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HILBERT'S SPACE-FILLING CURVE

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ABSTRACT. This paper introduces the notion of a certain type of space-filling curve. We will be talking about how a one-dimensional curve can be called "space-filling". The curve will begin at a one-dimensional unit interval and be mapped onto a unit square by a certain *correspondence*. This correspondence is said to be *one-to-one*. Later on in the paper it could be denoted as a function. There are different kinds of space-filling curves such as the Lebesque Curve, the Peano Curve, the Sierpinski Curve, and the Recursive space-filling curve. This amazing finding was first brought to everyone's attention by Henri Lebesque. He explained this finding geometrically and analytically in his paper as a function of "t". We will discuss a specific type of space-filling curve named after David Hilbert and how this correspondence that builds the curve is not one-to-one.

INTRODUCTION

In 1878, a man named Georg Cantor published a remarkable finding: "That there exists a *one-to-one correspondence* between any two finite-dimensional smooth manifolds". In simpler terms, he proposed that "it is possible for a mapping of all points in the unit interval $[0,1]$ *onto* the unit square $[0,1]X[0,1]$ be *injective* and *surjective*. Hence, the *one-to-one correspondence* claim [1]. In this paper, we plan to introduce what is called the Hilbert's Space-Filling curve and to disprove the statement of this curve being one-to-one. At the beginning of this paper we question the theory, *"What is a "space-filling" curve"?* A curve that is *"space-filling"* is a *surjective* function mapping the unit interval,

$$J = \{x \mid 0 \leq x \leq 1\} = [0,1],\ where\ x \in J$$

as the domain of the function and is mapped *onto* the unit square,

$$B = \{(x,y) \mid 0 \leq x \leq 1,\ \ 0 \leq y \leq 1\} = [0,1]\ \ X\ \ [0,1], where\ \ (x,y) \in B$$

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which is our range of the function. We must show this space-filling effect of the square is a *surjection* only.

Illustration

"The first point starts in the lower left corner to begin the illustration of the curve. This means that it will start at coordinate $(0, 0)$ when $x = 0$ and $(1, 0)$ when $x = 1$. The first enumeration or *iteration* determines the satisfaction of two conditions" [1]. We will show how this space-filling curve is a *surjection* only. This specific function is mapping our unit interval *onto* our unit square hitting all possible points.

Def: A function that does the mapping from the domain to the range is called a *surjection* such that for all $y \in B$, there exists an $x \in J$ such that $f(x) = y$. In mathematical notation it is defined as,

$$\forall y \in B, \exists x \in J \ \ s.t. \ \ f(x) = y.$$

This can be viewed as the *definition of a function*. The word *correspondence* comes from the *mapping between the unit interval and the unit square*.

Def: "Let J and B be sets. A *correspondence* that associates with each element $x \in J$ with an *unique element* $f(x) \in B$ is called a *function* from J to B, which we can write as

$$f : J \to B.\text{"[6]}$$

Def: "Let $f : J \to B$ be given. The *set* J is called the *domain* of f and the *set* B is called the *range* of f" [6].

This function f could not only be called a mapping but could also be called a "transformation" of J onto B. A nice statement that can be made is that this function "carries each point $x \in J$ *onto* its *corresponding* point $f(x) \in B$" [6].

Def: "A function $f : J \to B$ is called a "constant surjection" if there exists a point $y \in B$ such that $f(x) = y$ for all $x \in J$" [6]. This previous definition can also be named as a *constant surjection*. In notation we write,

$$\exists y \in B, \forall x \in J \ \ s.t. \ \ f(x_n) = y.$$

This previous definition is vital to this paper as we continue to look at the illustration of this specific curve.

