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THEORETICAL PROPERTIES OF THE WEIGHTED FELLER-PARETO AND RELATED DISTRIBUTIONS

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Abstract. In this paper, for the first time, a new six-parameter class of distributions called weighted Feller-Pareto (WFP) and related family of distributions is proposed. This new class of distributions contains several other Pareto-type distributions such as length-biased (LB) Pareto, weighted Pareto (WP I, II, III, and IV), and Pareto (P I, II, III, and IV) distributions as special cases. The pdf, cdf, hazard and reverse hazard functions, monotonicity properties, moments, entropy measures including Renyi, Shannon and s-entropies are derived.

1. Introduction

Pareto distributions provide models for many applications in social, natural and physical sciences and are related to many other families of distribution. A hierarchy of the Pareto distributions has been established starting from the classical Pareto (I) to Pareto (IV) distributions with subsequent additional parameters related to location, shape and inequality. A general version of this family of distributions is called the Pareto (IV) distribution. Pareto distribution has applications in a wide variety of settings including clusters of Bose-Einstein condensate near absolute zero, file size distribution of internet traffic that uses the TCP protocol, values of oil reserves in oil fields, standardized price returns on individual stocks, to mention a few areas.

Brazauskas (2002) determined the exact form of the Fisher information matrix for the Feller-Pareto distribution. Rizzo (2009) developed a new approach to goodness of fit test for Pareto distributions. Riabi et al. (2010) obtained entropy measures for the family of weighted Pareto-type distributions with the the general weight function $w(x;k,t,i,j) = x^k e^{tx} F^i(x)\bar{F}^j(x)$, where $F(x)$ and $\bar{F}(x) = 1 - F(x)$ are the cumulative distribution function (cdf) and survival or reliability function, respectively.

1.1. Some Basic Utility Notions.

Suppose the distribution of a continuous random variable X has the parameter set $\theta^* = \{\theta_1, \theta_2, \cdots, \theta_n\}$. Let the probability density function (pdf) of X be given by $f(x;\theta^*)$. The cumulative distribution function (cdf) of X, is defined to be $F(x;\theta^*) = \int_{-\infty}^{x} f(t;\theta^*)\,dt$. The hazard rate and reverse hazard rate functions are given by $h(x;\theta^*) = \frac{f(x;\theta^*)}{1-F(x;\theta^*)}$, and $\tau(x;\theta^*) = \frac{f(x;\theta^*)}{F(x;\theta^*)}$, respectively, where $\bar{F}(x;\theta^*)$ is the survival or reliability function. The following useful functions are applied in subsequent sections. The gamma function is given by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$. The first and the second derivative of the gamma function are given by: $\Gamma'(x) = \int_0^\infty t^{x-1}(\log t)e^{-t}\,dt$, and $\Gamma''(x) = \int_0^\infty t^{x-1}(\log t)^2 e^{-t}\,dt$,

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respectively. The digamma function is defined by $\Psi(x) = \Gamma'(x)/\Gamma(x)$. The lower incomplete gamma function and upper incomplete gamma function are $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt$ and $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt$, respectively.
Let $a, b > 0$, then

$$\int_0^1 x^{a-1}(1-x)^{b-1}\ln(x)\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\Psi(a) - \Psi(a+b)). \tag{1}$$

1.2. **Introduction to Weighted Distributions.** Statistical applications of weighted distributions, especially to the analysis of data relating to human population and ecology can be found in Patil and Rao (1978). To introduce the concept of a weighted distribution, suppose X is a non-negative random variable (rv) with its natural probability density function (pdf) $f(x;\theta)$, where the natural parameter is $\theta \in \Omega$ (Ω is the parameter space). Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x;\beta)$, so that the recording (weight) function $w(x;\beta)$ is a non-negative function with the parameter β representing the recording (sighting) mechanism. Clearly, the recorded x is not an observation on X, but on the rv X_w, having a pdf

