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ON AN ESSENTIAL CONNECTION OF THE RIEMANN HYPOTHESIS AND DIFFERENTIAL EQUATIONS

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Abstract. We state that if randomly distributed prime numbers p are building blocks of the natural numbers n, which are regularly distributed with respect to primes, then such a regularity should be far more complex than it is revealed via a distribution of n. We find such a regularity within the axioms of the calculus, winning attributes of "meromorphic" and "holomorphic" for the Riemann zeta function in this natural way. A differential equation with the right hand side as a distribution term is obtained together with the function x/lnx approximating the function $\pi(x)$. This (wave) equation is shown to be the model of the theory of distribution of prime numbers containing the Riemann hypothesis. Namely, there exists an unique solution of the obtained equation on Re s = ½ which coincidates with the nontrivial zeros of the Riemann zeta function $\zeta(s)$.

Preliminary conceptual considerations (an equivalence of "Part" and "Whole")

Before we take the distribution of the prime numbers p and the Riemann zeta function $\zeta(s)$ into a connection, we will try to outline a framework, within which a context of the complex nature of the variable s can be simply observed with respect to the nature of p.

a) If primes p can be seen as the buildings blocks of structure of natural numbers n, then we can say that they have such a *structural participation* on all $n \in \mathrm{N}$, via which that regulariry R is revealed which is mapped within a regular distribution of all n with respect to all p.

b) Let therefore as R be denoted a general property of such a function F_R , which binds an existence of numbers p with respect to n and a regularity R together.

c) Thus we consider firstly only $F(\exists p) = \exists n$ and not $F_R(\exists p) = \exists n$, since F_R cannot be any function of $\exists p$ and thus neither of $\exists n$. (F_R cannot namely depend on its own property R existencially revealed by a distribution of natural numbers.)

d) In this sense we can write

$$\partial F_R/\partial(\exists n) = 0 \,. \tag{1}$$

e) Now, let us introduce such a coordinate x, on which natural numbers n exist as its particular instances. Since x is not identically ∃n, we put

$$\partial \mathrm{FR}/\partial \mathrm{x} \neq 0 \ \text{ and } \mathrm{dFR}/\mathrm{dx} = 0 \tag{2}$$

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distinguishig ("differentiating") x and $\exists n$ from each other.

f) The given differential approach to F_R can indicate an existence of a curve $y = F_R(x)$, for which there exists no point $x = x_1$, at which the property R could be changed, i.e. a notion of "inflex points" is not permitted at

(3) $$\mathrm{d}^2 F_R(x)/\mathrm{d}x^2 \neq 0.$$

g) If we have such a variable t now, which is in no sense regular and it is not a curve at the same time, i.e. $t \neq y$ essentially, then an expression

(4) $$F_R(x, t) = 0$$

does not represent any curve (as it would be the case for $F_R(x, y) = 0$). The "curve case" cannot exist due to the nonexistence of inflex points for $y = F_R(x)$, the case (4) creates a necessary condition of existence of the given meaning of regularity of $F_R(x)$, provided that the irregular t is in some sense coupled with the distribution of primes p.

h) It has no sense to search for singularities for (4), which are searched in cases when prescriptions like (4) represent curves. Therefore we must strictly forbid the function $F_R(x)$ to take a part in the relations searching for singularities for the function $F_R(x, t) = 0$ which could be of the type: $(F_R)_{xt}^2 > (F_R)_{xx} \cdot (F_R)_{tt}$ (knot), $(F_R)_{xt}^2 = (F_R)_{xx} \cdot (F_R)_{tt}$ (cusp) and $(F_R)_{xt}^2 < (F_R)_{xx} \cdot (F_R)_{tt}$ (isolated point). For this goal we take $(F_R)_{xx}$ from these relations and fit it by (3), i.e. we have

(5) $$\partial^2 F_R/\partial x^2 = \mathrm{d}^2 F_R(x)/\mathrm{d}x^2 \text{ with respect to (4).}$$

i) **The relevance interpretation.**
In the relation (5) we have yielded a concept of "Part" $\Leftrightarrow$ "Whole".
Within this concept it can be contextually distinguished:

1. The set of all primes is in some infinite(simal) approach equivalent to the set N of all natural numbers. Since the both sets have the same cardinal number $\aleph_0$ that generally carry an idea of "infinity", we accept the relation (5) with respect to $\aleph_0$.

