ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2014, Article ID ama0127, 13 pages
ISSN 2307-7743
http://scienceasia.asia

SECOND HANKEL DETERMINANT FOR SUBCLASSES OF PASCU CLASSES OF ANALYTIC FUNCTIONS

M. S. SAROA, GURMEET SINGH AND GAGANDEEP SINGH

Abstract: We define two subclasses of the class of Pascu functions. For any real μ, we are interested in determining the upper bound of $\left|a_2a_4-\mu a_3^2\right|$ for an analytic function $f(z)=z+a_2z^2+a_3z^3+\ldots\ \left(|z|<1\right)$ belonging to these classes.

1. INTRODUCTION AND DEFINITION:

PRINCIPLES OF SUBORDINATION: Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc $E=\{z:|z|<1\}$. Then, $f(z)$ is said to be subordinate to $F(z)$ in the unit disc E if there exists an analytic function $w(z)$ in E satisfying the condition $w(0)=0$, $|w(z)|<1$ such that $f(z)=F(w(z))$ and we write as $f(z)\prec F(z)$. In particular if $F(z)$ is univalent in D, the above definition is equivalent to $f(0)=F(0)$ and $f(E)\subset F(E)$.

FUNCTIONS WITH POSITIVE REAL PART: Let Ƥ denotes the class of analytic functions of the form

(1.1) $$P(z)=1+p_1z+p_2z^2+\ldots\ldots.$$

with $\operatorname{Re}P(z)>0,\ z\in D$.

Let A denote the class of functions of the form

(1.2) $$f(z)=z+\sum_{k=2}^{\infty}a_kz^k$$

which are analytic in the open unit disc $D=\{z:|z|<1\}$.

S is the class of functions of the form (1.2) which are univalent.

The Hankel determinant: ([9],[10])

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be analytic in D. For $q\geq 1$, the qth Hankel determinant is defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1}\ldots\ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2}\ldots\ldots & a_{n+q} \\ a_{n+q-1} & a_{n+q}\ldots\ldots & a_{n+2q-2} \end{vmatrix}.$$

2010 Mathematics Subject Classification: 30C45.

Key words and phrases: Subordination, Hankel determinant, functions with positive real part, univalent functions and close to star functions.

The Hankel determinant was studied by various authors including Hayman[3] and Ch. Pommerenke ([13],[14]). For q= 2 and n = 2 , the second Hankel determinant for the analytic function $f(z)$ is defined by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left(a_2 a_4 - a_3^2\right)$$

R_0 represents the class of functions $f(z) \in A$ and satisfying the condition

(1.3) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 0,\ z \in D.$$

R_0 is a particular case of the class of close to star function defined by Reade[17]. The class R_0 and its subclasses were vastly studied by several authors including Mac-Gregor[7].
Let R be the class of functions $f(z) \in A$ and satisfying

(1.4) $$\operatorname{Re} f'(z) > 0,\ z \in D.$$

The class R was introduced by Noshiro [11] and Warschawski[18] (known as N-W class) and it was shown by them that R is a class of univalent functions. The class R and its subclasses were investigated by various authors including Goel and the author ([1], [2]).
For $\alpha \geq 0,$ $R_1(\alpha)$ and $R_2(\alpha)$ denote the classes of functions in A which satisfy, respectively, the conditions

(1.5) $$\operatorname{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > 0,\ z \in D$$

and

(1.6) $$\operatorname{Re}\left[f'(z) + \alpha z f''(z)\right] > 0,\ z \in D.$$

The classes $R_1(\alpha)$ and $R_2(\alpha)$ were introduced by Pascu [12] and are called Pascu classes of functions. It is obvious that $f(z) \in R_1(\alpha)$ implies that $zf'(z) \in R_2(\alpha)$.
We shall deal with the following classes

(1.7) $$R_1(\alpha; A, B) = \left\{ f \in A : \left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1+Az}{1+Bz}, \alpha \geq 0, -1 \leq B < A \leq 1, z \in D \right] \right\}$$

and

(1.8) $$R_2(\alpha; A, B) = \left\{ f \in A : \left[f'(z) + \alpha z f''(z) \prec \frac{1+Az}{1+Bz}, \alpha \geq 0, -1 \leq B < A \leq 1, z \in D \right] \right\}.$$

$R_1(\alpha;1,-1) \equiv R_1(\alpha)$ and $R_2(\alpha;1,-1) \equiv R_2(\alpha)$. $R_1(\alpha;A,B)$ is a subclass of $R_1(\alpha)$ and $R_2(\alpha;A,B)$ is a sublass of $R_2(\alpha)$. The classes $R_1(\alpha;A,B)$ and $R_2(\alpha;A,B)$ were studied by the author[8].Througout the paper, we assume that $\alpha \geq 0, -1 \leq B < A \leq 1$ and $z \in D$.