$$f_w(x;\theta,\beta) = \frac{w(x,\beta)f(x;\theta)}{\omega}, \tag{2}$$

where ω is the normalizing factor obtained to make the total probability equal to unity by choosing $0 < \omega = E[w(X,\beta)] < \infty$. The random variable X_w is called the weighted version of X, and its distribution is related to that of X. The distribution of X_w is called the weighted distribution with weight function w. Note that the weight function $w(x,\beta)$ need not lie between zero and one, and actually may exceed unity. For example, when $w(x;\beta) = x$, in which case $X^* = X_w$ is called the *size-biased version* of X. The distribution of X^* is called the *size-biased distribution* with pdf $f^*(x;\theta) = \frac{xf(x;\theta)}{\mu}$, where $0 < \mu = E[X] < \infty$. The pdf f^* is called the length-biased or size-biased version of f, and the corresponding observational mechanism is called *length-biased* or *size-biased sampling*. Weighted distributions have seen much use as a tool in the selection of appropriate models for observed data drawn without a proper frame. In many situations the model given above is appropriate, and the statistical problems that arise are the determination of a suitable weight function, $w(x;\beta)$, and drawing inferences on θ. Appropriate statistical modeling helps accomplish unbiased inference in spite of the biased data and, at times, even provides a more informative and economic setup. See Rao (1965), Patel and Rao (1978), Oluyede (1999), Nanda and Jain (1999), Gupta and Keating (1985) and references therein for a comprehensive review and additional details on weighted distributions.

Motivated by various applications of Pareto distribution in several areas including reliability, exponential tilting (weighting) in finance and actuarial sciences, as well as in economics, we construct and present some statistical properties of a new class of generalized Pareto-type distribution called the Weighted Feller-Pareto (WFP) distribution.

The aim of this paper is to propose and study a generalization of the Pareto distribution via the weighted Feller-Pareto distribution, and obtain a larger class of flexible parametric models with applications in reliability, actuarial science, economics, finance and telecommunications. This paper is organized as follows. Section 2 contains some utility notions and basic results. The weighted Feller-Pareto distribution is introduced in section 2, including the cumulative distribution function (cdf), pdf, hazard and reverse hazard functions and monotonicity properties. In section 3, moments of the WFP distribution are presented. The mean, variance, standard deviation, coefficients of variation, skewness, and kurtosis are readily obtained from the moments. Section 4 contains measures of uncertainty including Renyi, Shannon and $s-$entropies of the distribution. Some concluding remarks are given in section 5.

2. The Weighted Feller-Pareto Class of Distributions

In this section, the weighted Feller-Pareto class of distributions is presented. First, we discuss the Feller-Pareto distribution, its properties and some sub-models. Some sub-models of the FP distribution are given in Table 1 below.

2.1. **Feller-Pareto Distribution.** In this section, we take a close look at a more general form of the Pareto distribution called the Feller-Pareto distribution which traces it's root back to Feller (1971).

Definition 2.1. *The Feller-Pareto distribution called FP distribution for short, is defined as the distribution of the random variable* $Y = \mu + \theta(X^{-1} - 1)^{\gamma}$*, where* X *follows a beta distribution with parameters* α *and* β*,* $\theta > 0$ *and* $\gamma > 0$*, that is,* $Y \sim FP(\mu, \theta, \gamma, \alpha, \beta)$ *if the pdf of* Y *is of the form*

$$f_{FP}(y;\mu,\theta,\gamma,\alpha,\beta) = \frac{1}{B(\alpha,\beta)\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{\frac{\beta}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+\beta)}, \tag{3}$$

for $-\infty < \mu < \infty$, $\alpha > 0$, $\beta > 0$, $\theta > 0$, $\gamma > 0$, and $y > \mu$, where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

The transformed beta (TB) distribution is a special case of the FP distribution. The pdf of the transformed beta distribution is given by

$$f_{TBD}(x;\theta,\gamma,\alpha,\beta) = \frac{1}{B(\alpha,\beta)x}\frac{\gamma(x/\theta)^{\gamma\beta}}{[1+(x/\theta)^{\gamma}]^{\alpha+\beta}}, \quad x \geq 0. \tag{4}$$

Therefore, from (4), $TB(\theta, \gamma, \alpha, \beta) = FP(0, \theta, 1/\gamma, \alpha, \beta)$. The family of Pareto distributions (Pareto I to Pareto IV) can be readily obtained for specified values of the parameters μ, θ, γ, α, and β.