Def: *Let "f" be a function defined on B. It is said that f is "one-to-one" on B iff for every x and y in B* such that,

$$f(x) = f(y) \iff x = y.$$

It is said that a function that is one-to-one on a set B " assigns distinct values to specific members of B." [4]. These functions are vital to our correspondence because they possess "*inverses*" [4]. This makes things interesting because the previous statement is saying that our correspondence has inverses. This means that "two sets J and B which are called "equivalent", we write $J \sim B$ *if and only if* there exists a one-to-one function "f" whose domain is the set J and the range is the set B" [4]. The inverse to the previous statement would be "two sets J and B are also called equivalent, we also write $B \sim J$ *if and only if* there exists a one-to-one "inverse" function f^{-1} whose range is the same set B and domain is the same set J" [4]. Now, we shall recall that we must show that these sets J and B are equivalent to each other. The proof will be shown below. In order for our sets to be equivalent, we must define a one-to-one correspondence. Before we accumulate our correspondence we must analyze the sets. There is a question that should come up to our minds. Are these two sets "finite" or "infinite"? In order to find out that either sets are infinite of finite, we must define these sets.

A certain question should rise when reading the previous statement. *How do we show and know that these two sets are equivalent ?* A similar statement could be that this "f" establishes a *"one-to-one correspondence"* between the sets J and B.

Def: *"If "f" is a one-to-one correspondence which makes the set J "equivalent" to the set B, then "f^{-1}" will make the set B equivalent to the set J.* In notation we yield,

$$If \;\; f(x) = y, \;\; then \;\; f^{-1}(y) = x$$

$$where \;\; f \;\; makes \;\; J \sim B \;\; and \;\; f^{-1} \;\; makes \;\; B \sim J.\text{"}[4]$$

Recall back to your Real Analysis course working in set theory. There is a theorem where it states that we have to prove that the set of real numbers $\mathbb{R}$ is uncountable. This can be applied to our situation having our two sets of sub-intervals and sub-squares. Another question should rise up from this theorem. *Is the set of sub-intervals and the set of sub-squares countable or uncountable?* In a space-filling curve case, the domain is the set of intervals J and the range is the set of squares B. The inverse function f^{-1} maps the range onto the domain such that $f^{-1} : B \to J$. In order to illustrate the curve, the domain is mapped onto the range such that $f(x) = y$. The definition of a function is a surjection mapping certain elements of a set to another set.

We would like to show that a correspondence is defined as a *function* mapping the unit interval J *onto* the unit square B. Before we do that, we must first define the *sub-intervals* in the unit interval and define the *sub-squares* in the unit square such that the correspondence "f" maps these sub-intervals onto the sub-squares as,

$$\{J_{n,n_k}\} \rightarrow \{B_{n,m_k}\}$$

where n is the number of *iterations*, $\{n_k\}$ is the indexing notation for sub-intervals, and $\{m_k\}$ is the indexing notation for sub-squares. Iteration *zero*, or the 0^{th} iteration is where the given unit interval is mapped onto the given unit square denoted as,

$$J_{0,n_k} \rightarrow B_{0,m_k}$$

where as the sub-interval and sub-square indexing stays the same since we have not yet made iterations. In the given interval J, the $1st$ iteration is the unit interval quartered into four sub-intervals and mapped onto four corresponding sub-squares. Let this be denoted as,

$$J_{1,n_k} \rightarrow B_{1,m_k}.$$

The second iteration is the first iteration quartered again making sixteen sub-intervals that must correspond to sixteen sub-squares. Let this be denoted as,

$$J_{2,n_k} \rightarrow B_{2,m_k}.$$

This process is repeated for all iterations n to k where the *sequence* of sub-intervals is mapped onto the corresponding sequence of sub-squares until *the limit of all correspondences goes to a specific function filling all points in the unit square*. Let this be denoted as,

$$\{J_{k,n_i}\} \rightarrow \{B_{k,m_j}\}$$

where the indexing of sub-intervals goes from k to i such that $\{n_0, n_1, n_2, ..., n_i\}$ and the indexing of sub-squares goes from k to j such that $\{m_0, m_1, m_2, ..., m_j\}$. We can let the indexing of the sub-intervals and the sub-squares be equal to 4^n as desired such that $n_k = 4^n$ and $m_k = 4^n$. When we compute our indexed sets we would get $\{4^1, 4^2, ..., 4^k\}$ for each set of sub-intervals nd sub-squares. The limit of the *set* of sub-squares goes to a unique point.