2. "*Meros*" $\Leftrightarrow$ „*Holos*“ with respect to (4), where $F_R(x, t)$ carries a notion of shape in that sense that *it is not* a curve. Thus, consequently, it is induced a general notion of "morphic": We distinguish a meromorphic function F_R on some open subset D of the complex plane that is holomorphic as $F_R(x)$ on all D except its pole(s). Since the function $F_R(x, t) = 0$ has no poles, it is a very special instance of "meromorphism".

3. Now, we can already search for a "*part*" t of certain "*whole*" variable $s \in D$, that changes itself in such a way that another part of s (denoted contextually as σ) stays unchanged. - If namely σ could be changed too, than we would have no one changing "part" t with respect to one changing "whole" s as a particular case of the "part - whole"-equivalence-concept.

Thus any movements within t characterize completely a change of s as a "whole" provided, that σ can be tuned for an entire participation of R on the distribution of all n This "state of entire participation of R" will be cosidered as

$$(6) \qquad \sigma_{\mathrm{I}} \text{ for } F_R = F_R(x) \text{ in } (5)$$

with respect to the the "state"

$$(7) \qquad \sigma_{\mathrm{II}} \text{ for } F_R(x, t) = 0$$

avoiding singularities for "entire participation" of the distribution of primes p.

Hint. According with the given construction of (4), it is also $F_R(\exists n, t) \neq 0$. - If there is no function φ such that it could "absorb" the regularity R with respect to $\exists n$, i.e. if no "transformation"

$$F_R(\exists n, t) \to \varphi(t)$$

is permitted, then any $\varphi(t)$ cannot exist. Therefore, if there exist the "xi function" $\xi(t)$ (introduced firstly by Riemann in /1/), then naturally $\xi \neq \varphi$ in that meaning that ξ cannot "absorb" regularity R with respect to $\exists n$ and thus it depends *additively* on R and $\exists n$. In this sense t should become a complex nature within $\xi(t)$. (Consequently, if we write $\frac{1}{2} + i\alpha$ for a "part" t, then we indicate that real part of the "whole" s, at which the concept of equivalence of a „part“ and a "whole" is completely satisfied here.)

Above, we have outlined a "logico-mathematical model" of a possible connection between the Riemann zeta function and a distribution of primes based on the certain concept of regularity, within which a regular function can be referred as meromorphic, or holomorphic respectively. Via $s \in D$, we have naturally obtained its complex nature within this concept. Our task is to show in the following text, how (7) can be obtained for the Riemann zeta function within the mathematical model, when $\sigma_{\mathrm{II}} = \frac{1}{2}$.
However, the following text can be regarded as completely indepedent of the preliminary consideareations, which only bring a "plastic picture" of the relation (7) and imply a possible connection of the pure analysis and differential equations on the basis of Riemann hypothesis.

1. Introduction

The Riemann zeta function $\zeta(s)$ is for the complex variable s , Re $s > 1$ defined by the absolute convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} (1/n^s) \qquad (1.1)$$

and has an analytic continuation in the whole complex plane C. The behavior of the zeta function in the region where the series (1.1) is not convergent is expressed by the Riemann hypothesis:

RH. *All nontrivial zeros of* $\zeta(s)$ *lie on the critical straight line* Re $s = ½$.

The distribution of these zeros is supposed to be uniquely related to the distribution of prime numbers. In this article we will try to show that if it is the case, then one must consider more general concept of distribution within which the both distributions are uniquely emerged, namely the PDE with the right hand side as their common distribution term. Thus one can construct a mathematical model M of the theory T of distribution of prime numbers concerning quite naturally the Goedel theorem of completeness: *Any theory T is consistent just when it has a model.* Consequently, the RH must be consistent, if M exists.

With a purely analytical distribution concept within RH on the one side, we have purely empirically obtained distribution of primes on another side, namely the function $x/\ln x$. Such a disjoint nature of both distributions enables to require a certain exceeding of an appeiron of both of them within a PDE-approach outlining their common logic within M. This is the main idea of the article that will be consequently followed starting with the definition of T up to the unique finding of the model M.