Table 1. Some Sub-Models of the FP Distribution

Family name	Symbol	Density function
$FP(I)$	$FP(y;\mu,\theta,1,\alpha,1)$	$\frac{1}{B(\alpha,1)\theta}(\frac{y-\mu}{\theta})^{\frac{1}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+1)}$
$FP(II)$	$FP(y;\mu,\theta,1,\alpha,1)$	$\frac{1}{B(\alpha,1)\theta}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+1)}$
$FP(III)$	$FP(y;\mu,\theta,\gamma,1,1)$	$\frac{1}{\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-2}$
$FP(IV)$	$FP(y;\mu,\theta,\gamma,\alpha,1)$	$\frac{1}{B(\alpha,1)\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+1)}$

2.2. **Moments of Feller-Pareto Distribution.** The k^{th} moment of the random variable $\frac{Y-\mu}{\theta}$ under FP distribution is given by:

$$E\left(\frac{Y-\mu}{\theta}\right)^k = \int_\mu^\infty \frac{1}{B(\alpha,\beta)\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+\beta)} dy.$$

Let $\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{t}{1-t}$, $0<t<1$, then $dy = \theta\gamma\left(\frac{t}{1-t}\right)^{\gamma-1}\frac{dt}{(1-t)^2}$, and

$$\begin{aligned} E\left(\frac{Y-\mu}{\theta}\right)^k &= \int_0^1 \frac{\theta\gamma}{B(\alpha,\beta)\theta\gamma}\left(\frac{t}{1-t}\right)^{\gamma(k+\frac{\beta}{\gamma}-1)}\left[\frac{1}{1-t}\right]^{-(\alpha+\beta)}\left(\frac{t}{1-t}\right)^{\gamma-1}\frac{dt}{(1-t)^2} \\ &= \frac{1}{B(\alpha,\beta)}\int_0^1 t^{k\gamma+\beta-1}(1-t)^{\alpha-k\gamma-1}dt \\ &= \frac{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \end{aligned}$$

for $k=0,1,....,$ $\alpha-k\gamma\neq 0,-1,-2,....,$ and $\alpha-k\gamma>0$. If $\mu=0$, then the k^{th} moment reduces to

$$E(Y^k) = \frac{\theta^k\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}{\Gamma(\alpha)\Gamma(\beta)},$$

for $\frac{-\beta}{\gamma}<k<\frac{\alpha}{\gamma}$. Note that if $\mu=0$ and $\theta=1$, then we have,

$$E(Y^k) = \frac{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}{\Gamma(\alpha)\Gamma(\beta)},$$

$k=0,1,2,....,$ $\alpha-k\gamma\neq 0,-1,-2,.....$ The mean, variance, coefficient of variation (cv), coefficient of skewness (cs) and coefficient of kurtosis (ck) of the Feller-Pareto distribution (when $\mu=0$) are given by

$$\begin{aligned} \mu_{FP} &= \frac{\theta\Gamma(\gamma+\beta)\Gamma(\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \\ \sigma^2_{FP} &= \frac{\theta^2\Gamma(2\gamma+\beta)\Gamma(\alpha-2\gamma)}{\Gamma(\alpha)\Gamma(\beta)} - \left(\frac{\theta\Gamma(\gamma+\beta)\Gamma(\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right)^2, \\ CV_{FP} &= \frac{\sqrt{\sigma^2_{FP}}}{\mu_{FP}} = \frac{\sqrt{\frac{\theta^2\Gamma(2\gamma+\beta)\Gamma(\alpha-2\gamma)}{\Gamma(\alpha)\Gamma(\beta)} - \left(\frac{\theta\Gamma(\gamma+\beta)\Gamma(\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right)^2}}{\frac{\theta\Gamma(\gamma+\beta)\Gamma(\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}}, \\ CS_{FP} &= \frac{E(Y^3)-3\mu\sigma^2-\mu^3}{\sigma^3}, \\ CK_{FP} &= \frac{E(Y^4)-4\mu E(Y)+6\mu^2E(Y^2)-4\mu^3E(Y)+\mu^4}{\sigma^4}, \end{aligned}$$

where $\mu=\mu_{FP}, \sigma=\sqrt{\sigma^2_{FP}}$, $E(Y^3)=\frac{\Gamma(3\gamma+\beta)\Gamma(\alpha-3\gamma)}{\Gamma(\alpha)\Gamma(\beta)}$ and $E(Y^4)=\frac{\Gamma(4\gamma+\beta)\Gamma(\alpha-4\gamma)}{\Gamma(\alpha)\Gamma(\beta)}$. See Arnold (1983), Luceno (2006) and references therein.

