ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2014, Article ID ama0122, 8 pages
ISSN 2307-7743
http://scienceasia.asia

CRANK NICOLSON TYPE METHOD WITH MOVING MESH FOR BURGERS EQUATION

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Abstract: A Crank-Nicolson Type method with moving mesh is constructed for numerical solution of one dimensional nonlinear Burgers Equation with Homogeneous Dirichlets boundary conditions. The scheme is derived by linearizing Crank-Nicolson finite difference scheme. Numerical solutions are obtained in different domains for different values of t and constant of diffusivity k.

1. Introduction

Moving mesh methods have important applications in a variety of physical and engineering areas such as solid and fluid dynamics, Combustion, heat transfer ,material science etc. Moving mesh method has become an important tool for computing singular or nearly singular problems such as interfaces, shock waves and boundary layers [HUANG . W (2001), MADZVAMUSE (2005)] and reaction diffusion systems in growing domains [BAINE(2001)]. As the domain changes shape, it necessitates the use of a computational mesh that moves and deforms to the new spatial configuration. The numerical investigation of these problems may require extremely fine meshes over a small portion of the physical domain to resolve the large solution variations. Successful implementation of this strategy can increase the accuracy of the numerical approximation and decrease the computational cost.

The analysis of numerical methods using moving meshes has received limited attention. [Ferreira (1997)] analysed a moving mesh implicit Euler method for linear convection reaction diffusion problems in 1-d using an energy type analysis . It was shown that the method was only conditionally stable depending on the temporal and spatial smoothness of the moving mesh. An important issue related to the use of moving mesh techniques is to solve conservation laws called Geometric conservation Laws [THOMAS P.D(1979)]. [R. M. Fuzzeland(1990)] and J. G. Verwer gave their moving mesh techniques which is due to the moving finite difference method proposed [Dorfi & Drury (1987)]. Some practical aspects of formulations and solutions of moving mesh PDE's are reviewed in [Huang(1991)]. One of the successful applications of the PDE's with moving mesh is given in [C.Budd(1996)] to solve one dimensional PDE's with blow up solution. [Dupont and co-workers (2002) Lie(2003)] analysed certain moving mesh finite element methods for a model advection diffusion equation . [Zhijum Tan,Zhengru Zhang (2004)] solved physical PDE on a fixed mesh and the effect of the mesh movement is achieved through the grid restructuring. [J.A.Mackenzie (2007)] analysed

2010 Mathematics Subject Classification: 65N06, 33F05.
Key words and phrases: Burgers equation, Moving Mesh, Crank-Nicolson.

the stability and convergence of a finite difference discretization of a convection diffusion equation in 1D using moving mesh.

Analytical solution and numerical solutions of Burgers equation with fixed domain are discussed by many authors in [Alireza,Hashemian (2008), Edris Dag Ali sahin (2007), E.Hopf(1950), H.Beteman(1915), J.D.Cole(1951), J. M. Burger(1948), Kanti Pandey and Lajja Verma(2011), N.Su.Australia(2004), Ronald E.mickens(2005), Sachin S.Wani,Sarita Thakar(2009), Skell & Berzins (1990)].Till now the solutions of Burgers equationwith moving mesh are not reported. In this paper we construct a finite difference scheme for one dimensional nonlinear Burgers equation with Moving Mesh. The equation is discretized by using Crank-Nicolson finite difference scheme. Since the resulting scheme is non-linear, the linearization of the scheme is achieved by defining an increment function.

The paper is arranged as follows. In section 2 we present the model problem and its reformulation with respect to moving coordinate frame. In section 3, the moving mesh discretization and the linearized Crank Nicolson scheme is given. In section 4 we present numerical solution of Crank Nicolson Type method with moving mesh in different domains for different values of time t and constant of diffusivity k. Finally in section 5 conclusion is given.

2. Model Problem

Let T>0 and for each $t \in [0,T]$,Ω_t be a domain in R. We shall use the notation

$Q_T = \{(x,t) \in R^2 : x \in \Omega_t, t \in (0,T)\}$

We consider one dimensional non linear Burger's Equation

$$\begin{aligned} & u_t + uu_x = ku_{xx} \quad (x,t) \in Q_T \\ & u = u_0, \qquad x \in \Omega_0, \quad t = 0 \\ & u = 0, \qquad x \in \partial\Omega_t, \quad t > 0 \end{aligned} \tag{1}$$

Where k is a constant of diffusivity.