2. PRELIMINARY LEMMAS

Lemma 2.1 [15]. Let $P(z) \in \mathrm{P}(z)$, then

$|p_n| \le 2 \quad (n = 1,2,3,..)$

Lemma 2.2 [5]. Let $P(z) \in \mathrm{P}(z)$, then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some x and z with $|x| \le 1, |z| \le 1$ and $p_1 \in [0,2]$.

3. MAIN RESULTS

Theorem 3.1: Let $f \in R_1(\alpha; A, B)$, then

$|a_2 a_4 - \mu a_3^2| \le$

$$\begin{cases}
(3.1)\quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{\left\{3(1+2\alpha)^2 - 2\mu(1+\alpha)(1+3\alpha)\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right\}} - \dfrac{4\mu}{(1+2\alpha)^2}\right] if\, \mu \le 0; \\[2ex]
(3.2)\quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{\left\{3(1+2\alpha)^2 - 4\mu(1+\alpha)(1+3\alpha)\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right\}} + \dfrac{4\mu}{(1+2\alpha)^2}\right] \\
\quad if\, 0 \le \mu \le \dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)}; \\[2ex]
(3.3)\quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{4\mu}{(1+2\alpha)^2}\right] if\, \dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)} \le \mu \le \dfrac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}; \\[2ex]
(3.4)\quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{\left\{2\mu(1+\alpha)(1+3\alpha) - 3(1+2\alpha)^2\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{\mu(1+\alpha)(1+3\alpha) - (1+2\alpha)^2\right\}} + \dfrac{4\mu}{(1+2\alpha)^2}\right] \\
\quad if\, \mu \ge \dfrac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}.
\end{cases}$$

Proof. By definition of subordination,

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

Taking real parts,

$$\mathrm{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] = \mathrm{Re}\left[\frac{1 + Aw(z)}{1 + Bw(z)}\right]$$

$$\geq \frac{1-Ar}{1-Br} > \frac{1-A}{1-B} \qquad (|z|=r)$$

which implies that

(3.5) $$1+\frac{1-B}{A-B}\left[(1+\alpha)a_2 z+(1+2\alpha)a_3 z^2+(1+3\alpha)a_4 z^3+\ldots\right]=P(z) .$$

Equating the coefficients in (3.5), we get

(3.6) $$\begin{cases} a_2 = \left(\dfrac{1-B}{A-B}\right)\dfrac{p_1}{(1+\alpha)} \\ a_3 = \left(\dfrac{1-B}{A-B}\right)\dfrac{p_2}{(1+2\alpha)} \\ a_4 = \left(\dfrac{1-B}{A-B}\right)\dfrac{p_3}{(1+3\alpha)} \end{cases}$$

System (3.6) ensures that

(3.7) $$C(\alpha)\left(a_2 a_4-\mu a_3^2\right)=(1+2\alpha)^2 p_1(4p_3)-\mu(1+\alpha)(1+3\alpha)(2p_2)^2 ,$$

(3.8) $$C(\alpha)=4\left(\frac{A-B}{1-B}\right)^2(1+\alpha)(1+3\alpha)(1+2\alpha)^2 .$$

Using Lemma 2.2 in (3.7), we obtain

$$C(\alpha)\left(a_2 a_4-\mu a_3^2\right)=(1+2\alpha)^2 p_1\left[p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]$$
$$-\mu(1+\alpha)(1+3\alpha)\left[p_1^2+\left(4-p_1^2\right)x\right]^2$$

for some x and z with $|x|\leq 1$, $|z|\leq 1.$ or

(3.9) $$\begin{aligned} C(\alpha)\left(a_2 a_4-\mu a_3^2\right)=&\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p_1^4 \\ &+2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p_1^2\left(4-p_1^2\right)x \\ &-\left(4-p_1^2\right)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p_1^2+4\mu(1+\alpha)(1+3\alpha)\right]x^2 \\ &+2(1+2\alpha)^2 p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z \end{aligned}$$