2.3. **Weighted Feller-Pareto Distribution.** In this section, we present the weighted Feller-Pareto (WFP) distribution. Some WFP sub-models are also presented in this section. Table 2 contains the pdfs of the sub-models for the WFP distribution. First, consider the weight function $w(y;k) = y^k$. The WFP pdf $f_{WFP}(y)$, when $\mu = 0$ and $\beta = 1$, is given by

$$f_{WFP}(y) = \frac{y^k f_{FP}(y)}{E(Y^k)} \;=\; \frac{\Gamma(\alpha+1)}{\theta\gamma\Gamma(1+k\gamma)\Gamma(\alpha-k\gamma)}\left(\frac{y}{\theta}\right)^{k+\frac{1}{\gamma}-1}\left[1+\left(\frac{y}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-\alpha-1},$$

for $\frac{-1}{\gamma} < k < \frac{\alpha}{\gamma}$. The length-biased ($k = 1$) Feller-Pareto (LBFP) pdf is

$$f_{LBFP}(y) = \frac{\Gamma(\alpha+1)}{\theta\gamma\Gamma(1+\gamma)\Gamma(\alpha-\gamma)}\left(\frac{y}{\theta}\right)^{\frac{1}{\gamma}}\left[1+\left(\frac{y}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-\alpha-1},$$

for $\gamma < \alpha$ and $\mu = 0$. In general, with the weight function $w(y;\mu,\theta,k) = \left(\frac{y-\mu}{\theta}\right)^k$, we obtain the WFP pdf as:

$$(5) \qquad f_{WFP}(y) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)B(\alpha,\beta)\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+\beta)},$$

for $k = 0,1,2,....;\ y > \mu,\ \alpha - k\gamma \neq 0,-1,-2,.......,$ and $\alpha - k\gamma > 0$. The cdf of WFP distribution is given by:

$$\begin{aligned} F(y;\alpha,\beta,\mu,\gamma,\theta,k) &= \int_0^y \frac{\Gamma(\alpha+\beta)}{\theta\gamma\Gamma(\beta+k\gamma)\Gamma(\alpha-k\gamma)} \frac{\left(\frac{x-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}}{\left[1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{\alpha+\beta}}dx \\ (6) \qquad &= \frac{\Gamma(\alpha+\beta)B\left(\frac{(\frac{y-\mu}{\theta})^{(1/\gamma)}}{[1+(\frac{y-\mu}{\theta})^{(1/\gamma)}]};k\gamma+\beta,\alpha+\gamma-k\gamma\right)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)} \end{aligned}$$

for $\alpha - k\gamma \neq 0,-1,-2,......, \alpha+\gamma-k\gamma \neq 0,-1,-2,......,$ where $B(t;a,b) = \int_0^t y^{a-1}(1-y)^{b-1}dy$ is the incomplete beta function. The plots of the pdf and cdf of the WFP distribution in Figures 1 and 2 suggest that the additional parameters k controls the shape and tail weight of the distribution.