The conservative form of Burgers equation is as follows

$$u_t + \left(\frac{1}{2}u^2 - ku_x\right)_x = 0\,. \tag{2}$$

We assume that Ω_0 is bounded and u_0 is sufficiently smooth. Let A_t be a family of mappings, which at each $t \in [0,T]$ maps the points ξ of a reference or computational domain Ω_c in to the points of domain Ω_t at time t. Then for each $t \in [0,T]$,

$\boldsymbol{A}_t : \Omega_c \to \Omega_t, \quad x(\xi,t) = A_t(\xi).$

We assume that A_t is a bijective and $\Omega_t = A_t(\Omega_c)$ is bounded. For a function $g : Q_T \to R$ defined on the physical domain, the time derivative in the reference domain is

$$\dot{g} \equiv \frac{\partial g}{\partial t}|_{\xi} : Q_T \to R$$

If $u : Q_T \to R$ is regular enough, then by chain rule $\dot{u} = \frac{\partial u}{\partial t}|_x + \dot{x}\frac{\partial u}{\partial x}|_x$.

The governing equation (2) in the computational domain therefore takes the form

$$\dot{u} - \dot{x}u_x + \frac{1}{2}\left(u^2\right)_x - ku_{xx} = 0 .$$

On rewriting the spatial derivatives with respect to computational coordinate, we get

$$x_\xi \dot{u} - \left(\dot{x} - \frac{u}{2}\right) u_\xi = k\left(\frac{u_\xi}{x_\xi}\right)_\xi$$

The conservative form of above equation is

$$(x_\xi u)^{\cdot} - \left[\left(\dot{x} - \tfrac{u}{2}\right) u\right]_\xi = k\left(\frac{u_\xi}{x_\xi}\right)_\xi \tag{3}$$

3. Finite Difference Scheme for Burgers Equation

We assume that the domain $\Omega_t = [x_l(t), x_r(t)]$ is covered by a nonuniform mesh of N cells with

$$x_l(t) = x_0(t) < x_1(t) < ------- < x_{N-1}(t) < x_N(t) = x_r(t)$$

The nonuniform moving mesh in physical space is assumed to be the image of a fixed uniform mesh covering the computational domain $\Omega_0 = [0,1]$.via the mapping $x(\xi,t)$,so that $x_j(t) = x(\xi_j, t) = x\left(j/N, t\right)$, j=0,1,-----N.

The magnitude of each physical cell will be denoted by $h_j(t) = x_j(t) - x_{j-1}(t)$.j=1, 2------N and the midpoints of the cell are defined as

$$x_{j-1/2}(t) = \frac{1}{2}\left(x_j(t) + x_{j-1}(t)\right). \text{ j=1,2------N.}$$

The location of the physical mesh points at time level $t = t_n$ and $t = t_{n+1}$ is well defined through the mapping $x(\xi,t)$.To obtain the numerical approximation of (3) we require an approximation $x^h(\xi,t)$ of $x(\xi,t)$.We will assume that $x^h(\xi,t)$ is piecewise linear in space and time and hence

$$x^h(\xi,t) = x_{j-1/2}(t) + \left(\xi - \xi_{j-1/2}\right)\left(x_{j+1/2}(t) + x_{j-1/2}(t)\right)N \text{ ,for } \xi_{j-1/2} \le \xi \le \xi_{j+1/2} \text{ and}$$

$$x_{j-\frac{1}{2}}(t)=x_{j-\frac{1}{2}}(t_n)+(t-t_n)\left(\frac{x_{j-1/2}(t_{n+1})-x_{j-1/2}(t_n)}{t_{n+1}-t_n}\right), t_n \le t \le t_{n+1}.$$

Assuming this form for the mapping we have

$$x_\xi^h(t_n)=\left(\frac{x_{j+1/2}(t_n)-x_{j-1/2}(t_n)}{\Delta\xi}\right)=\frac{1}{\Delta\xi}\left(\frac{h_{j+1}^n+h_j^n}{2}\right) \quad ,\xi_{j-1/2}\le\xi\le\xi_{j+1/2}. \tag{4}$$

And

$$\dot{x}_{j-1/2}^h(t)=\left(\frac{x_{j-1/2}(t_{n+1})-x_{j-1/2}(t_n)}{t_{n+1}-t_n}\right), t_n \le t \le t_{n+1}. \tag{5}$$

With these definitions, it is easy to see that $x^h(\xi,t)$ satisfies

$$\left(x_\xi^h\right)_j^{n+1}=\left(x_\xi^h\right)_j^n+\frac{(t_{n+1}-t_n)}{\Delta\xi}\left(\dot{x}_{j+1/2}^h-\dot{x}_{j-1/2}^h\right) \tag{6}$$

To define the semidiscretization of (3), we use the notation u_j^n to denote the approximation of $u(x_j^n,t_n)$, the solution u at $(x(\xi_j,t_n),t_n)$ and $\mathbf{u}^n=(u_0^n,u_1^n---u_{N-1}^n,u_N^n)^T$. Forward and Backward divided differences in computational space are denoted by

$$(D_+\mathbf{u})_j=\frac{u_{j+1}-u_j}{h_{j+1}},(D_-\mathbf{u})_j=\frac{u_j-u_{j-1}}{h_j},$$