Replacing p_1 by $p\in[0,2]$ and applying triangular inequality to (3.9), we get

$$C(\alpha)\left|a_2 a_4-\mu a_3^2\right|\leq\begin{cases}\left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right|p^4+2\left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right|p^2\left(4-p^2\right)\delta \\ +\left(4-p^2\right)\left[\ \left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\ \right|p^2+|4\mu|(1+\alpha)(1+3\alpha)\right]\delta^2 \\ +2(1+2\alpha)^2 p\left(4-p^2\right)\left(1-\delta^2\right),\left(\delta=|x|\leq 1\right)\end{cases}$$

which can be put in the form

$$(3.10)\quad C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2(1+2\alpha)^2p(4-p^2) \\ +2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2)\delta \\ +(4-p^2)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2-4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\,\mu\le 0; \\ \\ \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2(1+2\alpha)^2p(4-p^2) \\ +2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2)\delta \\ +(4-p^2)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2+4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\,0\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}; \\ \\ \left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^4+2(1+2\alpha)^2p(4-p^2) \\ +2\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^2(4-p^2)\delta \\ +(4-p^2)\left[\left\{\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right\}p^2+4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\,\mu\ge\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)} \end{cases}$$

$$= F(\delta).$$

$F'(\delta)>0$ and therefore $F(\delta)$ is increasing in [0,1]. $F(\delta)$ attains its maximum value at $\delta=1$. (3.10) reduces to

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2) \\ +(4-p^2)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2-4\mu(1+\alpha)(1+3\alpha)\right] if\,\mu\le 0; \\ \\ \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2) \\ +(4-p^2)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2+4\mu(1+\alpha)(1+3\alpha)\right] if\,0\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}; \\ \\ \left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^4+2\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^2(4-p^2) \\ +(4-p^2)\left[\left\{\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right\}p^2+4\mu(1+\alpha)(1+3\alpha), if\,\mu\ge\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}\right] \end{cases}$$

$= G(p)$, or

(3.11) $$C(\alpha)\left|a_2a_4-\mu a_3^2\right|\le \max G(p),$$

Case (i) $\mu \le 0$

$$G(p)=-2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]p^2-16(1+\alpha)(1+3\alpha)\mu$$

$G(p)$ is maximum for

$$G'(p)=-8\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^3+8\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]p=0$$

which implies that $p=\sqrt{\dfrac{\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]}{\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]}}$.

Putting the corresponding value of $G(p)$ along with $C(\alpha)$ from (3.8) in (3.11), we get (3.1)

Case (ii) $0\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$$G(p)=-2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[3(1+2\alpha)^2-4\mu(1+\alpha)(1+3\alpha)\right]p^2+16(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $0\le\mu\le\dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)}$

It is easy to see that $G(p)$ is maximum at

$$p=\sqrt{\frac{\left[3(1+2\alpha)^2-4\mu(1+\alpha)(1+3\alpha)\right]}{\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]}}.$$

Substituting the corresponding value of $G(p)$ and the value of $C(\alpha)$ in (3.11), (3.2) follows

Sub-case (b) $\dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)}\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$G'(p)<0$ and $G(p)$ is maximum at p=0

In this sub-case $\max G(p)=16(1+\alpha)(1+3\alpha)\mu$.

Case (iii) $\mu\ge\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$$G(p)=-2\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^4+4\left[2\mu(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2\right]p^2+16(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}\le\mu\le\dfrac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}$

$G'(p)<0$ and maximum $G(p)=G(0)=16(1+\alpha)(1+3\alpha)\mu$

Combining the cases (ii)-(b) and (iii)-(a) we arrive at (3.3)

Sub-case (b) $\mu \geq \dfrac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}$

A simple calculus shows that $G(p)$ is maximum at $p=\sqrt{\dfrac{\left[2\mu(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2\right]}{\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]}}$

Substituting the corresponding value of $G(p)$ and the value of $C(\alpha)$ in (3.11), (3.4) follows

Remark 3.1 Put A=1 and B= -1 in the theorem we get the estimates for the class $R_1(\alpha)$.

