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ON THE ALEKSANDROV-RASSIAS PROBLEM IN 2-NORMED SPACE

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Abstract. Let X and Y be 2-normed linear spaces. If a mapping $f : X \times X \to Y$ preserves two distances with a noninteger ratio, f must be an isometry. In this paper, we provide some results of the Aleksandrov-Rassias problem for mappings which preserves three distances in 2-normed space.

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1. INTRODUCTION

The theory of isometry had its beginning in the important paper by Mazur and Ulam in 1932. He proved that every isometry mapping of a normed real linear space onto a normed real linear space is a linear mapping up to translation. When the target space Y is a strictly convex real normed space, for into mapping, Baker[1] proved that every isometry of a normed real linear space into a strictly convex normed real linear space is also a linear isometry up to translation. What happens if we require, instead of one conservative distance for a mapping between normed vector spaces, two conservative disrtances? Aleksandrov and Rassias give some results about this problem. Aleksandrov-Rassias problem has obtained some results in Hilbert spaces, X and Y are Hilbert spaces with $dimX \geq 2$, if $T : X \to Y$ preserves two distances with a noninteger ratio, then T is linear isometry up to translation.

In 2001,Xiang Shuhuang[7] introduced that if f preserves two distances and X, Y are real normed vector spaces such that Y is strictly convex and $dimY \geq 2$, it is an open problem whether or not f must be an isometry, however, if f preserves three distances, we have the result about isometry.

Aleksandrov-Rassias problem: If T preserves two distances with a noninteger ratio, and X and Y are real normed vector spaces such

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that Y is strictly convex and $dimX \geq 2$,whether or not T must be an isometry?

Benz and Berens[2] proved the following theorem and pointed out that the condition that Y is strictly convex can not be relaxed. Other authors proved Aleksandrov-Rassias problem in [3],[4]and[5].

Let X and Y be real normed vector spaces, assume that $dimX \geq 2$ and Y is strictly convex, suppose $T : X \to Y$ satisfies that : T preserves the two distances ρ and $\lambda\rho$ for some integer $\lambda \geq 2$, that is, for all $x, y \in X$ with $\|x - y\| = \rho$, then $\|T(x) - T(y)\| \leq \rho$, and for all $x, y \in X$ with $\|x - y\| = \lambda\rho$, then $\|T(x) - T(y)\| \geq \lambda\rho$, so T is linear isometry up to translation.

It is easy to verify rhat the condition in above result is equivalent to that T preserves the distances ρ and $\lambda\rho$. Next we need the following definitions we will use in our main result.

Definition 1.1. Let X be a real linear space with $\|\cdot, \cdot\| : X^2 \to R$, then $(X, \|\cdot, \cdot\|)$ is called a 2-normed space if
(N_1) $\|x, y\| = 0 \Longleftrightarrow x$ and y are linearly dependent.
(N_2) $\|x, y\| = \|y, x\|$.
(N_3) $\|\alpha x, y\| = |\alpha|\|x, y\|$.
(N_4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

For $\alpha \in R$ and $x, y, z \in X$.The function $\|\cdot, \cdot\|$ is called the 2-norm on X.

Definition 1.2. [6] we called f a 2-isometry if $\|x - y, y - z\| = \|f(x) - f(y), f(y) - f(z)\|$ for all $x, y, z \in X$.

Definition 1.3. [6] (AOPP) Let $x, y, z \in X$ with $\|x - y, y - z\| = 1$, then $\|f(x) - f(y), f(y) - f(z)\| = 1$.

Definition 1.4. [6] We call f a 2-Lipschitz mapping if there is a $k \geq 0$ such that $\|f(z) - f(x), f(y) - f(x)\| \leq k\|z - x, y - x\|$ for all $x, y, z \in X$, the smallest such k is called the 2-Lipschitz constant.