We must show that such a correspondence is possible. There exists a correspondence between sub-intervals and sub-squares that satisfy two conditions: The *Adjacency* and *Nesting* Conditions. The *Adjacency* Condition states that if two sub-intervals share an *endpoint*, then two corresponding sub-squares must share a common *edge*. In mathematical terms for example we yield,

$$\begin{array}{l} J_{0,r_0}, J_{1,r_1}, J_{2,r_2} \quad ,\ldots, \quad J_{k,r_i}, J_{k+1,r_{j+1}} \\ f_0\downarrow,\ \ f_1\downarrow,\ \ f_2\downarrow \quad ,\ldots, \quad f_k\downarrow, f_{k+1}\downarrow \\ B_{0,p_0}, B_{1,p_1}, B_{2,p_2} \quad ,\ldots, \quad B_{k,p_i}, B_{k+1,p_{j+1}}. \end{array}$$

The *Nesting* Condition states that if a *sequence* of sub-intervals is *nested*, then the corresponding *sequence* of sub-squares must also be *nested*. Then for all iterations of the correspondence $\{f_n\}$ from n to k we yield,

$$\begin{array}{ccccccc} J_{0,r_0} \supseteq & J_{1,r_1} \supseteq & J_{2,r_2} \supseteq & \ldots & \supseteq J_{k,r_i} & \supseteq & J_{k+1,r_{j+1}} \\ f_0\downarrow & f_1\downarrow & f_2\downarrow & \ldots & f_k\downarrow & & f_{k+1}\downarrow \\ B_{0,p_0} \supseteq & B_{1,p_1} \supseteq & B_{2,p_2} \supseteq & \ldots & \supseteq B_{k,p_i} & \supseteq & B_{k+1,p_{j+1}} \end{array}$$

which shows pictorially that the following correspondence is nested. We must prove that this correspondence *exists* and is possible.

Theorem 1 (Adjacency Condition)**.** *If for all iterations "n" two sub-intervals share a common endpoint, then the function "f_n" maps two sub-intervals to two sub-squares that "share a common edge".[1]*

Proof. Let two sub-intervals be denoted as J_{n,r_n} and J_{n,s_n} then let two sub-squares be denoted as B_{n,p_n} and B_{n,q_n}. Let $n = k$ and r and s be the indexing for sub-intervals and p and q be the indexing for the sub-squares. Let the k^{th} sub-intervals be denoted as J_{k,r_k} and J_{k,s_k} and the k^{th} sub-squares be denoted as B_{k,p_k} and B_{k,q_k}. We must show that $n = k+1$ holds for the *sequences* of sub-intervals and sub-squares so that both sequences satisfy the adjacency condition. So, for all iterations f_n to f_k, the *infinite* sequences of sub-intervals $\{J_{n,r_n}\}$ and $\{J_{n,s_n}\}$ satisfy the Adjacency Condition as long as they correspond to the *infinite* sequences of sub-squares $\{B_{n,p_n}\}$ and $\{B_{n,q_n}\}$. So for all iterations f_{k+1} of every sequence we yield,

$$\begin{array}{l} J_{0,r_0}, J_{1,r_1}, J_{2,r_2} \quad ,\ldots, \quad \{J_{n,r_k}\}, \{J_{k,r_i}\}, \{J_{k+1,r_{j+1}}\} \\ f_0\downarrow,\ \ f_1\downarrow,\ \ f_2\downarrow \quad ,\ldots, \quad \{f_n\}\downarrow,\ \{f_k\}\downarrow,\ \{f_{k+1}\}\downarrow \\ B_{0,p_0}, B_{1,p_1}, B_{2,p_2} \quad ,\ldots, \quad \{B_{n,p_i}\}, \{B_{k,p_i}\}, \{B_{k+1,p_{j+1}}\} \end{array}$$

$$\begin{array}{l} J_{0,s_0}, J_{1,s_1}, J_{2,s_2} \quad ,\ldots, \quad \{J_{n,s_k}\}, \{J_{k,s_i}\}, \{J_{k+1,s_{j+1}}\} \\ f_0\downarrow,\ \ f_1\downarrow,\ \ f_2\downarrow \quad ,\ldots, \quad \{f_n\}\downarrow, \{f_k\}\downarrow,\ \{f_{k+1}\}\downarrow \\ B_{0,q_0}, B_{1,q_1}, B_{2,q_2} \quad ,\ldots, \quad \{B_{n,q_i}\}, \{B_{k,q_i}\}, \{B_{k+1,q_{j+1}}\}. \end{array}$$

Hence, by using the Induction Hypothesis the sequences of sub-intervals $\{J_{k,n_k}\}$ and $\{J_{k+1,n_{k+1}}\}$ share a common endpoint and the sequences of squares $\{B_{k,m_k}\}$ and $\{B_{k+1,m_{k+1}}\}$ share a common edge.