The central divided difference and average operator in space variables are given by

$$(\delta\mathbf{u})_j=(u_{j+1}-u_{j-1}) \text{ and } \mu\mathbf{u}_j=\frac{u_{j+1}+u_{j-1}}{2}$$

Using these notations, the semidiscretization of (3) takes the form

$$\left(x_\xi^h u\right)^{\cdot}_j-\left\{\left[\left(\dot{x}-\frac{u}{2}\right)u\right]_\xi\right\}_j=\frac{k}{\Delta\xi}\{[(D_+-D_-)u]_\xi\}_j \tag{7}$$

Using Crank-Nicolson scheme for equation (3) we get

$$\left(x_\xi^h u\right)_j^{n+1}-\left(x_\xi^h u\right)_j^n-\frac{\Delta t}{\Delta\xi}\mu\delta\left\{\frac{\left(\left(\dot{x}-\frac{u}{2}\right)u\right)_j^{n+1}+\left(\left(\dot{x}-\frac{u}{2}\right)u\right)_j^n}{2}\right\}$$

$$=k\frac{\Delta t}{\Delta\xi}\left[\frac{((D_+-D_-)u)_j^{n+1}+((D_+-D_-)u)_j^n}{2}\right] \tag{8}$$

In equation (8) we put

$$u_j^{n+1}=u_j^n+v_j^n.$$

For smalll Δt the difference between the two consecutive solutions will be small and hence the quadratic terms in $v_{i,j}$ can be neglected. And we get linear form of equation (8) as

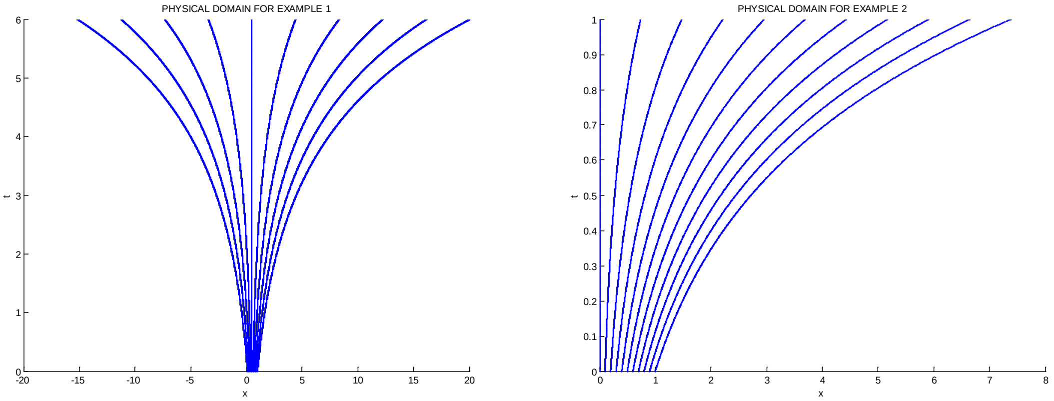
$$v_{j+1}^{n}\left[-r\dot{x}_{j+1}^{n+1}+ru_{j+1}^{n}-2krq\right]+v_{j}^{n}\left[\left(x_{\xi}^{h}\right)_{j}^{n+1}+2rkq+2rkp\right]$$
$$+v_{j-1}^{n}\left[r\dot{x}_{j-1}^{n+1}-ru_{j-1}^{n}-2krp\right]$$
$$=u_{j+1}^{n}\left[r\dot{x}_{j+1}^{n+1}-ru_{j+1}^{n}+4krq+r\dot{x}_{j+1}^{n}\right]+u_{j}^{n}\left[\left(x_{\xi}^{h}\right)_{j}^{n}-\left(x_{\xi}^{h}\right)_{j}^{n+1}-4krq-4krp\right]$$
$$+u_{j-1}^{n}\left[-r\dot{x}_{j-1}^{n+1}+ru_{j-1}^{n}+4krp-r\dot{x}_{j-1}^{n}\right]$$
(9)

where $r=\frac{\Delta t}{4\Delta\xi}, p=\frac{1}{h_j}, q=\frac{1}{h_{j+1}}$

We call scheme (9) as Crank-Nicolson Type method with moving mesh.

