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CHIP-FIRING GAME AS PROBABILITY ABACUS BY CHARACTERIZATION OF ULD LATTICE

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Abstract: Based on the graph and distribution of chips on its vertices three types of dynamic models are available. Out of these three models, CFGs (Chip-Firing Game) have received extensive attention on account of their application in many areas of mathematics such as algebra, combinatory, dynamical systems, statistics, algorithms, and computational complexity. In this paper, we use characterization properties of set of configurations called configuration space which are ordered by predecessor relations. We use CFG as a procedure called "Probability abacus" to determine absorbing probability in the form of vector addition language. Furthermore, by using characterization property of ULD lattice we show that termination state of any CFG on absorbing Markova chain with rational transition matrix does not depend on order of firing and at termination state critical loading re-occurred.

1. Introduction

There are number of classical discrete dynamic models. One such dynamic model Chip-firing game is defined on directed graphs which are called support graphs. It was introduced by Bjorner and Lovasz [1].Later study of Generalization of CFGs on undirected graph was done by Bjorner, Lovasz, and Shor [2]. [6] Magnien, Phan, and Vuillon found that every distributive lattice can be represented as a CFG. Moreover, they found that every generalized CFG can be viewed as ULD. Kimmo Eriksson [10] studied that every CFG is strongly convergent which implies that, "for an initial configuration either a given game can be played forever, or it reaches a unique fixed point where no firing is possible" called termination state" In this paper we consider only those CFGs that reach up to the unique fixed point.

Generally, random walks on graphs are approximated by computing the expected hitting time, or probable number of random moves required to go from one vertex to another. Although random walks are useful in mathematics and computer science, yet probabilistic systems do not offer sufficient precision for some applications. There are, however, several emerging methods of deterministically simulating random walks which can be used to compute position of the object at a given stage more efficiently [11,9].

Engel [3,4] considered a chip-firing as a procedure called the "probabilistic abacus" to determine the absorption probabilities and should be access times of certain Markov chains by combinatorial means. Here we deal with CFGs which can be interpreted in vector-addition languages. Results produced by vector-addition languages are same as ULD lattices. Hence the languages are called generalized chip-firing games.

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Vector addition languages were introduced by Karp and Miller [14]. They are also known as general Petri nets (Reisig[15]) and are one of the most popular formal methods for analysis and representation of parallel processes [16]. We will only use them for splitting absorbing probability.
In this paper we define CFG on absorbing Markov chain with rational transition probabilities ordered by predecessor relations with three absorbing states. We will use results of [16] and show that termination state of game does not depend on order of firing to vertex. Furthermore, with the same initial configuration a game played with two different strategies game reaches to critical loading which will be in the form of vector addition language.

2. Preliminaries and terminologies

The model (G, L (μ),and ≤) is a game which consists of a directed multi-graph G, the set of configurations on G i.e. L (μ) and an evolution rule ≤ called firing rule on set of configurations. Here, a configuration μ on G is a map from the set V (G) of vertices of G to non-negative integers associate a weight to each vertex, which can be considered as a number of chips stored in the vertex. In a configuration μ, vertex v is firable if v has at least one outgoing edge and μ (v) is at least the out-degree of v called evolution rule [6]. When v is firable in μ, μ can be transformed into another configuration $\mu'$by moving chips stored in v along each outgoing edge of v. We call this process firing v, and write $\mu \rightarrow \mu'$ An execution is a sequence of firing and is often written in the form $\mu_1 \xrightarrow{v_1} \mu_2 \xrightarrow{v_2} \mu_3 \xrightarrow{v_3} \mu_4 \xrightarrow{v_4} \dots\dots\dots\dots\dots \xrightarrow{v_{k-1}} \mu_k$ .The set of configurations which can be obtained from CFG by a sequence of firing is called configuration space, and denoted by L (μ).Set of all configurations reachable from initial configuration (μ_0) to final configuration (μ_k) ordered by the reflexive and transitive closure of the predecessor relation, is a lattice [13] .When a game is convergent, its configuration space is a ULD lattice [6]. This is a very strong property, because ULD lattices are very structured sets. For instance, an immediate consequence of this is the fact that, in any convergent CFG all the firing sequences from the initial configuration to the final configuration have the same length i.e. constructed lattice is ranked. We say that two convergent games are equivalent if corresponding lattice of their configuration space are isomorphic. Shot-set s(μ) of a configuration μ is the set of vertices fired to reach μ_k from the initial configuration μ_0[6].CFG with given support graph we say that a subset X ⊆ V is a valid shot-set if there exists a configuration μ_i reachable from the initial configuration such that s(μ_i) = X. A list $\{v_1, v_2, v_3, \dots\dots. v_n\}$ of vertices is a valid firing sequence if, for each i $\{v_1, v_2, v_3, \dots\dots. v_i\}$ is a valid shot-set [13]. The configuration space of a CFG is isomorphic to the lattice of the shot-sets of its configurations, ordered by inclusion.