□

Theorem 2 (Nesting Condition). *If for all iterations "n" "two sub-intervals of a sequence of sub-intervals are nested, then two corresponding sub-squares of a sequence of sub-squares are nested."[1]*

Proof. Let the sequences of sub-intervals and sub-squares be denoted as $\{J_{n,r_n}\}$ and $\{B_{n,p_n}\}$. Supposed that $n = k$ holds for the infinite sequences of sub-intervals and sub-squares. Then we have $\{J_{n,r_k}\} = \{J_{k,r_k}\}$ and $\{B_{n,r_k}\} = \{B_{k,r_k}\}$. We must show that $n = k+1$ holds for both infinite sequences and that for $n = k+1$, both sequences satisfy the nesting condition. Thus, by using the Induction Hypothesis,

$$\begin{array}{ccccccc} J_{0,r_0} \supseteq & J_{1,r_1} \supseteq & J_{2,r_2} \supseteq & \dots & \supseteq \{J_{n,r_i}\} & \supseteq \{J_{k,r_i}\} & \supseteq \{J_{k+1,r_{i+1}}\} \\ f_0 \downarrow & f_1 \downarrow & f_2 \downarrow & \dots & \{f_n\} \downarrow & \{f_k\} \downarrow & \{f_{k+1}\} \downarrow \\ B_{0,p_0} \supseteq & B_{1,p_1} \supseteq & B_{2,p_2} \supseteq & \dots & \supseteq \{B_{n,p_i}\} & \supseteq \{B_{k,p_i}\} & \supseteq \{B_{k+1,p_{i+1}}\} \end{array}$$

holds for every iteration n to $k+1$.

□

Theorem 3. *Any correspondence "f_n" between sub-intervals and sub-squares that satisfy the Adjacency and Nesting conditions determine an* ***unique continuous function*** *"f" which maps the unit interval J onto the unit square B.[1]*

Proof. Let f be a function such that $f : J \to B$. Let $x \in J$ be an element of the sequences of closed nested sub-intervals $\{J_{n,n_k}\}$ to $\{J_{k+1,n_{i+1}}\}$ such that,

$$J_{0,r_0} \supseteq J_{1,r_1} \supseteq J_{2,r_2} \supseteq \dots \supseteq \{J_{n,r_i}\} \supseteq \{J_{k,r_i}\} \supseteq \{J_{k+1,r_{i+1}}\} \supseteq \dots$$

where one sub-interval is from each *partition*. So for all iterations, the limit of the diameter of squares approaches a specific point determining a unique point y. This unique point y is used to define our function setting $f(x)$ equal to y. The Adjacency Condition explains the *continuity* of our correspondence between sub-intervals ans sub-squares. From the Nesting Condition, we have noticed that our correspondence does the mapping from a nested sub-interval to a nested sub-square. Since any correspondence does the mapping for all iterations n, it is safe to say that this continuous function $\{f_n\}$ is *unique* due to the correspondences shown in the previous conditions.

To explain how the curve is called "space-filling" is that every correspondence between sub-intervals and sub-squares for every iteration n determines a *unique continuous function* "f" where the limit of sub-intervals and sub-squares goes to a specific point. More specifically, *if the limit of the sequence of sub-intervals gives us a point "x", then the* *limit of the corresponding sequence of sub-squares gives us an unique point "y"* Thus, by the Induction Hypothesis the *same* function maps the sub-intervals onto the sub-squares.