2. MAIN RESULTS

Lemma 2.1. [6] *For $b, c \in X$, if b and c are linearly dependent with the same direction, that is $c = \alpha b$ for some $\alpha > 0$, then $\|a, b + c\| = \|a, b\| + \|a, c\|$ for all $a \in X$.*

Lemma 2.2. *Let X, Y are 2-normed space and $f : X \times X \to Y$ is a surjection and satisfied:*
(1) $\|x - y, p - q\| \leq 1$, then $\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|$.
(2) $\|x - y, p - q\| \geq \alpha$, then $\|f(x) - f(y), f(p) - f(q)\| \geq \alpha$.
For all $x, y, p, q \in X$ and then f is an 2-isometry.

Proof. (a) First, we proof $\|f(x)-f(y), f(p)-f(q)\| \leq \|x-y, p-q\|$ for all $x, y, p, q \in X$.
Let $\|x-y, p-q\| \leq \frac{m}{n}$, if $m=1$, the result is obvious.
We suppose that $m \geq 2$. Define $q_i = q + \frac{i}{m}(p-q), (i = 0, 1, 2, \ldots, m)$, and

$$q_{i+1} - q_i = \frac{1}{m}(p-q), p-q = \sum_{i=0}^{m-1}(q_{i+1}-q_i)$$

then

$$\|x-y, q_{i+1}-q_i\| = \|x-y, \frac{1}{m}(p-q)\| = \frac{1}{m}\|x-y, p-q\| \leq \frac{1}{n},$$

$$(i = 0, 1, 2, \ldots, m-1),$$

$$\begin{aligned} \|f(x)-f(y), f(p)-f(q) &\leq \sum_{i=0}^{m-1}\|f(x)-f(y), f(q_{i+1})-f(q_i)\| \\ &\leq \sum_{i=0}^{m-1}\|x-y, q_{i+1}-q_i\| \\ &\leq \frac{m}{n}. \end{aligned}$$

By Lemma 2.1

$$\|x-y, p-q\| = \sum_{i=0}^{m-1}\|x-y, q_{i+1}-q_i\|$$

Thus $\|f(x)-f(y), f(p)-f(q)\| \leq \|x-y, p-q\|$.
(b) We proof f perserves α. we suppose $\|x-y, p-q\| = \alpha$, Then there are $m, n \in N$, and $\alpha \leq \frac{m}{n}$, by (a), we have

$$\|f(x)-f(y), f(p)-f(q)\| \leq \|x-y, p-q\|$$

by the condition (2) $\|f(x)-f(y), f(p)-f(q)\| \geq \|x-y, p-q\|$, so

$$\|f(x)-f(y), f(p)-f(q)\| = \|x-y, p-q\| = \alpha.$$

(c) $\|f(x)-f(y), f(p)-f(q)\| = \|x-y, p-q\|$ as $\|x-y, p-q\| < \alpha$. For $\alpha > 0$, there must be $m, n \in N$, so $\alpha < \frac{m}{n}$, by (a), $\|f(x)-f(y), f(p)-f(q)\| \leq \|x-y, p-q\|$
Assume that

$$\|f(x)-f(y), f(p)-f(q)\| < \|x-y, p-q\|.$$

Let $z = x + \frac{\alpha}{\|x-y,p-q\|}(y-x)$, so

$$\|z-x,p-q\| = \alpha, \|z-y,p-q\| = \alpha - \|x-y,p-q\|.$$

Then by (b) and (a)

$$\begin{aligned}\alpha &= \|f(z)-f(x),f(p)-f(q)\| \\ &\leq \|f(z)-f(y),f(p)-f(q)\| + \|f(x)-f(y),f(p)-f(q)\| \\ &< \alpha - \|x-y,p-q\| + \|x-y,p-q\| = \alpha.\end{aligned}$$