2. Construction of M

The first step showing the connection between $x/\ln x$ and $\zeta(s)$ was made by Hadamard and, independently, by de la Vallee Poussin in 1896. It concerns the fact that the information about the position of nontrivial zeros of $\zeta(s)$ can be directly transferred onto a behavior of the function $\pi(x)$ such that, when we only know that $\zeta(s)\neq 0$ on Re s $=1$, then we obtain the relation

$$\pi(x) \sim x/\ln x\text{, i.e. } \lim_{x\to+\infty} (\pi(x)\ln x)/x = 1, \qquad (2.1)$$

or more precisely

$$\pi(x) \sim \mathrm{Li}\, x = \int d\tau/\ln\tau\,. \qquad (2.2)$$

This implies that $\zeta(s) = 0$ cannot be considered independently without its relation to $\zeta(s)\neq 0$ on Re $s=1$ with respect to the relation $\pi(x)\sim x/\ln x$ within T.

2.2. Definition *As the theory of distribution of prime numbers we regard such a theory* T*, which contains the relation* $\pi(x)\sim x/\ln x$ with respect to $\zeta(s)=0$ on Re $s = ½$ and $\zeta(s)\neq 0$ *on*
Re $s = 1$ *uniquely.*

2.3. Theorem *The* RH *is consistent since the theory* T *has the model* M.

Proof of Theorem. (The construction of M). Let

$$w = x^{-(\sigma + it)},\ \sigma = \mathrm{Re}\ s \text{ for } x = n \tag{2.3}$$

just if and only if the function $x/\ln x$ can be expressed within the model M with the function w as its solution. Knowing that we tend to the PDE with a distribution term, it is not hard to find the searched distribution $x/\ln x$. We simply put in the elementary formula

$$\mathrm{d}^2 w/\mathrm{d}x^2 = \partial^2 w/\partial x^2 + 2(\partial^2 w/\partial x\partial t)\mathrm{d}t/\mathrm{d}x + (\partial^2 w/\partial t^2)(\mathrm{d}t/\mathrm{d}x)^2 \tag{2.4}$$

of differential calculus that

$$2(\partial^2 w/\partial x\partial t) + (\partial^2 w/\partial t^2)\mathrm{d}t/\mathrm{d}x = 0\ . \tag{2.5}$$

Since

$$\left.\begin{aligned}&\partial^2 w/\partial x\partial t = i/x[(\sigma + it)\ln x - 1]\,w\\ \text{and}\quad &\\ &\partial^2 w/\partial t^2 = -(\ln x)^2 w.\end{aligned}\right\} \tag{2.6}$$

Then, substituting from (2.6) into (2.5) we obtain the equation

$$2i/x[(\sigma + it)\ln x - 1]\,w\,\mathrm{d}x = (\ln x)^2 w\,\mathrm{d}t, \tag{2.7}$$

which can be also written as

$$2i[(\sigma + it)(1/x) - 1/(x\ln x)]\,w\,\mathrm{d}x = (\ln x)\,w\,\mathrm{d}t \text{ for } x>1\quad . \tag{2.8}$$

The differential equation

$$\mathrm{d}^2 w/\mathrm{d}x^2 = \partial^2 w/\partial x^2 \quad \text{for } x>1 \tag{2.9}$$

is yielded from (2.4), if (2.8) is satisfied. If further (2.8) is observed from the building blocks x and $\ln x$ of the function ($x/\ln x$) point of view, then some close correspondence

$$\pi(x) \leftrightarrow w \tag{2.10}$$

should be expected during a process of yielding of ($x/\ln x$). The main features of (2.10) are:

• Differential equation (2.9) is deterministic, the distribution $\pi(x)$ not. (The situation in (2.10) reminds thus the analogous situation in quantum mechanics, where the probability wave function satisfies the deterministic Schrödinger equation.)
• Unlikely w, $\pi(x)$ is not a function, but a distribution, so that the function w as the solution of M must be independent on $x/\ln x$ (which approximates the distribution $\pi(x)$ only). In that way the function w cannot take a part in the yielding of $x/\ln x$ by the process of integration of the relation (2.8).