□

Proof of Surjection. An injective correspondence states that, If $f(x) = f(y)$, then $x = y$ and inversely, If $f(x) \neq f(y)$, then $x \neq y$. A surjection again states that,

$$\forall y \in B, \exists x \in J \ \ s.t. \ \ f(x) = y.$$

Going back to our *"constant surjection"* definition we found out that our point y is *unique* in a sense that,

$$\forall x \in J, \exists! y \in B \ \ s.t. \ \ f(x) = y.[6]$$

We must show that this specific point y is *unique*. Now, suppose we have a set of points $S = \{x_1, x_2, ..., x_k\}$ and the set of points $T = \{y_1, y_2, ..., y_k\}$. Let these set of points S equal to the set of sub-intervals where $S \in J$ and the set of points T equal to the set of sub-squares. When we map our sub-intervals onto our corresponding sub-squares we have,

$$f(x) = y.$$

Since the set of points S and T gives us our *unique* point as defined,

$$\begin{gathered}(x_1, y_1)\\(x_2, y_2)\\.\\.\\.\\(x, y)\end{gathered}$$

our corresponding points will be,

$$\begin{gathered}(x_1, f_1(x_1))\\(x_2, f_2(x_2))\\.\\.\\.\\(x, f(x)) \quad .\end{gathered}$$

Thus proves that our correspondence "f_n" is a *surjection* only and hence $f(x) = y$. In mathematical notation we confirm that our constant *surjection* maps our sub-intervals onto our sub-squares such that,

$$\begin{aligned} f_1(x_1) &= y_1 \\ f_2(x_2) &= y_2 \\ f_3(x_3) &= y_3 \\ &. \\ &. \\ &. \\ f_k(x_k) &= y_k \end{aligned}$$

For the correspondence to be both injective and surjective,

$$f(x) = f(y) = y \quad or \quad f(x) \neq f(y) \neq y.$$

As the surjection states $f(x) = y$ but $f(y) \neq y$. In this case of our space-filling curve, $f(x) = y$ which is a *surjection only* and not an injection. Since the same function is mapping the sub-intervals onto the sub-squares, the function "f_n" is *unique* and *continuous*. Therefore, the limit of sub-squares as n goes to infinity determines a unique point y such that,

$$\begin{aligned} f_0(J_{0,n_0}) &= B_{0,m_0} \\ f_1(J_{1,\{n_4\}}) &= B_{1,m_1} \\ f_2(J_{2,\{n_{16}\}}) &= B_{1,m_1} \\ &. \\ &. \\ &. \\ f_k(J_{k,\{n_k\}}) &= B_{1,m_1}. \end{aligned}$$

If the limit of the sequence of sub-squares gives us a point "y", then the limit of the corresponding sequence of sub-intervals gives us an unique point "x".

□

In order for this proof of our surjection to hold true, we have to verify it by showing that there are subsets in our unit square.

Theorem 4. *If $J_{n,\{n_k\}} \subseteq B_{n,\{m_k\}}$ and $B_{n,\{m_k\}} \subseteq B$, then $J_{n,\{n_k\}} \subset B$.*

Proof. Let the set of sub-intervals of the first iteration be $J_{1,\{n_4\}}$ and let the subset of the first iteration of sub-squares be $B_{1,\{m_4\}}$. In order to map a certain domain onto a specific range, we must show that these are subsets of a particular superset. We notice that there are four sub-intervals that correspond to the first iteration of sub-squares $B_{1,\{m_4\}}$ meaning that the set of sub-intervals must be a subset of the first iteration of sub-squares such that

$$J_{1,\{n_4\}} \subseteq B_{1,\{m_4\}}.$$

Then as we take more iterations, we notice that the previous iteration becomes a *proper subset* of $B_{2,\{m_{16}\}}$ such that

$$B_{1,\{m_4\}} \subset B_{2,\{m_{16}\}}.$$

Since $J_{1,\{n_4\}} \subseteq B_{1,\{m_4\}}$ and $B_{1,\{m_4\}} \subset B_{2,\{m_{16}\}}$, then we can proceed to confirm that $J_{1,\{n_4\}} \subset B_{2,\{m_{16}\}}$.