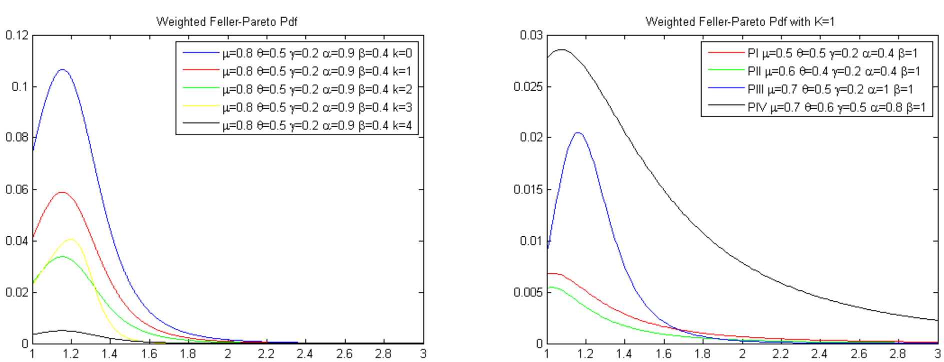

FIGURE 1. PDF of the Weighted Feller-Pareto with k=1 and different values of k

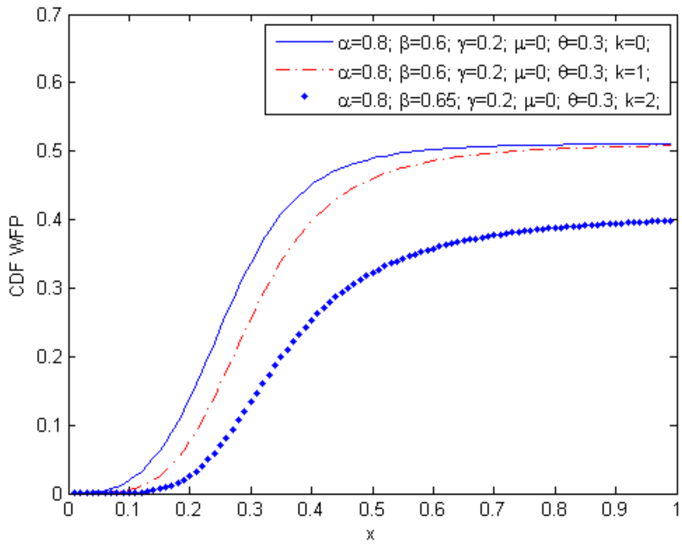

FIGURE 2. CDF of the Weighted Feller-Pareto with k=1 and different values of k

TABLE 2. Sub-Models of the WFP Distribution

Family name	Symbol	Density function
$WP(I)$	$FP(\theta,\theta,1,\alpha,1,k)$	$\frac{\Gamma(\alpha)}{\Gamma(k\gamma+1)\Gamma(\alpha-k)B(\alpha,1)\theta}\left[1+\left(\frac{y-\theta}{\theta}\right)\right]^{-(\alpha+1)}$
$WP(II)$	$FP(\mu,\theta,1,\alpha,,k)$	$\frac{\Gamma(\alpha)}{\Gamma(k\gamma+1)\Gamma(\alpha-k)B(\alpha,1)\theta}\left[1+\left(\frac{y-\mu}{\theta}\right)\right]^{-(\alpha+1)}$
$WP(III)$	$FP(\mu,\theta,\gamma,1,1,k)$	$\frac{\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\Gamma(k\gamma+1)\Gamma(1-k\gamma)B(1,1)\theta\gamma}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-2}$
$WP(IV)$	$FP(\mu,\theta,\gamma,\alpha,1,k)$	$\frac{\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}\Gamma(\alpha+1)}{\Gamma(k\gamma+1)\Gamma(\alpha-k\gamma)\theta\gamma}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+1)}$

2.4. **Hazard and Reverse Hazard Functions.** In this section, hazard and reverse hazard functions of the WFP distribution are presented. Graphs of the hazard function for selected values of the model parameters are also given. The hazard and reverse hazard functions are given by:

$$
\begin{aligned}
h_F(y;k,\alpha,\beta,\mu,\gamma,\theta,k) &= \frac{f(y;\alpha,\beta,\mu,\gamma,\theta,k)}{1-F(y;\alpha,\beta,\mu,\gamma,\theta,k)} \\
&= \frac{\frac{\Gamma(\alpha+\beta)}{\theta\gamma\Gamma(\beta+k\gamma)\Gamma(\alpha-k\gamma)}\left(\frac{y-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-\alpha-\beta}}{1-\frac{\Gamma(\alpha+\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}B\left(\frac{(\frac{y-\mu}{\theta})^{(1/\gamma)}}{[1+(\frac{y-\mu}{\theta})^{(1/\gamma)}]};k\gamma+\beta,\alpha+\gamma-k\gamma\right)}
\end{aligned}
$$

and

$$\tau_F(y;\alpha,\beta,\mu,\gamma,\theta,k) = \frac{f(y;\alpha,\beta,\mu,\gamma,\theta,k)}{F(y;\alpha,\beta,\mu,\gamma,\theta,k)}$$
$$= \frac{\frac{\Gamma(\alpha+\beta)}{\theta\gamma\Gamma(\beta+k\gamma)\Gamma(\alpha-k\gamma)}\left(\frac{y-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-\alpha-\beta}}{\frac{\Gamma(\alpha+\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}B\left(\frac{(\frac{y-\mu}{\theta})^{(1/\gamma)}}{[1+(\frac{y-\mu}{\theta})^{(1/\gamma)}]};k\gamma+\beta,\alpha+\gamma-k\gamma\right)},$$

for $\alpha-k\gamma\neq 0,-1,-2,......,\alpha+\gamma-k\gamma\neq 0,-1,-2,.......$, respectively. Graphs of the WFP hazard rate function for selected values of the model parameters are given in Figure 3. The graphs shows unimodal and upside down bathtub shapes.