This is a contradiction, that implies that $\|f(x)-f(y),f(p)-f(q)\| = \|x-y,p-q\|$.
(d) f preserve that the distance $\frac{n}{2}\alpha$.
Let $\|x-z,p-q\| = \frac{n}{2}\alpha$, by (a), then $\|f(x)-f(z),f(p)-f(q)\| \leq \frac{n}{2}\alpha$, let

$$u = f(x) + \frac{\alpha}{2}\frac{f(z)-f(x)}{\|f(z)-f(x),f(p)-f(q)\|}$$

there exsits a $v \in X$, such that $f(v) = u$ by f is a surjection. then

$$\|u-f(x),f(p)-f(q)\| = \frac{\alpha}{2} < \alpha,$$

by (2), $\|v-x,p-q\| < \alpha$. by (c), $\|v-x,p-q\| = \|u-f(x),f(p)-f(q)\| = \frac{\alpha}{2}$.
Then

$$\|u-f(z),f(p)-f(q)\| \geq \frac{\alpha}{2}(n-1).$$

Otherwise if $\|u-f(z),f(p)-f(q)\| < \frac{\alpha}{2}(n-1)$, we can find a sequence $v_i \in X, (i = 1,2,\cdots,n-1)$, such that $v_0 = v, v_{n-1} = z$, which implies $\|f(v)-f(z),f(p)-f(q)\| < \frac{\alpha}{2}(n-1)$, by f is surjection, there exsits $v_i \in X$, then

$$f(v_i) = f(v) + \frac{i}{n-1}(f(z)-f(v)), (i = 0,1,2,\cdots,n-1),$$

so $f(v_i) - f(v_{i+1}) = \frac{1}{n-1}(f(z)-f(v))$ and $f(v_i) - f(v_{i+1})$ collinear, hence

$$f(v) - f(z) = \sum_{i=0}^{n-2}(f(v_i) - f(v_{i+1}))$$

by Lemma2.1,

$$\|f(v)-f(z),f(p)-f(q)\| = \sum_{i=0}^{n-2}\|f(v_i)-f(v_{i+1}),f(p)-f(q)\| < \frac{(n-1)\alpha}{2}$$

So

$$\|f(v_i) - f(v_{i+1}), f(p) - f(q)\| < \frac{\alpha}{2} < \alpha,$$

by the condition (2), thus $\|v_i - v_{i+1}, p - q\| < \alpha (i = 0, 1, 2, \cdots, n-1)$ and by (c)

$$\|v_i - v_{i+1}, p - q\| = \|f(v_i) - f(v_{i+1}), f(p) - f(q)\| < \frac{\alpha}{2}(i = 0, 1, 2, \cdots, n-1),$$

that implies

$$\begin{aligned} \|v - z, p - q\| = \|\sum_{i=0}^{n-1}(v_i - v_{i+1}), p - q\| &\leq \sum_{i=1}^{n-1}\|v_i - v_{i+1}, p - q\| \\ &< \sum_{i=0}^{n-1}\frac{\alpha}{2} = \frac{(n-1)\alpha}{2}. \end{aligned}$$

by(c),

$$\|v - x, p - q\| = \|f(v) - f(x), f(p) - f(q)\| = \|u - f(x), f(p) - f(q)\| = \frac{\alpha}{2}$$

Moreover

$$\|x - z, p - q\| \leq \|x - v, p - q\| + \|v - z, p - q\| < \frac{\alpha}{2} + \frac{n-1}{2}\alpha = \frac{n}{2}\alpha$$

.

This is a contradiction with $\|x - z, p - q\| = \frac{n}{2}\alpha$. and

$$\begin{aligned} u - f(z) &= f(x) - f(z) + \frac{\alpha}{2}\frac{f(z) - f(x)}{\|f(z) - f(x), f(p) - f(q)\|} \\ &= (f(x) - f(z))(1 - \frac{\alpha}{2\|f(z) - f(x), f(p) - f(q)\|}) \end{aligned}$$

then

$$\begin{aligned} &\|u - f(z), f(p) - f(q)\| \\ = &\|f(x) - f(z), f(p) - f(q)\|(1 - \frac{\alpha}{2\|f(z) - f(x), f(p) - f(q)\|}) \\ = &\|f(x) - f(z), f(p) - f(q)\| - \frac{\alpha}{2} \end{aligned}$$

So

$$\frac{\alpha}{2}(n-1) \leq \|u - f(z), f(p) - f(q)\| = \|f(z) - f(x), f(p) - f(q)\| - \frac{\alpha}{2}$$

that implies $\|f(z) - f(x), f(p) - f(q)\| = \frac{\alpha}{2}n$.