So, integrating (2.8) we obtain

$$2i\sigma\ln x - 2t\ln x - 2i\ln|\ln x| + G(t) = t\ln x + F(x)\,, \tag{2.11}$$

where $F(x)$ and $G(t)$ are arbitrary independent functions. We rewrite the last equation as

$$2\sigma\ln x - 2\ln|\ln x| = -i[3t\ln x + (F(x) - G(t))]\,, \tag{2.12}$$

whence it is immediately evident that there are only two ways how to express the function $x/\ln x$ uniquely, namely

$$x/\ln x = \exp[-i/2(3t\ln x + (F(x) - G(t))_{\mathrm{I}}] \quad \text{for } \sigma_{\mathrm{I}} = 1 \tag{2.13}$$

and

$$x/\ln x = \exp[-i(3t\ln x + (F(x) - G(t))_{\mathrm{II}}) + \ln|\ln x|] \quad \text{for } \sigma_{\mathrm{II}} = ½\,. \tag{2.14}$$

This means that we can obtain the equation (2.9) only in two unique ways with respect to the function $x/\ln x$. We see at the same time that this function cannot be determined by any particular choice of the arbitrary functions $F(x)$ and $G(t)$ such that it would lead to a vanishing of the imaginary unit i in the both last relations. In such a way, this fact establishes an indeterministic nature of the function $x/\ln x$ with respect to (2.9). Consequently, if the theory T has the model M given by the differential equation (2.9) satisfying uniquely the correspondence (2.10), then the functions $F(x)$ and $G(t)$ cannot remain arbitrary in T and there must exist their particular choice following the distribution $\pi(x)$. Thus the equation (2.9) has particular solutions

$$w_{\mathrm{I}} \text{ on } \sigma_{\mathrm{I}} = 1 \text{ and } w_{\mathrm{II}} \text{ on } \sigma_{\mathrm{II}} = ½ \qquad (2.15)$$

at the points t_{I}, t_{II} at which the arbitrary functions F, G can take some particular forms typical for $x_{\mathrm{I}} = x_{\mathrm{II}} = n = 2, 3, 4, \ldots$

Note that the model M does not require any concrete values for t_{I} and t_{II}, since $\pi(x)$ does not explicitly depend on t. In thus indicated existence of $\pi(x)$, we reveal its nature consequently: having (2.15) as particular solutions of M, then, according with the superposition principle, we have their sums

$$\sum_{n=2}^{\infty} w_{\mathrm{I}} \text{ and } \sum_{n=2}^{\infty} w_{\mathrm{II}} \qquad (2.16)$$

as the solutions too, provided that the single particular solutions are mutually linearly independent within σ_{I} and σ_{II} . So the distribution $\pi(x)$ must concern the distribution of mutually independent objects, or, relevantly, the distribution of prime numbers with none of them being any linear combination of another arbitrary ones.

On the other hand, the both solutions (2.16) are mutually dependent with respect to (2.10), so they cannot commutate with each other as

$$\sum_{n=2}^{\infty} w_{\mathrm{I}} \sum_{n=2}^{\infty} w_{\mathrm{II}} \neq \sum_{n=2}^{\infty} w_{\mathrm{II}} \sum_{n=2}^{\infty} w_{\mathrm{I}} \,. \qquad (2.17)$$

This mutual dependence makes impossible to apply the superposition principle for them within M. Therefore, we will use only one way proposed by M how to let (2.17) vanish, namely, to put one of the solutions equaled to zero. Thus we firstly arrive at the paradox

$$\sum_{n=2}^{\infty} w_{\mathrm{I}}\, 0 \neq 0 \sum_{n=2}^{\infty} w_{\mathrm{I}} \,, \qquad (2.18)$$

that can be removed again in one possible way. Instead of $n \geq 2$ for "$\neq$" within (2.18) to put $n \geq 1$ for "$=$" there. In such a way we uniquely arrive at two typical cases of the zeta function

$$\zeta(s_{\mathrm{I}}) \equiv \sum_{n=1}^{\infty} w_{\mathrm{I}} \neq 0 \text{ on } \sigma_{\mathrm{I}} = \mathrm{Re}\, s_{\mathrm{I}} \equiv 1 \qquad (2.19)$$

