, x_{i+1}]$ for i=0,.....,n. Since $s \in C^2[0,1]$, so applying continuity condition of first derivative of s in (2.5) we have

$$h_{i-1}(\theta^4 - 3\theta^3 + 3\theta^2 - \theta)s''_{i-1} + \{h_{i-1}(2\theta^4 - 3\theta^3 + \theta)$$

$$+ h_i(3\theta^2 - 2\theta^4 - 5\theta^3)\}\, s''_i + h_i(\theta^4 - \theta^3)s''_{i+1} = F_i \qquad \text{(Say)}$$

where $F_i = \left[\left\{ \frac{(-6\theta^4 - 6 + 12\theta^3)}{h_i} - \frac{\theta(\theta^3 - 2\theta^2 + 12)}{h_{i-1}} \right\} f_{\theta i} \right.$

$$- \frac{(-6\theta^4 + 6 + 12\theta^3 - 12\theta) f_{i-1}}{h_{i-1}} + \frac{6}{h_{i-1}} f(\alpha_{i-1})$$

$$\left. + \frac{6}{h_i} f(\alpha_i) + \frac{6\theta^3(\theta - 2)}{h_i} f(x_{i+1}) \right] \qquad (2.6)$$

Clearly the coefficient matrix of the system of equation (2.6) diagonally dominant under the conditions of Theorem 2.1 and hence invertible. This complete proof of theorem 2.1.

3. ERROR BOUNDS

Following the method of Hall and Meyer [6] in this solution, we shall obtain bounds of error function $e(x) = f(x) - s(x)$ for the spline interpolant of Theorem 2.1 which are best possible. Let $s(x)$ be the second time continuously differentiable quartic spline function satisfying the conditions of theorem 2.1. Now considering $f \in C^5[0,1]$ and writing $L_i[f, x]$ for the unique quartic which agree with $f(x_i)$, $f(x_{i+1})$ $f(\alpha_i)$ $f''(x_i)$ and $f''(x_{i+1})$, we see that for $x \in [x_i, x_{i+1}]$, we have

$$|f(x) - s(x)| = |f(x) - s_i(x)|$$

$$\leq |f(x) - h_i[f, x]| + |L_i[f, x] - s_i(x)] \qquad (3.1)$$

First we have to obtain the bounds of right hand side of (3.1) by Cauchy Theorem (See Davis [2]). Thus, we have

$$|f(x)L_i[f, x]| \leq \frac{h_i^5}{5!}|t^2(1-t)^2(t-\theta)| \qquad (3.2)$$

where $t = \frac{x - x_i}{h_i}$ and $F = \max_{0 \le x \le 1} | f^{(5)}(x) |$.

To get the bounds of $| L_i[f, x] - s_i(x) |$,we have from (2.4)

$$h_i[f, x] - s_i(x) = [\{ h_{i-1}^2 e''(x_{i-1}) q_4(t) + h_i^2 e''(x_i) q_5(t)] \tag{3.3}$$

Thus,

$$| L_i[f, x] - \le i\{(x) |$$

$$\le | h_{i-1}^2 e''(x_{i-1}) q_4(t) + h_i^2 e''(x_i) q_5(t) | \tag{3.4}$$

Let $\max_{0 \le t \le 1} | e(x_i) |$ exists for i= j then the inequality (3.4).

$$| L_i[f, x] - s_i(x) |$$

$$\le h_j^2 | e''(x_j) |[| q_4(t)\} + q_5(t) |]$$

Let $k(t) = \max_{0<0<1} [\theta | \theta^2 + 6\theta - 3 | + 3 | \theta^3 - 2\theta^2 + 1 | \ t + 6 | \theta - 1 | t^2 + 3(1 - \theta) t^3]$.