(e) f is isometry from X to Y. That is $\|f(x) - f(y), f(p) - f(q)\| = \|x - y, p - q\|$.

For any $x, y, p, q \in X$ and $\alpha > 0$, there exsits n such that $\|x - y, p - q\| <$

$\frac{\alpha}{2}n$. For $\frac{\alpha}{2}n$, there exsit $m, n\prime \in N$, and $\frac{\alpha}{2}n < \frac{m}{n\prime}$, by (a), we have $\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|$.
Assume that

$$\|f(x) - f(y), f(p) - f(q)\| < \|x - y, p - q\|.$$

Let

$$z = x + \frac{\frac{\alpha}{2}n}{\|x - y, p - q\|}(y - x).$$

So $\|z - x, p - q\| = \frac{\alpha}{2}n$, $\|z - y, p - q\| = \frac{\alpha}{2}n - \|x - y, p - q\|$. So by (d),

$$\|f(z) - f(x), f(p) - f(q)\| = \|z - x, p - q\| = \frac{\alpha}{2}n.$$

By (c),(d) and the assumption

$$\begin{aligned} \frac{\alpha}{2}n &= \|f(z) - f(x), f(p) - f(q)\| \\ &\leq \|f(z) - f(y), f(p) - f(q)\| + \|f(x) - f(y), f(p) - f(q)\| \\ &< \frac{\alpha}{2}n - \|x - y, p - q\| + \|x - y, p - q\| = \frac{\alpha}{2}n \end{aligned}$$

that is a contradiction, that implies $\|f(x) - f(y), f(p) - f(q)\| = \|x - y, p - q\|$. □

we can say that f preserves the two distance 1 and α in above Lemma.

Theorem 2.3. *Let X and Y be real 2-normed space. Assume that Y is strictly convex, suppose $f : X \to Y$ satisfied AOPP and f is a 2-Lipschitz mapping with $k = 1$, that is $\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|$ for all $x, y, p, q \in X$. Then f is a 2-isometry.*

Proof. Let $x, y, p, q \in X$, and $\|x - y, p - q\| = \frac{1}{2}$, set $z = x + 2(y - x)$
Then $\|x - z, p - q\| = 1, \|z - y, p - q\| = \frac{1}{2}$
And by the condition 2-Lipschitz and AOPP,

$$\begin{aligned} & \|x - y, p - q\| \\ \geq\ & \|f(x) - f(y), f(p) - f(q)\| \\ \geq\ & \|f(z) - f(x), f(p) - f(q)\| - \|f(z) - f(y), f(p) - f(q)\| \\ \geq\ & 1 - \|z - y, p - q\| \geq \frac{1}{2} \end{aligned}$$

By the condition, $\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\| = \frac{1}{2}$
Hence

$$\|f(x) - f(y), f(p) - f(q)\| = \frac{1}{2}$$

Similarly

$$\|f(z)-f(y),f(p)-f(q)\|=\frac{1}{2}$$

And

$$\|f(z)-f(x),f(p)-f(q)\|$$
$$=\|f(z)-f(y),f(p)-f(q)\|+\|f(y)-f(x),f(p)-f(q)\|=1$$

Because Y is strictly convex, then $f(y)=\frac{f(x)+f(z)}{2}$ and $\|f(y)-f(x),f(p)-f(q)\|=\frac{1}{2}$, so f preserves distances 1 and $\frac{1}{2}$, so f is an isometry due to lemma2.2. □