Taking A = 1, B = -1 and $\alpha=0$ in the theorem we have

Corollary 3.1 If $f \in R_0$, then

$$\left|a_2a_4-\mu a_3^2\right| \leq \begin{cases} \dfrac{(3-2\mu)^2}{2(1-\mu)}-4\mu, \mu \leq 0; \\ \dfrac{(3-4\mu)^2}{2(1-\mu)}+4\mu, 0 \leq \mu \leq \dfrac{3}{4}; \\ 4\mu, \dfrac{3}{4} \leq \mu \leq \dfrac{3}{2}; \\ \dfrac{(2\mu-3)^2}{2(\mu-1)}+4\mu, \mu \geq \dfrac{3}{2}. \end{cases}$$

Letting A = 1,B = -1 and $\alpha=1$ we get

Corollary 3.2 If $f \in R$, then

$$\left|a_2a_4-\mu a_3^2\right| \leq \begin{cases} \dfrac{(27-16\mu)^2}{144(9-8\mu)}-\dfrac{4\mu}{9}\mu \leq 0; \\ \dfrac{(27-32\mu)^2}{144(9-8\mu)}+\dfrac{4\mu}{9}, 0 \leq \mu \leq \dfrac{27}{32}; \\ \dfrac{4\mu}{9}, \dfrac{27}{32} \leq \mu \leq \dfrac{27}{16};; \\ \dfrac{(16\mu-27)^2}{144(8\mu-9)}+\dfrac{4\mu}{9}, \mu \geq \dfrac{27}{16}. \end{cases}$$

This results was proved by Janteng et al [4]

Theorem 3.2 Let $f \in R_2(\alpha;A,B)$, then

$\left|a_2a_4-\mu a_3^2\right| \leq$

$$
\begin{cases}
\left(\frac{1-B}{A-B}\right)^2\left(\frac{\left\{27(1+2\alpha)^2-16\mu(1+\alpha)(1+3\alpha)\right\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}}-\frac{4\mu}{9(1+2\alpha)^2}\right) & (3.12)\\
\text{if } \mu\le 0; & \\
\left(\frac{1-B}{A-B}\right)^2\left[\frac{\left\{27(1+2\alpha)^2-32\mu(1+\alpha)(1+3\alpha)\right\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}}+\frac{4\mu}{9(1+2\alpha)^2}\right] & (3.13)\\
\text{if } 0\le\mu\le\frac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}; & \\
\left(\frac{1-B}{A-B}\right)^2\left[\frac{4\mu}{9(1+2\alpha)^2}\right] & (3.14)\\
\text{if } \frac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}\le\mu\le\frac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}; & \\
\left(\frac{1-B}{A-B}\right)^2\left[\frac{\left\{16\mu(1+\alpha)(1+3\alpha)-27(1+2\alpha)^2\right\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right\}}+\frac{4\mu}{9(1+2\alpha)^2}\right] & (3.15)\\
\text{if } \mu\ge\frac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}. &
\end{cases}
$$

Proof. We have

$$f'(z)+\alpha z f''(z)=\frac{1+Aw(z)}{1+Bw(z)}$$

Taking real parts,

$$\operatorname{Re}\left[f'(z)+\alpha z f''(z)\right]=\operatorname{Re}\left[\frac{1+Aw(z)}{1+Bw(z)}\right]\ge\frac{1-Ar}{1-Br}>\frac{1-A}{1-B}\qquad (|z|=r)$$

This implies that

$$1+\left(\frac{1-B}{A-B}\right)\left[2(1+\alpha)a_2z+3(1+2\alpha)a_3z^2+4(1+3\alpha)a_4z^3+\dots\dots\right]=P(z)\tag{3.16}$$

Identifying the terms in (3.16), we get

$$
\begin{cases}
a_2=\left(\frac{A-B}{1-B}\right)\frac{p_1}{2(1+\alpha)}\\
a_3=\left(\frac{A-B}{1-B}\right)\frac{p_2}{3(1+2\alpha)}\\
a_4=\left(\frac{A-B}{1-B}\right)\frac{p_3}{4(1+3\alpha)}
\end{cases}\tag{3.17}
$$

System (3.17) yields

$$C(\alpha)\left(a_2a_4-\mu a_3^2\right)=9(1+2\alpha)^2p_1(4p_3)-8\mu(1+\alpha)(1+3\alpha)(2p_2)^2,\tag{3.18}$$

(3.19) $C(\alpha)=\left(\frac{A-B}{1-B}\right)^2\left[288(1+\alpha)(1+3\alpha)(1+2\alpha)^2\right]$