Thus $| L_i[f, x] - s_i(x) | \le h_j^2 | e''(x_j) | k(t)$ (3.6)

Now, we proceed to obtain $| e''(x_i) |$ replacing $s''(x_i)$ by $e''(x_i)$ in equation (2.6), we have

$$h_{i-1}(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta) e''_{i-1} + \{ h_{i-1}(2\theta^4 - 3\theta^3 + \theta)$$

$$+ h_i(3\theta^2 - 2\theta^4 - 5\theta^3)\} e''_i + h_i(\theta - 1)\theta^3 e''_{i+1}$$

$$= F_i - h_{i-1}(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta) f''_{i-1}$$

$$- \{ h_{i-1}(2\theta^4 - 3\theta^3 + \theta) + h_i(3\theta^2 - 2\theta^4 - 5\theta^3)\}$$

$$f''_i - h_i(\theta - 1)\theta^3 f''_{i+1} = E(f) \quad \text{(say).} \tag{3.7}$$

In view of that E(*f*) is linear functional which is zero for polynomials of degree 4 or less we can apply Peano Theorem (see Davis [2]) to obtain

$$E(f) = \int_{x_{j-1}}^{x_{j+1}} \frac{f^{(5)}(y)}{4!} E\left[(x - y)_+^4\right] dy \tag{3.8}$$

Thus, from (3.8) we have

$$|E(f)| \le \frac{1}{4!} F \int_{x_{j-1}}^{x_{j+1}} |E(x-y)_+^4|\, dy \qquad (3.9)$$

Where $F = \max |f^5(y)|$.

Further, it can observed that from (3.9) in $x_{i-1} \le x \le x_{j+1}$, we get

$$E[(x_i - y)_+^4] = [\{\frac{-6\theta^4 - 6 + 12\theta^3}{h_i} - \frac{\theta(\theta^3 - 2\theta^2 + 12)}{h_{i-1}}\}(x_i - y)_+^4$$

$$+\frac{6}{h_{i-1}}(\alpha_{i-1} - y)_+^4 + \frac{6}{h_i}(\alpha_i - y)_+^4 + \frac{6\theta^3(\theta-2)}{h_i}(x_{i+1} - y)_+^4$$

$$-12\{h_{i-j}\}2\theta^4 - 3\theta^2 + \theta)$$

$$+h_i(3\theta^2 - 2\theta^4 - 5\theta^3)(x_i - y)_+^2 - 12\, h_i(\theta^4 - \theta^3)(x_{i+1} - y)_+^2 . \qquad (3.10)$$

In order to estimate the internal of r.h.s. of (3.10) we rewrite the above expression in the following symmetric form about x_i to see

$$6\theta^3\{(\theta-2)(x_i - y + h_i)^2 = 2h_i^2(\theta-1)\} \quad [x_i - y + h_i]^2 \qquad \alpha_i \le y \le x_{i+1}$$

$$= \frac{6}{h_i}[(\theta^4 - 2\theta^3 + 1)(x_i - y)^2 + (4\theta^4 - 8\theta^3 + \theta)h_i(x_i - y) + 2\theta^2$$

$$(2\theta^2 - 5\theta + 3)h_i^2] \quad [(x_i - y)^2] \qquad x_i \le y \le \alpha_i$$

$$= \frac{6}{h_{i-1}}[(\theta^4 - 2\theta^3 + 1)(x_i - y)^2 + \theta(4\theta^3 - 8\theta^2 + 1)$$

$$h_{i-1}(x_i - y) + 2\theta^2(2\theta^2 - 5\theta + 3)h_{i-1}^2](x_i - y)^2 , \qquad \alpha_{i-1} \le y \le x_i$$

$$= \frac{6\theta^3}{h_{i-1}}\{(\theta-2)(x_i - y - h_{i-1})^2 + 2h_{i-1}^2(\theta-1)\}[x_i - y - h_{i-1}]^2 ,$$

$$x_{i-1} \le y \le \alpha_{i-1} .$$

Thus, it is clear from above expression that $E[(x-y)_+^4]$ is non negative for $x_{i-1} \le x \le x_{i+1}$ given by

$$\int_{x_{j-1}}^{x_{j+1}} | E(x-y)_+^4 | dy = \frac{K_1(\theta)}{5}\left(h_{i-1}^4 + h_i^4\right), \tag{3.11}$$

where $K_1(\theta) = \left[\frac{\theta^3}{2}\left(24\theta^5 - 124\theta^4 + 400\theta^3 - 543\theta^2 - 84\right)\right]$.