4. Numerical Results

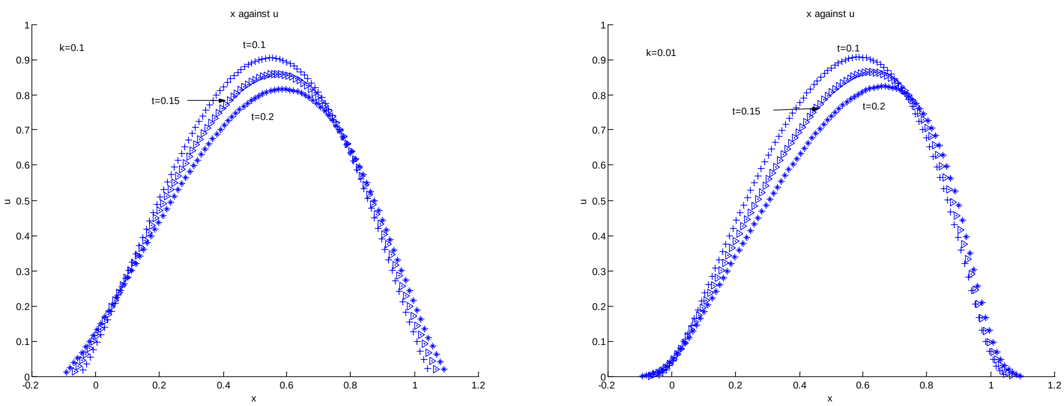
Numerical solution of one-dimensional nonlinear Burgers equation (1) is obtained by Crank Nicolson Type Method (9) with moving mesh. The solutions are obtained for k=0.1 and k=0.01 , t=0.1,0.15,0.2.The increment in time and space are $\Delta t=0.001\ \&\ \Delta\xi=0.01$ respectively. We assume that the computational mesh points are equispaced so that $h_{j+1}=h_j\ \forall\ j$. The values of x_{ξ}^{h} *and* $\dot{x}$ are evaluated from equation (4),(5) and (6). The solutions are obtained using Matlab.



(a) (b)

Fig. 1 Physical domain for Example 1 is (a) and for Example 2 is (b)

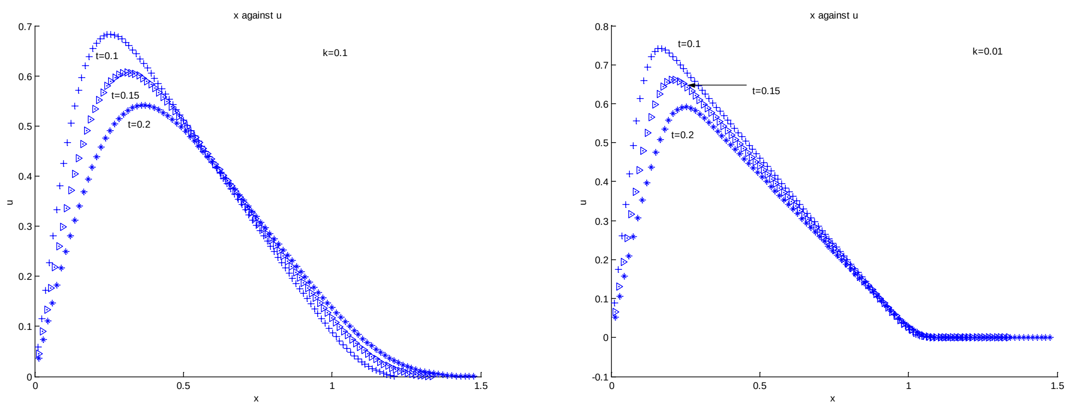
Example1:-Consider one dimensional non linear Burgers equation (1) in the region bounded by $x_l(t)=1-e^{t/2}$ and $x_r(t)=e^{t/2}$ with Homogeneous Dirichlets boundary conditions and initial condition $u(x,0)=u_0(x)=\sin\pi x$. The physical domain $\{[x_l(t),x_r(t)]\ ,t\epsilon[0,6]\}$ is shown in fig.1(a).The solutions obtained from scheme (9) are shown in fig.(2) for k=0.1 and k=0.01 .



(a) (b)

Fig.2 Solution of Burgers equation for example (1) with $\Delta t = 0.001, \Delta\xi = 0.01$ for a) k=0.1 b) k=0.01

Example2:-Consider one dimensional non linear Burgers equation (1) in the region bounded by $x_l(t) = 0$ and $x_r(t) = e^{2t}$ with Homogeneous Dirichlets boundary conditions and initial condition is $u_0(x) = \dfrac{x_r - x}{x_r - x_l}$. The physical domain $\{[x_l(t), x_r(t)]\ , t\epsilon[0,1]\}$ is shown in fig.1(b).The solutions obtained from scheme (9) are shown in fig.(3) for k=0.1 and k=0.01 .



(a) (b)

Fig.3 Solution of Burgers equation for example (2) with $\Delta t = 0.001, \Delta\xi = 0.01$ for a) k=0.1 b) k=0.01

5. Conclusion

A Crank Nicolson Type method with Moving Mesh is constructed for one dimensional nonlinear Burgers equation with Homogeneous Dirichlets boundary conditions. For this method discretization of the equation is obtained by Crank-Nicolson finite difference scheme. Numerical solutions are obtained for different domains for different values of t and constant of diffusivity k using Matlab.

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