Throughout the paper we use terminology of [6]

Lattice (2.1)

Let L = (X, ≤) be a finite partial order set ,L is a lattice if any two elements of L have a least upper bound (join) and a greatest lower bound (meet). For every x, y ∈ X, x ∨ y and x ∧ y denote the join and the meet of x, y, respectively. For x ∈ X, x is a meet-irreducible if it has exactly one upper cover. The element x is a join-irreducible if x has exactly one lower cover. Let M and J denote the collections of the meet-irreducible and the join-irreducible of L, respectively. Let $M_{\mathrm{x}} = \{\mathrm{m} \in M: \mathrm{x} \leq \mathrm{m}\}$ and $\mathrm{J_x} = \{\mathrm{j} \in \mathrm{J}: \mathrm{j} \leq \mathrm{x}\}$.

Preposition 2.1 [6] Let L be a lattice. Any element x of L is the join of the join-irreducible that are smaller than itself, and the meet of the meet-irreducible that are greater than itself:

$x = \vee\{j \in J, j \leq x\} = \wedge\{m \in M, x \leq m\}$. $x = \vee J_x = \wedge M_x$

ULD Lattice (2.2)

Let L= (X,≤) be a poset. L is an upper locally distributive lattice (ULD) if L is a lattice and each element has a unique minimal representation as meet of meet-irreducible, i.e., there is a mapping M: L → L= {m ∈ L: m is meet-irreducible} with the properties:

$$* \, x = \wedge M_x (representation) \quad \star \, x = \wedge A \Longrightarrow M_x \subseteq A (minimal)$$

If L be a ULD lattice with M be a set of meet-irreducible then consider the map $\uparrow x_M := \uparrow x \cap$ M. The definition of meet-irreducible implies that $x = \wedge \uparrow x_M$ for all x, i.e., where $\uparrow x$ is meet-irreducible above x.

Random walks (2.3)

A walk in a graph or digraph is a sequence of vertices $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \dots \dots . \mathrm{v}_\mathrm{k}\}$, not necessarily distinct. Now, if we place some objects corresponding to each stage on each vertex and edge shoes probability of moving objects from one vertex to other given by Markova transition matrix $\mathrm{m_{ij}}$,at each stage occurs a sequence of adjacent vertices. This sequence represents the position of the object at a given stage, which is called random walk.

Absorbing Markov chain (2.4)

A state s_i of a Markov chain is called absorbing if it is impossibleto leave it (i.e., $p_{ij} = 1$). A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state.

In this paper we define chip-firing game as a process called Engel [7,8] "probabilistic abacus "on supported graph, then we will prove some results of absorbing probability by properties of ULD lattice generated by configuration space with predecessor relation followed by firing sequence. For which we create one node for each state and put some chips at the nodes corresponding to the non-absorbing called transitions states. Transitions probability of moving chip from one vertex to another is $p_{ij} = \frac{r_{ij}}{r_i} \; \forall \, j$, where $\mathrm{S}r_i, r_{i1}, r_{i2}, \dots \dots r_{in}$ are integers. If there were r_i chips at node *i* we could 'fire' or 'make a move' in node *i*.To begin the game we require initial configuration in probability abacus, called critical loading.

Critical loading (2.5)

Critical loading is one in which each node has one less chip that it needs to fire, i.e. $c_i = r_i - 1$.

Vector addition language (2.6)

A vector-addition language is a language L (M, μ) given by an alphabet M $\subset R^d$and its starting configuration is $\mu \in R^d \geq 0$. A word s = $(x_1, x_2, x_3, \dots \dots . x_k)$ is in L (M, μ) if$x_i \in$ M and $\mu + x_1 + \dots + x_i \geq 0$ for all $1 \leq i \leq k$.

3. Results used

For terminology, notations and properties of ↑refer [12] and for absorbing Markova chain properties refer [16].

Theorem 3.1[6]. A lattice is distributive if and only if it is isomorphic to the lattice of the ideals of the order induced by its meet-irreducible.

Theorem 3.2[6] The lattice of the configuration space of a CFG is ULD.