Combining (3.9) and (3.11), we have

$$| E_i(f) | \le F\,\frac{K_1(\theta)}{5!}\left(h_{i-1}^4 + h_i^4\right). \tag{3.12}$$

Thus, from (3.8) and (3.11)

$$| e''(x_i) | \le | e_i'' | \le \frac{FK_1(\theta)(h_{i-1}^4 + h_i^4)}{5![(-3\theta^4 - 3\theta^2 + 2\theta)h_{i-1} + \left(6\theta^3 + \theta^4 - 3\theta^2\right)h_i]} \tag{3.13}$$

Now using (3.2), (3.7) along with (3.13) in (3.1) we have

$$| e(x) | \le \frac{h^5}{5!} | t^2(1-t)^2(t-\theta) | F + h^2 | e''(x_i) | k(t) F$$

$$= \frac{h^5}{5} | t^2(-t)^2(t-\theta) | F + \frac{FK_1(\theta)h^5 k(t)}{(6\theta^3 - 2\theta^4 - 6\theta^2 + 2\theta)} \tag{3.14}$$

$$\le \frac{h^5}{5!} F\, | C(t) | \tag{3.15}$$

where $C(t) = \left[t^2(1-t)^2(t-\theta) + K_1^*(\theta)K(t)\right]$

and $K_1^*(\theta) = \dfrac{K_1(\theta)}{(6\theta^3 - 2\theta^4 - 6\theta^2 + 2\theta)}$

Thus we prove the following.

Theorem 3.1. Suppose s(x) is the quartic spline of Theorem 2.1 interpolating a function *f(x)* and $f \in C^5[0,1]$. Then

$$| e(x) | \le K\,\frac{h^5}{5!} \max_{0 \le x \le 1} | f^{(5)}(x) |, \tag{3.17}$$

where $K = \max_{0 \le t \le 1} | C(t) | = \max_{0 \le t \le 1} | t^2(1-t^2)(t-\theta) + K_1^*(\theta)k(t) |$,

and

$$| e''(x_i) | \leq \frac{1h^5}{5!} K_1^*(\theta) \max_{\theta \leq x \leq 1} | f^{(5)}(x) |. \tag{3.18}$$

Further more, it can be seen easily that K in (3.17) can not be improved for an equally spaced mesh. Inequality (3.18) is also best possible. Also, we have

$$| e''(x_i) | \leq K_1 \frac{h^5}{5!} \max_{\theta \leq x \leq 1} | f^5(x) | \quad , \tag{3.19}$$

where K_1 is positive constant.

Equation (3.15) proves inequality (3.17) where as inequality (3.18) is direct consequence of (3.13).

Now we turn to prove that the inequality (3.17) is best possible in the limit. Considering $f(x) = \frac{x^5}{5!}$ and using Cauchy formula Davis [2] we have

$$L_i \left[\frac{x^5}{5!*} x \right] - \frac{x^5}{5!} = \frac{h^5 t^2 (1-t)^2 (t-\theta)}{5!} \tag{3.20}$$

For the function with equally spaced knots from equation (3.7), the following relation holds

$$\left[\frac{x^5}{5!} \right] = \frac{K_1(\theta) h^3}{5!} = \left(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta\right) e''_{i-1}$$

$$+ \left(\theta - 8\theta^3 + 3\theta^2\right) e''_i + (\theta - 1)\theta^3 e''_{i+1} \tag{3.21}$$

Consider for a moment

$$e''_j = \frac{K_1(\theta) h^3}{6!\theta^2 (1-\theta)^2} = e''_{j-1} = e''_{j+1}. \tag{3.22}$$