By Lemma 2.2, (3.19) can be written as

$$C(\alpha)(a_2a_4-\mu a_4^2)=9(1+2\alpha)^2p_1\left[p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z\right]$$

$$-8\mu(1+\alpha)(1+3\alpha)\left[p_1^2+(4-p_1^2)x\right]^2$$

for some x and z with $|x|\le 1,|z|\le 1.$

or

(3.20) $$C(\alpha)(a_2a_4-\mu a_3^2)=\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p_1^4+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p_1^2(4-p_1^2)x$$

$$-(4-p_1^2)\left[\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)+32\mu(1+\alpha)(1+3\alpha)\right\}\right]x^2$$

$$+18(1+2\alpha)^2p_1(4-p_1^2)\left(1-|x|^2\right)z\,.$$

Replacing p_1 by $p\in[0,2]$ and applying triangular inequality to (3.20), we get

$$C(\alpha)|a_2a_4-\mu a_3^2|\le\left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right|p^4$$

$$2\left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right|p^2(4-p^2)\delta$$

$$+(4-p^2)\left[\ \left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\ \right|p^2+|32\mu|(1+\alpha)(1+3\alpha)\right]\delta^2$$

$$+18(1+2\alpha)^2p(4-p^2)\left(1-|\delta|^2\right)\,.\quad(\delta=|x|\le 1)$$

which can be put in the form

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+18(1+2\alpha)^2p(4-p^2) \\ +2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2)\delta \\ +(4-p^2)\begin{bmatrix}\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2-32\mu(1+\alpha)(1+3\alpha) \\ -18(1+2\alpha)^2p\end{bmatrix}\delta^2 if\,\mu\le 0; \\ \\ \left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+18(1+2\alpha)^2p(4-p^2) \\ +2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^2(4-p^2)\delta \\ (4-p^2)\begin{bmatrix}\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2+32\mu(1+\alpha)(1+3\alpha) \\ -18(1+2\alpha)^2p\end{bmatrix}\delta^2 \\ if\,0\le\mu\le\dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}; \\ \\ \left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^4+18(1+2\alpha)^2p(4-p^2) \\ +2\left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^2(4-p^2)\delta \\ (4-p^2)\begin{bmatrix}\left\{8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right\}p^2+32\mu(1+\alpha)(1+3\alpha) \\ -18(1+2\alpha)^2p\end{bmatrix}\delta^2 \\ if\,\mu\ge\dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}. \end{cases}$$

$$= F(\delta)$$

$F'(\delta)>0$ which means that $F(\delta)$ is increasing in [0 , 1] and $F(\delta)$ attains maximum value at $\delta=1$

(3.21) reduces to

$$C(\alpha)\left|\left(a_2a_4-\mu a_3^2\right)\right| \le \begin{cases} \left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right) \\ +\left(4-p^2\right)\left[\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2-32\mu(1+\alpha)(1+3\alpha)\right], if\, \mu \le 0; \\ \\ \left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)p^2\left(4-p^2\right)\right] \\ +(4-p^2)\left[\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2+32\mu(1+\alpha)(1+3\alpha)\right] \\ if\, 0 \le \mu \le \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}; \\ \\ \left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^4+2\left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^2\left(4-p^2\right) \\ +\left(4-p^2\right)\left[\left\{8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right\}p^2+32\mu(1+\alpha)(1+3\alpha)\right] \\ if\, \mu \ge \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}. \end{cases}$$

Or

(3.22) $C(\alpha)\left|\left(a_2a_4-\mu a_3^2\right)\right| \le G(p)$.

Case (i) $\mu \le 0$

$$G(p) = -2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[27(1+2\alpha)^2-16\mu(1+\alpha)(1+3\alpha)\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu \ .$$

$G(p)$ is maximum for

$$G'(p) = -8\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^3+8\left[27(1+2\alpha)^2-16\mu(1+\alpha)(1+3\alpha)\right]p = 0$$

which gives

$$p = \sqrt{\frac{27(1+2\alpha)^2-16\mu(1+\alpha)(1+3\alpha)}{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)}} \ .$$

Putting the corresponding value of $G(p)$ along with the value of $C(\alpha)$ from (3.19) in (3.22), we get (3.12).

Case (ii) $0 \le \mu \le \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$$G(p) = -2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[27(1+2\alpha)^2-32\mu(1+\alpha)(1+3\alpha)\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $0 \le \mu \le \dfrac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}$

An elementary calculus shows that $G(p)$ is maximum at

$$p = \sqrt{\frac{27(1+2\alpha)^2 - 32\mu(1+\alpha)(1+3\alpha)}{9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)}}\,.$$

With the corresponding value of $G(p)$ along with the value of $C(\alpha)$ in (3.22) , we arrive at (3.13).