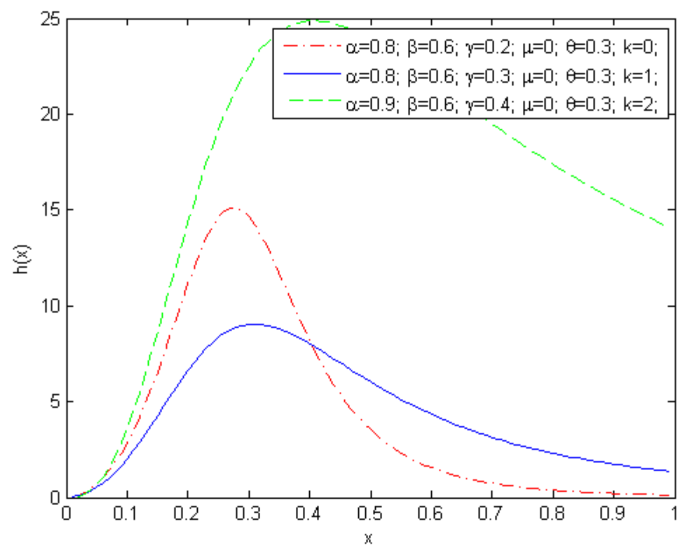

FIGURE 3. Hazard function of the Weighted Feller-Pareto with k=0 and different values of k

2.5. **Monotonicity Properties.** In this section, monotonicity properties of the WFP distribution are presented. The log of the WFP pdf is given by:

$$\ln(f_{WFP}(y)) = n\log\Gamma(\alpha+\beta)-n\log\Gamma(k\gamma+\beta)-n\log\Gamma(\alpha-k\gamma)-n\log\theta\gamma$$
$$+ \left(\frac{\beta}{\gamma}+k-1\right)\log\left(\frac{y-\mu}{\theta}\right)-(\alpha+\beta)\log\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right].$$

Now differentiating $\ln(f_{WFP}(y))$ with respect to y, we have

$$\frac{\partial lnf_{WFP}(y)}{\partial y} = \frac{\left(k-1+\frac{\beta}{\gamma}\right)}{y-\mu} - \frac{(\alpha+\beta)\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\gamma\theta\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]},$$

and solving for y, we have

$$\frac{\partial ln f_{WFP}(y;\theta,\gamma,\alpha,\beta,k)}{\partial y} = 0 \Rightarrow y = \theta\left(\frac{\gamma-\beta-k\gamma}{k\gamma-\gamma-\alpha}\right)^{\gamma} + \mu.$$

Note that

$$\frac{\partial ln f_{WFP}(y;\theta,\gamma,\alpha,\beta,k)}{\partial y} < 0 \Longleftrightarrow y > \theta\left(\frac{\gamma-\beta-k\gamma}{k\gamma-\gamma-\alpha}\right)^{\gamma} + \mu,$$

and

$$\frac{\partial ln f_{WFP}(y;\theta,\gamma,\alpha,\beta,k)}{\partial y} > 0 \Longleftrightarrow y < \theta\left(\frac{\gamma-\beta-k\gamma}{k\gamma-\gamma-\alpha}\right)^{\gamma} + \mu.$$

The mode of the WFP distribution is given by $y_0 = \theta\left(\frac{\gamma-\beta-k\gamma}{k\gamma-\gamma-\alpha}\right)^{\gamma} + \mu$.

3. Moments of WFP Distribution

Recall that the k^{th} moment of the random variable $\frac{Y-\mu}{\theta}$ under the FP distribution are given by

$$E\left(\frac{Y-\mu}{\theta}\right)^{k} = \frac{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}{\Gamma(\alpha)\Gamma(\beta)},$$

for $k = 0, 1, ..., \alpha - k\gamma \neq 0, -1, -2, ..., \alpha - k\gamma > 0$. Now, we derive the r^{th} moments of the random variable $\frac{Y-\mu}{\theta}$ under WFP distribution. This is given by

$$E\left(\frac{Y-\mu}{\theta}\right)^{r} = \int_{\mu}^{\infty} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)B(\alpha,\beta)\theta\gamma} \frac{\left(\frac{y-\mu}{\theta}\right)^{r+k+\frac{\beta}{\gamma}-1}}{\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{(\alpha+\beta)}} dy.$$