□

Def: "Let $\{f_n\}$ be a sequence of correspondences. We say that $\{f_n\}$ *converges* to a limit L and write,

$$\lim_{n\to\infty} f_n = L$$

$$\forall\epsilon > 0, \exists N < n \ \ such \ \ that \mid f_n - L \mid \ < \epsilon, \ \ whenever \ \ n \geq N."[7]$$

"Let $\{f_n\}$ be a sequence defined on the range B. If $\lim\limits_{n\to\infty} f_n(x)$ exists (in the set of real numbers $\mathbb{R}$) $\forall x \in J$, it is safe to say that the sequence $\{f_n\}$ can also *converge pointwise* on B. This limit can also define a function f on B by the equation,

$$\lim_{n\to\infty} f_n(x) = f(x)".[7]$$

If the limit of the sequence of sub-squares gives us a point "y", then the limit of the corresponding sequence of sub-intervals gives us an unique point "x". Conversely, if the limit of the sequence of sub-intervals gives us a point "x", then the limit of the corresponding sequence of sub-intervals gives us an unique point "y". In mathematical terms,

$$\lim_{n\to\infty} \{J_{n,n_k}\} = x$$

$$\lim_{n\to\infty} \{B_{n,m_k}\} = y.$$

How do we come about building Hilbert's Curve? Since we defined our correspondence f to be a sequence $\{f_n\}$ as $n \to \infty$ we can now confirm our goal of our illustration. This unique continuous curve becomes Hilbert's Curve "f" when we take *the limit of all correspondences* $\{f_n\}$ *as* n *goes to* ∞ as defined earlier in the paper such that,

$$\lim_{n\to\infty} \{f_n(x_n)\} = f(x). \quad [2], [3], [5]$$

Summary of Illustration. "We have defined a sequence of correspondence f_n where each correspondence maps a sub-interval onto a sub-square. For each $x \in J$, we have accumulated another sequence $\{f_n(x_n)\}$ where it possess corresponding function values" [4]. There is a set of points $S = \{x_1, x_2, ..., x_k\}$ which is in the set J that was not defined intentionally until now. It is said that the set of points S makes the sequence $\{f_n(x_n)\}$ *converge*. Our function "f" is also defined by the equation,

$$\lim_{n \to \infty} \{f_n(x_n)\} = f(x), \ \ \forall x \in J.$$

This function can also be named the "*limit function* of the sequence $\{f_n\}$ and that this sequence *converges pointwise* in the unit square" [4]. Since we have shown that there is a specific correspondence between the sub-intervals and the sub-squares, both the Adjacency and Nesting conditions hold. To sum up the illustration, the limit of the squares determines a unique space-filling curve mapping a limit of a sequence of correspondences $\{f_n\} : J \to B$, which came out to be a unique continuous function "f" and let this function be named the Hilbert's Space-Filling Curve.

Relations

Now, we will explain how the *iterations relate to the theorem*. The correspondences in theorem 4 are the mappings between every sub-interval and sub-square for every iteration "n". Another thing to explain is the *relation of the correspondence "f_n" to the theorem*. The limit of correspondence "f_n" determines the unique continuous function "f" which was later denoted as Hilbert's Space-Filling Curve. How would *the iterations relate to Hilbert's Curve f*? Earlier in the paper we found out that the limit of our correspondence "f_n" as n goes to infinity between sub-intervals and sub-squares determines our Hilbert's Curve. Due to the definition of a surjection, each sub-square of the range is mapped onto by at least one sub-interval of the domain. Thus, the limits of the domain and range *converge* to a certain point by the Nesting and Adjacency Conditions.

Why is this correspondence *not one-to-one*? Lets go back to our question about our sets being countable or uncountable. The set of Real Numbers $\mathbb{R}$ is *uncountable* by definition and was proven by Georg Cantor. A specific set is *countable* when it has *finite* elements in it. The set is uncountable when it has *infinite* elements in it. We have defined that our set of sub-intervals and set of sub-squares are infinite. "When you have a correspondence between two *finite* sets it is said that it is injective or "one-to-one"" [4]. On the contrary, when you have a correspondence between two infinite sets, the state of being one-to-one does not exist. We noticed that for every iteration n, our two sets *converges* to the set of real numbers $\mathbb{R}$ which is *uncountable*.