Sub-case (b) $\dfrac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)} \le \mu \le \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$G'(p) < 0$ and $G(p)$ is maximum at $p = 0$.

$\max G(p) = G(0) = 128(1+\alpha)(1+3\alpha)\mu$.

Case (iii) $\mu \ge \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$$G(p) = -2\left[8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2\right]p^4 + 4\left[16\mu(1+\alpha)(1+3\alpha) - 27(1+2\alpha)^2\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu\,.$$

Sub-case (a) $\dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)} \le \mu \le \dfrac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}$

$G'(p) < 0$ and $\operatorname{Max} G(p) = G(0) = 128(1+\alpha)(1+3\alpha)\mu$.

Combining the cases (ii-b) and (iii-a), (3.14) follows.

Sub-case (b) $\mu \ge \dfrac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}$

An easy calculation shows that $G(p)$ is maximum at $p = \sqrt{\dfrac{16\mu(1+\alpha)(1+3\alpha) - 27(1+2\alpha)^2}{8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2}}$.

Substituting the corresponding value of $G(p)$ along with the value of $C(\alpha)$ in (3.22), we obtain (3.15).

Remark 3.2 Putting A=1 and B= -1 in the theorem we get the estimates for the class $R_2(\alpha)$.

Remark 3.3 Letting $A = 1, B = -1$ and $\alpha = 0$ in the theorem , corollary 3.2 follows.

REFERENCES

[1] R.M.Goel and B.S.Mehrok, A subclass of univalent functions, Houston J. Math., 8(1982), 343-357.

[2] R.M.Goel and B.S.Mehrok, A subclass of univalent functions, J.Austral.Math.Soc. (Series A), 35(1983), 1-17.

[3] W.K. Hayman, Multivalent functions, Cambridge Tracts in math. and math-physics, No. 48 Cambridge university press, Cambridge (1958).

[4] A.Janteng , S.A.Halim and M.Darus, Estimates on the second Hankel functional for functions whose derivative has a positive real part, J. Quality Measurement and analysis, JQMA(1) (2008) 189-195.

[5] R.J.Libera and E.J.Zlotkiewics, Early coefficients of the inverse of a regular convex function, Proceeding Amer. Math.Soc., 85(1982) , 225-230.

[6] J.E. Littlewood , On inequalties in the theory of functions. Procc. London Math. Soc. 23 (1925) ,481-519 .

[7] T.H. MacGregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14 (1963) 514-520.

[8] B S Mehrok. Two subclasses of certain analytic functions (to appear) .

[9] J W Noonan and D K Thomas , Coefficient differences and Hankel Determinant of really p-val;ent functions , Proc. London Math Soc.[3],25 (1972), 503-524.

[10] J W Noonan and D K Thomas , On the second Hankel determinant of a really mean p-valent functions. Trans. American Math Society (223) 1976, 337-346.

[11] K.Noshiro, On the theory of schlicht functions,J. Fac. Soc. Hokkaido univ.Ser.1,2 (1934-1935), 129-155.

[12] N N Pascu, Alpha- starlike functions, Bull. Uni Brasov CXIX, 1977.

[13] C H Pommerenke, On the coefficients and Hankel determinant of univalent functions, J. London math. Soc. 41, 1966, 111-122.

[14] C H Pommerenke , On the Hankel determinant of univalent functions, Mathematica 14 (C) , 1967, 108-112.

[15] C.H. Pommerenke , Univalent functions wadenheoch and Ruprech, Gotingen, 1975

[16] W.W. Rogosinski, On coefficients of subordinate functions, Proc. London Math. Zeith (1932), 92-123.

[17] M O Reade, On close to convex univalent functions, Michhigan Math. J. 1955, 59-62.

[18] S.S.Warschawski, On the higher derivative at the boundary in conformal mapping, Tran. Amer. Math.Soc., 38(1935), 310-340.

M. S. SAROA, MAHARISHI MARKANDESHAWAR UNIVERSITY, MULLANA

GURMEET SINGH (CORRESPONDING AUTHOR), KHALSA COLLEGE, PATIALA

GAGANDEEP SINGH, AMRITSAR