Lemma 3.3 [6]. Let L be a ULD and x, y ∈ L. We have $x \prec y$ if and only if $|\uparrow x_M \backslash \uparrow y_M| = 1$.

4. Main Results

Theorem 1.

If defined CFG on absorbing Markov chain with rational transition probabilities ordered by predecessor relations and its configuration space forms ULD lattice then termination state of CFG does not depend on the order of firing in which moves are made.

***Pr*oof**: Suppose, if possible, termination state depends on the order of moves- Suppose game be played by two strategy and we get C $(L_1) = \{\mu_0 \ , \ \mu_1, \mu_2 \dots \dots \dots \dots . \mu_m \ , \leq_p\}$ configuration space with predecessors relations $\leq_p$ from first strategy &C $(L_2) = \{\varphi_0, \varphi_1, \varphi_2 \dots \dots \dots \dots . \varphi_n \ , \leq_p\}$ configuration space from strategy second. And their respective shot sets ordered by inclusion order will be $S(L_1) = \{v_0 \ , v_1 \ , v_3 \ , \dots \dots \dots v_m, <\}$ and $S(L_2) = \{u_0, u_1 \ , u_2 \ , \dots \dots \dots u_n, <\}$ .Each shot set will be isomorphic to their corresponding configuration space i.e. C $(L_1) \cong S(L_1)$ and C$(L_2) \cong S(L_2)$. Both configuration spaces are ULD lattice . Now we compare both strategies-let for $x, y \in S_1$Let $x < y$ i.e. $x \ covers \ y$ since set of configurations are ULD with predecessors relations and shot set are isomorphic to respective configuration space. Hence by lemma [2.2] $|\uparrow x_M \backslash \uparrow y_M| = 1$, because lattice are ranked so number of meet irreducible for respective elements will be same. Now, let by strategy two we get shot set S_2 for which $x, y \in S_2$ such that $x \nprec y$ then by lemma [2.2] $|\uparrow x_M \backslash \uparrow y_M| \neq 1$ let us suppose $|\uparrow x_M \backslash \uparrow y_M| \geq 2$.Since $\wedge M_x < \wedge \uparrow y_M$ there has to be some$m_1 \in \frac{M_x}{\uparrow y_M}$ .Let $z = \wedge (\uparrow x_M - m_1)$.Since in ULD lattice for every element there is unique inclusion-minimal set $M_x \subseteq M(S_1)$ such that $x = \wedge M_x$ so we have $z = \wedge (\uparrow x_M - m_1) > x$ .Since $(\uparrow x_M - m_1) \supseteq \uparrow y_M$ therefore we have z ≤ y. Let m_2 be an element which differs from m_1 *and* belongs to $\uparrow x_M \backslash \uparrow y_M$, follows $m_2 \in \uparrow z_M$ and $m_2 \notin \uparrow y_M$ hence $z \neq y$.This implies that $x < z < y$ i.e. pair x, y is not in covering relation, which is in contradiction to strategy one which follows lemma [2.2]. Hence we conclude that our assumption that termination state depends on order of firing is wrong, and we can say it does not depend on order of firing.

Theorem 2.

Let G be any digraph, let$\mu_0 \ , \ \mu_1, \mu_2 \dots \dots \dots \dots . \mu_m$*be a sequence of chip configurations on*

G, ordered each of which is a predecessors of the one before, and let $\varphi_0, \varphi_1, \varphi_2 \ldots\ldots\ldots\ldots \varphi_n$ *be another such sequence with* $\mu_0 = \varphi_0$

1. *If configuration space with predecessors relations* $\leq_p$ *is a ULD then, and* μ_m *is a terminating configuration then* $n \leq m$ *and moreover no firing sequence may have more thanm meet-irreducible.*
2. *If* μ_m *and* φ_n *both terminating configuration then* $n = m$*,* $\mu_m = \varphi_n$ *and in each firing sequences game terminates when critical loading reoccurs.*

Proof: Part 2 is an immediate corollary of part 1; let part 1 fail so we prove our claim by taking an opposite assumption with $m + n$ minimal. Suppose in strategy one the vertex v_i is fired when configuration μ_{i-1} becomes μ_i(because $\leq_p$ is a predecessor relation) moreover in strategy two the vertex u_j will be fired when configuration φ_{j-1} becomes φ_j .In strategy two the vertex u_1 must be fired at some stage in the sequence of its configuration .Since μ_m is the configuration in strategy one at which game is terminated ,then v_i must be equal to u_1 then $v_i, v_1, v_2 \ldots\ldots\ldots.. v_{i-1}, v_{i+i}, v_m$ be the valid firing sequence which turns configuration μ_0 into μ_m from our first main result with the same number of firing in different order game can be terminated at same configuration i.e. μ_m. So we can see that the firing sequence $v_1, v_2 \ldots\ldots\ldots.. v_{i-1}, v_{i+i}, v_m$and $u_2, u_3 \ldots\ldots\ldots u_n$will be contradicting to the minimalty of lemma starting with same initial configurations, which proves part 2.