Theorem 2.4. *Let X and Y be real 2-normed spaces. Assume that $dimX\geq 2$ and Y is strictly convex, suppose $f:X\times X\to Y$ satisfies the property that f preserves the three distances 1,a and 1+a, where a is any positive constant. Then f is a 2-isometry.*

Proof. (1) Let $x,y\in X$, $\|x-y,p-q\|=2+a$, set $x_1=x+\frac{1}{2+a}(y-x)$, $x_2=x+\frac{1+a}{2+a}(y-x)$
Then

$$\|x_1-x,p-q\|=1,\|x_1-x_2,p-q\|=a,$$
$$\|y-x_1,p-q\|=1+a,\|x_2-x,p-q\|=1+a,\|y-x_2,p-q\|=1$$

It follows that

$$\|f(x_1)-f(x),f(p)-f(q)\|=1,\|f(x_1)-f(x_2),f(p)-f(q)\|=a$$

$\|f(y)-f(x_1),f(p)-f(q)\|=1+a$, $\|f(x_2)-f(x),f(p)-f(q)\|=1+a$, $\|f(y)-f(x_2),f(p)-f(q)\|=1$. Since Y is strictly convex, let

$$f(x_1)-f(x)=\alpha(f(x_2)-f(x))(\alpha>0)$$

Then

$$\begin{aligned}1=\|f(x_1)-f(x),f(p)-f(q)\| &= \alpha\|f(x_2)-f(x),f(p)-f(q)\|\\ &= \alpha(a+1)\end{aligned}$$

So $\alpha=\frac{1}{a+1}$ and $f(x_1)-f(x)=\frac{1}{a+1}(f(x_2)-f(x))$
We have

$$f(x_1)=f(x)+\frac{1}{1+a}(f(x_2)-f(x))$$

And

$$f(x)=\frac{1+a}{a}f(x_1)-\frac{1}{a}f(x_2).$$

Since Y is strictly convex, let

$$f(y)-f(x_2)=\alpha^{'}(f(x_2)-f(x_1))$$

Then

$$1 = \|f(y)-f(x_2), f(p)-f(q)\| = \alpha^{'}\|f(x_2)-f(x_1), f(p)-f(q)\| = \alpha^{'}a$$

So $\alpha^{'} = \frac{1}{a}$ and $f(y)-f(x_2) = \frac{1}{a}(f(x_2)-f(x_1))$
We have

$$f(x_2) = f(x_1) + \frac{a}{1+a}(f(y)-f(x_1)$$

And

$$f(y) = \frac{1+a}{a}f(x_2) - \frac{1}{a}f(x_1).$$

Thus $\|f(x)-f(y), f(p)-f(q)\| = 2+a$ for all $x, y \in X$ with $\|x-y, p-q\| = 2+a$, so f preserves the distance 2+a.

(2) Let $\|x-y, p-q\| = 2a$, set $x_1 = x + \frac{a-1}{2+2a}(y-x)$, $x_2 = x + \frac{a}{2+2a}(y-x)$
Then

$$\|x_1-x, p-q\| = 1+a, \|x_1-x_2, p-q\| = 1,$$

$$\|y-x_1, p-q\| = 1+a, \|x_2-x, p-q\| = 2+a, \|y-x_2, p-q\| = a,$$

since f preserves distances 1,a,1+a and 2+a, in a similar way, we obtain that

$$\|f(y)-f(x), f(p)-f(q)\| = 2a.$$

By (1),(2) and Lemma 2.2, we have f preserves 1 and 2a, then f is a 2-isometry. □

Corollary 2.5. *Let X and Y be real 2-normed spaces. Assume that $dimX \geq 2$ and Y is strictly convex, suppose $f : X \times X \to Y$ satisfies the property that f preserves the three distances a,b and a+b, where a,b is any positive constant. Then f is a 2-isometry.*

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