We have from (3.4)

$$s(x) - L_i [f, x] = \frac{K_1(\theta) h^5}{6!\theta^2 (1-\theta)^2} \left(q_4(t) + q_5(t)\right)$$

$$= \frac{K_1(\theta) h^5}{6!\theta^2 (1-\theta)^2} [\theta^2 (\theta^2 + 6\theta - 3)t + 2\theta (2\theta^3 + 1 - 3\theta^2) t^2$$

$$+ 6\theta(\theta - 1)t^3 + 3\theta(1-\theta)t^4]. \tag{3.23}$$

Now combining (3.20) and (3.23) we get for

$x_i \le x \le x_{i+1}$

$$s(x) - f(x) = \frac{h^5}{5!}\Big[\frac{K_1(\theta)}{6\theta^2(1-\theta)^2}\{\theta^2(\theta^2 + 6\theta - 3)t$$

$$+ 2\theta(2\theta^3 + 1 - 3\theta^2)t^2 + 6\theta(\theta - 1)t^3$$

$$+ 3\theta(1-\theta)t^4\} + t^2(1-t)^2(t-\theta)\Big]. \qquad (3.24)$$

From (3.24) it is clearly observed that (3.17) is best possible, provided we would prove that

$$e''_{i-1} = e''_i = e''_{i+1} = \frac{k_1(\theta)h^5}{6!\theta^2(1-\theta)^2} \qquad (3.25)$$

In fact (3.25) is attained only in the limit, the difficulty will appear in the use of boundary condition i.e. $e''(x_0) = e''(x_n) = 0$.

However, it can be shown that as we move many sub-intervals away from the boundaries

$e''(x_i) \to \dfrac{K_1(\theta)h^5}{6!\theta^2(1-\theta)^2}$. For that we shall apply (3.20) indutively to move away from the end conditions $e''(x_0) = e''(x_n) = 0$.

The first step in this direction is to establish that $e''(x_i) \ge 0$ for some i=1,2,....n which can be shown by contradictory result. Let $e''(x_i) \le 0$ for i=1,2,....n-1.

Now making use of (3.18) we have

$$\frac{K_1(\theta)h^3}{6!\theta^2(1-\theta)^2} \ge \max_{\theta \le x \le 1} |e''(x_i)| \ge \frac{1}{2}[(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta)e''_{i-1} + (\theta - 1)\theta^3 e''_{i+1}$$

$$> \frac{1}{2}[(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta)e''_{i-1} + (\theta - 8\theta^3 + 3\theta^2)e''_i + (\theta - 1)\theta^3 e''_{i+1}$$

$$> \frac{1}{2}\left[\frac{K_1(\theta)h^3}{5!}\right].$$

This is contradiction, hence $e''(x_i) > 0$

Now from (3.21)

$$(\theta - 8\theta^3 + 3\theta^2)e_i'' = \frac{K_1(\theta)h_i^3}{5!} - \left(5\theta^4 - 3\theta^3 + 3\theta^2 - \theta\right)$$

$$e_{i-1}'' - (\theta - 1)\theta^3 e_{i+1}''$$

$$-(\theta - 8\theta^3 + 3\theta^2)e_i'' \le \frac{K_1(\theta)h^3}{5!}. \qquad (3.26)$$

$$e_i'' \le \frac{1}{(\theta - 8\theta^3 + 3\theta^2)}\frac{K_1(\theta)h^3}{5!}.$$

Now again using (3.26) in (3.21) we have

$$e''(x_i) \le \frac{K_1(\theta)h^3}{(\theta - 8\theta^3 + 3\theta^2)5!}$$

$$\le \left[1 - \frac{6\theta^4 - 4\theta^2 - \theta}{(\theta - 8\theta^3 + 3\theta^2} + \left(\frac{6\theta^4 - 4\theta^3 + 3\theta^2 - \theta}{\theta - 8\theta^3 + 3\theta^2}\right)^2 + \ldots\ldots\ldots\right] \qquad (3.28)$$

Now it can be seen easily that r.h.s. of (3.28), $\to \frac{K_1(\theta)h^3}{6\theta(1-\theta)^2}$ and hence in the limiting case $e''(x_i) \to \frac{K_1(\theta)h^3}{6\theta(1-\theta)^2}$ which verify proof of inequality (3.19).

Thus, corresponding to the function $f(x) = \frac{x^5}{5!}$ (3.27) and (3.28) tends to $\frac{K_1(\theta)h^3}{6\theta(1-\theta)^2}$ in the limit for equally spaced knots. This complete proof of Theorem 3.1.

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