For better explanation of our results, we take one example

Let us define a chip-firing game on absorbing Markova chain with rational transition matrix. Corresponding to given transition matrix, we create some nodes-{1,4} which are the transition state (vertex) from where chip can move on other adjusting vertices according to given firing rule (transition probability) and each transition node has one less chip that it needs to fire i.e.$c_i = r_i - 1$.{2,3,5}which are absorbing states from where once a chip enters in this state it is impossible to leave this state, and {0} is firing node which contains large number of chips follow firing rule that node(vertex) 0 may fire only if no other node(vertex) can fire.

As per defined Chip-firing game critical loading by placing chips at transition nodes {1.4} will be (9,0,0,9,0):

Now we play defined game by two different strategies, and will compare results.

Strategy 1

We start with critical loading (9,0,0,9,0) then fire node (vertex) 0and then node(vertex) 1.

$$(9,0,0,9,0) \xrightarrow{0} (10,0,0,9,0) \xrightarrow{1} (4,2,3,9,1) \xrightarrow{0} (4,2,3,10,1) \xrightarrow{4} (5,7,4,1,3) \xrightarrow[9\ times]{0} (5,7,4,10,3) \xrightarrow{4} (6,12,5,1,5) \xrightarrow[9\ times]{0} (6,12,5,10,5) \xrightarrow{4} (7,17,6,1,7) \xrightarrow[9\ times]{0} (7,17,6,10,7) \xrightarrow{4} (8,22,7,1,9) \xrightarrow[9\ times]{0} (8,22,7,10,9) \xrightarrow{4} (9,27,8,1,11) \xrightarrow[8\ times]{0} (9,27,8,9,11)$$

Strategy 2

We start with critical loading (9,0,0,9,0) then fire node (vertex) 0and then node(vertex)

4.

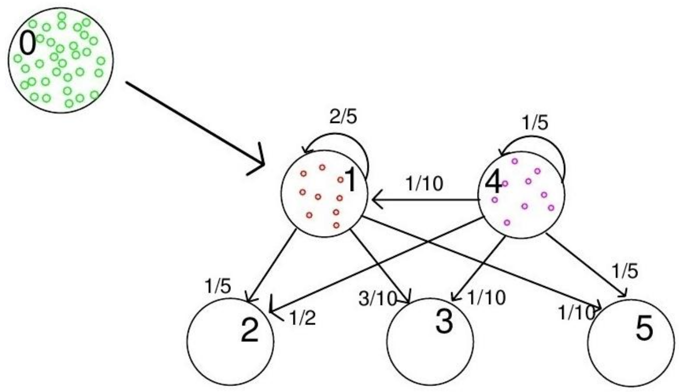
$(9,0,0,9,0) \xrightarrow{0} (9,0,0,10,0) \xrightarrow{4} (10,5,1,1,2) \xrightarrow{1} (4,7,4,1,3) \xrightarrow[9\ times]{0} (4,7,4,10,3) \xrightarrow{4} (5,12,5,1,5) \xrightarrow[9\ times]{0} (5,12,5,10,5) \xrightarrow{4} (6,17,6,1,7) \xrightarrow[9\ times]{0} (6,17,6,10,7) \xrightarrow{4} (7,22,7,1,9) \xrightarrow[9\ times]{0} (7,22,7,10,9) \xrightarrow{4} (8,27,8,1,11) \xrightarrow[8\ times]{0} (8,27,8,9,11) \xrightarrow{0} (9,27,8,9,11)$

From both firing sequence it is clear that after firing node (vertex) 1 one time, node (vertex) 4 five times and node zero 46 times we stop, because transition node {1,4} have exactly the same loading as at the start. Which can be read with the help of vector addition language i.e. (9,0,0,9,0)+(0,27,8,0,11) from final configuration at which game terminates critical loading reoccurs. Thus absorbing probabilities will be $p_{12} = \frac{27}{46}$, $p_{13} = \frac{8}{46}$, $p_{15} = \frac{11}{46}$

All firing rules can be understood from figures given below:

$$\begin{pmatrix} 2/5 & 1/5 & 3/10 & 0 & 1/10 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/10 & 1/2 & 1/10 & 1/10 & 1/5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Rational transition probability matrix



0 is firing node,(1,4) are transition state and (2,3,5) are absorbing states.