Since we have two infinite sets, a one-to-one correspondence does not exist because each set is uncountable. A *one-to-one* correspondence is a *bijection* meaning that it is *injective* and *surjective*. This correspondence is *not* one-to-one because every sub-square in the unit square *cannot* be mapped onto by *at most one* corresponding sub-interval in the unit interval. Note that in a one-to-one function, each x is paired with *only one* y and each y is paired with *only one* x. Also, there has to be *identity functions of injection* **defined** such that $f(x) = x$ and $f(y) = y$ so that $f(x) = y = x$ for a function to be defined as *one-to-one* [4]. In the topic of space-filling curves, I did not come across any research so far where there was an identity function **defined** to determine a curve being space-filling. Some mathematicians may state that an identity function is obvious therefore it will not be stated or defined. On the contrary, every sub-square *can be* mapped onto by *at least one* corresponding sub-interval. The first iteration yields,

$$f_1 : J_{1,\{n_4\}} \to B_{1,\{m_4\}}$$

where $\{m_4\}$ = the set of sub-squares $\{m_1, m_2, m_3, m_4\}$. In the first iteration, *at most one* sub-interval corresponds to *at most one* sub-square as stated earlier to satisfy the *injection* definition. Then as we take more iterations of the correspondence, f_n, we notice that we can map the next iterations of sub-intervals onto the first iteration of sub-squares $B_{1,\{m_4\}}$. To confirm this statement we show the next iterations such that,

$$f_2 : J_{2,\{n_{16}\}} \to B_{1,\{m_4\}}$$

$$f_3 : J_{3,\{n_{64}\}} \to B_{1,\{m_4\}}$$

$$\vdots$$

$$f_k : J_{k,\{n_k\}} \to B_{1,\{m_4\}}$$

More specifically, for the second iteration,

$$f_2 : J_{2,\{n_4\}} \to B_{1,m_1}$$

$$f_2 : J_{2,\{n_4\}} \to B_{1,m_2}$$

$$f_2 : J_{2,\{n_4\}} \to B_{1,m_3}$$

$$f_2 : J_{2,\{n_4\}} \to B_{1,m_4}$$

and for the third iteration,

$$f_3 : J_{3,\{n_{16}\}} \to B_{1,m_1}$$

$$f_3 : J_{3,\{n_{16}\}} \to B_{1,m_2}$$

$$f_3 : J_{3,\{n_{16}\}} \to B_{1,m_3}$$

$$f_3 : J_{3,\{n_{16}\}} \to B_{1,m_4}$$

and so forth. In other words, four sub-intervals from the set of sixteen are mapped onto a corresponding sub-square. This process is repeated for every iteration. Since we quarter the unit interval for every iterative correspondence, we can map multiple sub-intervals onto the first iteration of sub-squares. Due to this mapping being possible, this disproves the statement that "*at most one* element in the range can be mapped onto by *at most one* element in the domain. Thus, our *one-to-one* correspondence is a *surjection only* which is a contradiction! Also, from the definition of the injective function having inverses we have noticed that we can *not* map our set of squares unto the set of intervals with the inverse function f^{-1} because the set of intervals is strictly defined as our *one-dimensional* space $x \in [0, 1]$. Mapping the range to our domain will simply put our two dimensional squares in our one-dimensional interval which can not happen, thus confirming another statement bringing us to a contradiction yet again.

References

[1] Rose, J. Nicholas, *"Hilbert-Type Space-Filling Curves,"* pp. 1-2

[2] Sherbert, D., Bartle, R. (2000). *"Introduction to Real Analysis"*: (3^{rd} edition). Danvers, MA : John Wiley and Sons, Inc.. pp. 120, 227.

[3] Trench, W. (2002). *"Introduction to Real Analysis,"*. Pearson Education, Inc. pp. 54, 234

[4] Apostol, T. (1957). *"Mathematical Analysis"*: (1^{st} edition). Reading, MA : Addison-Wesley Publishing Company. pp. 29-31, 390, 392.

[5] Kirkwood, J. (1995). *"An Introduction to Analysis,"*. Boston, MA : PWS Publishing Company. pp. 200-201.

[6] Mendelson, B. (1975). *"Introduction to Topology"*: (3^{rd} edition). Toronto : General Publishing Company. pp. 12-14.

[7] Thompson, B., Bruckner, J., Bruckner, A. (2008). *"Elementary Real Analysis"*: (2^{nd} edition). Prentice Hall(Pearson). pp. 32-34.

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