Let $\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{t}{1-t}$, $0 < t < 1$, then $dy = \theta\gamma\left(\frac{t}{1-t}\right)^{\gamma-1}\frac{dt}{(1-t)^2}$, and

$$\begin{aligned} E\left(\frac{Y-\mu}{\theta}\right)^{r} &= \int_0^1 \left[\frac{\Gamma(\alpha)\Gamma(\beta)\theta\gamma}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)B(\alpha,\beta)\theta\gamma}\left(\frac{t}{1-t}\right)^{\gamma(r+k+\frac{\beta}{\gamma}-1)}\right. \\ &\times \left.\left[1+\frac{t}{1-t}\right]^{-(\alpha+\beta)}\left(\frac{t}{1-t}\right)^{\gamma-1}\frac{1}{(1-t)^2}\right]dt \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)B(\alpha,\beta)}\int_0^1 t^{r\gamma+k\gamma+\beta-1}(1-t)^{\alpha-r\gamma-k\gamma-1}dt \\ &= \frac{\Gamma(r\gamma+k\gamma+\beta)\Gamma(\alpha-r\gamma-k\gamma)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)}, \qquad (7) \end{aligned}$$

for $\alpha - k\gamma \neq 0, -1, -2, ..., \alpha - r\gamma \neq 0, -1, -2, ..., \alpha - \gamma(r+k) \neq 0, -1, -2, ...,$ $r = 0, 1, 2, ...$ The mean, variance, coefficients of variation (cv), skewness (cs) and kurtosis (ck) can be readily obtained from equation (7).

4. Measures of Uncertainty for the Weighted Feller-Pareto Distribution

The concept of entropy was introduced by Shannon (1948) in the nineteenth century. During the last couple of decades a number of research papers have extended Shannon's original work. Among them are Park (1995), Renyi (1961) who developed a one-parameter extension of Shannon entropy. Wong and Chen (1990) provided some results on Shannon entropy for order statistics. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present Renyi entropy, Shannon entropy and $s-$entropy for the WFP distribution.

4.1. **Shannon Entropy.** Shannon entropy (1948) for a continuous random variable Y with WFP pdf $f_{WFP}(y)$ is defined as

$$E(-\log(f_{WFP}(Y))) = \int_{\mu}^{\infty}(-\log(f_{WFP}(y)))f_{WFP}(y)dy. \tag{8}$$

Note that

$$\begin{aligned} -log(f_{WFP}(y)) &= -log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\theta\gamma}\right) + log\Gamma(\alpha) + log\Gamma(\beta) - log\Gamma(k\gamma+\beta) \\ &- log\Gamma(\alpha-k\gamma) + \left(k+\frac{\beta}{\gamma}-1\right)log\left(\frac{y-\mu}{\theta}\right) \\ &- (\alpha+\beta)log\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]. \end{aligned}$$

Now, Shannon entropy for the weighted Feller-Pareto distribution is

$$\begin{aligned} E(-logf_{WFP}(Y)) &= -log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\theta\gamma}\right) + log\Gamma(\alpha) + log\Gamma(\beta) - log\Gamma(k\gamma+\beta) \\ &- log\Gamma(\alpha-k\gamma) + \left(k+\frac{\beta}{\gamma}-1\right)E\left[log\left(\frac{Y-\mu}{\theta}\right)\right] \\ &- (\alpha+\beta)E\left[log\left[1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]\right]. \end{aligned} \tag{9}$$

Now, with the substitution $\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{t}{1-t}$ for $0 < t < 1$, we can readily obtain both $E\left[log\left(\frac{Y-\mu}{\theta}\right)\right]$ and $E\left[log\left[1+\left(\frac{Y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]\right]$, so that Shannon entropy for the WFP distribution is given by

$$\begin{aligned} E(-logf_{WFP}(Y)) &= -logB(\alpha,\beta) + log(\theta\gamma) + \Gamma(\alpha) + \Gamma(\beta) - log\Gamma(k\gamma+\beta) \\ &- log\Gamma(\alpha-k\gamma) + (k\gamma+\beta-\gamma)[\psi(\beta)-\psi(\alpha)] \\ &- (\alpha+\beta)(\psi(\alpha)-\psi(\alpha+\beta)), \end{aligned} \tag{10}$$

where $\psi(.) = \frac{\Gamma'(.)}{\Gamma(.)}$ is the digamma function.