and

$$\zeta(s_{\mathrm{II}}) \equiv \sum_{n=1}^{\infty} w_{\mathrm{II}} = 0 \quad \text{on } \sigma_{\mathrm{II}} = \operatorname{Re} s_{\mathrm{II}} \equiv ½ \tag{2.20}$$

as the unique solutions of the equation (2.9) with respect to the relation $\pi(x) \sim x/\ln x$. (Now, the fact that $x = n = 1$ is not possible to substitute into $x/\ln x$ is logically coupled with the fact that this function is not determinable within M for $x>1$.)
Consequently, the last step is to rewrite the equation (2.9) in terms of the zeta function. At first, all its solutions of the (2.3)-type are nonzero ones, which fact can be globally expressed, if the equation (2.9) is rearranged into the form

$$-\partial^2 w/\partial x^2 = -\mathrm{d}^2 w/\mathrm{d}x^2 \tag{2.21}$$

with respect to its zero-solution $\zeta(s) = 0$ in

$$-\partial^2\zeta(\mathrm{s})/\partial x^2 = -\mathrm{d}^2\zeta(s)/\mathrm{d}x^2 \quad \text{for } \operatorname{Re} s = 1, ½\ . \tag{2.22}$$

Since the both last linear equations are nonhomogenous, containing the differentials of a different type, the zero-solution of (2.22) (unlikely trivial solutions in the case of homogenous PDE) is logically nontrivial, as it is also required by the RH.
Following further the formal requirements of the PDE-approach, we obtain the right hand side-source term (having a meaning of distribution) within (2.22) in the form

$$\mathrm{D} \equiv -\mathrm{d}^2\zeta(s)/\mathrm{d}x^2 \quad \text{for } \operatorname{Re} s = 1, ½ \tag{2.23}$$

and thus we obtain the model M in the final form

$$-\partial^2\zeta(s)/\partial x^2 = \mathrm{D}\ , \tag{2.24}$$

where $\mathrm{D} \neq 0$ for $\operatorname{Re} s = 1$ and $\mathrm{D} = 0$ for $\operatorname{Re} s = ½$. **Q.E.D.**

Remark. The equation (2.24) reminds the form of a stationary wave equation with $\nabla^2 = \partial^2/\partial x^2$. Searching for the relevant waves within M, we can write, instead of

$$\sum_{n=1}^{\infty} w_{\mathrm{I}}\, 0 = 0 \sum_{n=1}^{\infty} w_{\mathrm{I}}$$

within (2.18) – (2.20), the expression

$$\zeta(s_{\mathrm{I}})\zeta(s_{\mathrm{II}}) \equiv 0\ . \tag{2.25}$$

Thus we immediately see that there is no (privileged) value of $\zeta(s_{\mathrm{I}}) \neq 0$ transferring (2.25) into the form $\zeta(s_{\mathrm{II}})=0$ of the unique nontrivial zero-solution of M. So, if $\zeta(s_{\mathrm{I}})$ is unique and not single-valued at the same time, then it can be interpreted via some vibrational process within M following strictly the distribution of prime numbers. In other words, there is the collective source D $\neq 0$ of elementary waves w_{I} building the stationary wave $\zeta(s_{\mathrm{I}})$ over $\zeta(s_{\mathrm{II}})$ within M.

3. Conclusion

We have demonstrated that the theory T of distribution of prime numbers has the model M which contains the RH in the form of $\zeta(s_{\mathrm{II}}) = 0$ as its unique nontrivial solution. Thus the Riemann hypothesis is completely logically consistent. The model M works only if the relation $\pi(x)\sim x/\ln x$ holds, which symptomatically supports the proof of this prime number theorem.

The main idea building the model is to unify the purely empirical approach to the distribution of primes mediated by the function $x/\ln x$ and purely analytical approach mediated by the RH via the common logical source contained in the stationary wave-like PDE. We can point out that this equation possesses a form that can be “translated” via the equation (2.9) as a “part” being equaled to a “whole”. Thus we have landed at (5) and can define (from the preliminary considerations): $F_R \equiv \zeta(s)$ and $F_R(x, t) = 0$ as $\zeta(s_{\mathrm{II}}) = 0$, what we have also wanted to show.

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