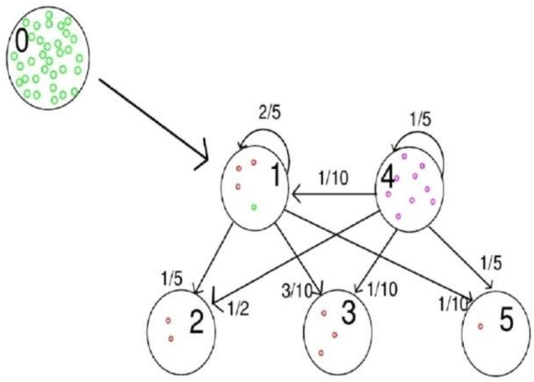


fig1: Firing 1.

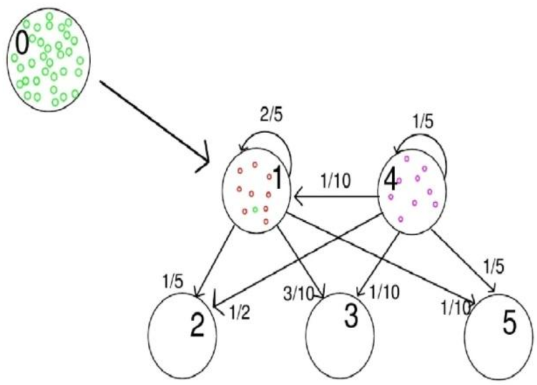


fig 2:Firing from 0 to 1.

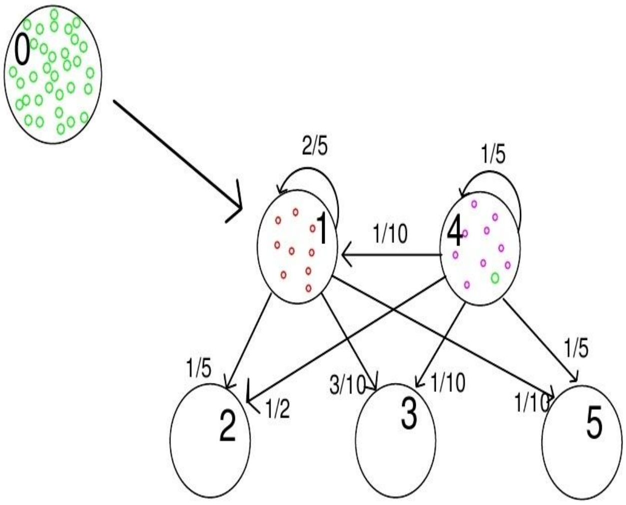


fig 3:Firing from 0 to 4.

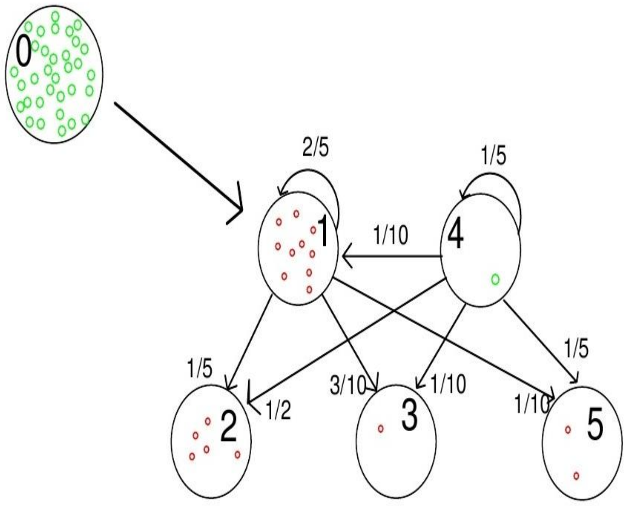


fig 4:Firing 4.

Conclusion:

From the above examples it is clear that if a chip-firing game is played with same initial configuration by different strategy then CFG reaches to a fixed point called final configuration and the state is called termination state. Also, in each firing sequences each vertex fires the same number of times to terminate the game in both strategies.

Since during the firing sequence configuration follows the predecessor relation which is reflexive, transitive and closed under predecessor relation and hence constructs lattice. In this lattice all the finite chains among fixed end points have same length which shows that lattice is ranked hence configuration space with predecessor relation called ULD lattice. In this paper we have proved some results which are based on absorbing probability by using properties of ULD lattice.

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