4.2. **s-Entropy for Weighted Feller-Pareto Distribution.** The s-entropy of the WFP distribution is defined as

$$H_s(f_{WFP}) = \begin{cases} \frac{1}{s-1}\left[1 - \int_\mu^\infty f_{WFP}^s(y)dy\right], & \text{if } s \neq 1,\ s > 0, \\ E(-log f_{WFP}(Y)), & \text{if } s = 1. \end{cases}$$

Note that

$$\begin{aligned} \int_\mu^\infty f_{WFP}^s(y)dy &= \int_\mu^\infty \left[\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)B(\alpha,\beta)\theta\gamma}\left(\frac{y-\mu}{\theta}\right)^{k+\frac{\beta}{\gamma}-1}\right. \\ &\times \left.\left[1+\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right]^{-(\alpha+\beta)}\right]^s dy, \end{aligned}$$

and using the substitution $\left(\frac{y-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{t}{1-t}$ for $0 < t < 1$, so that $dy = \theta\gamma\left(\frac{t}{1-t}\right)^{\gamma-1}\frac{dt}{(1-t)^2}$, we get

$$\begin{aligned} \int_\mu^\infty f_{WFP}^s(y)dy &= \int_0^1 \left[\frac{1}{B(\alpha,\beta)\theta\gamma}t^{k+\beta-\gamma}(1-t)^{\gamma+\alpha-k}\right]^s \theta\gamma t^{\gamma-1}(1-t)^{-\gamma-1}dt \\ &= (\theta\gamma)^{1-s}\frac{[\Gamma(\alpha+\beta)]^s\Gamma(ks+s\beta-s\gamma+\gamma)\Gamma(s\alpha-ks+s\gamma-\gamma)}{[\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)]^s\Gamma(s\beta+s\alpha)}, \end{aligned} \tag{11}$$

for $s > 0$, $s \neq 1$. Consequently, s-entropy for the WFP distribution is given by

$$H_s(f_{WFP}) = \frac{1}{s-1}\left[1 - (\theta\gamma)^{1-s}\frac{[\Gamma(\alpha+\beta)]^s\Gamma(ks+s\beta-s\gamma+\gamma)\Gamma(s\alpha-ks+s\gamma-\gamma)}{[\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)]^s\Gamma(s\beta+s\alpha)}\right],$$

for $s > 0$, $s \neq 1$, $\alpha - k\gamma \neq 0, -1, -2, ...,$ $s\alpha + s\gamma - ks - \gamma \neq 0, -1, -2, ...,$ $ks + s\beta - s\gamma + \gamma > 0$, and $s\alpha - ks + s\gamma - \gamma > 0$.

4.3. **Renyi Entropy.** Renyi entropy (1961) for the WFP distribution is presented in this section. Note that Renyi entropy is given by

$$H_R(f_{WFP}) = \frac{1}{1-s}\log\left(\int_\mu^\infty (f_{WFP}(x))^s dx\right), \quad s > 0,\ s \neq 1. \tag{12}$$

From equation (10), we obtain Renyi entropy as follows:

$$H_R(f_{WFP}) = \frac{1}{1-s}\log\left[\frac{[\Gamma(\alpha+\beta)]^s\Gamma(ks+s\beta-s\gamma+\gamma)\Gamma(s\alpha-ks+s\gamma-\gamma)}{(\theta\gamma)^{s-1}[\Gamma(k\gamma+\beta)\Gamma(\alpha-k\gamma)]^s\Gamma(s\beta+s\alpha)}\right],$$

for $s > 0$, $s \neq 1$, $\alpha - k\gamma \neq 0, -1, -2, ...,$ $s\alpha + s\gamma - ks - \gamma \neq 0, -1, -2, ...,$ $ks + s\beta - s\gamma + \gamma > 0$, and $s\alpha - ks + s\gamma - \gamma > 0$.

5. Concluding Remarks

In this paper, a new six-parameter class of distributions called weighted Feller-Pareto (WFP) distribution is constructed and studied. The pdf, cdf, hazard and reverse hazard functions, monotonicity properties are presented. Measures of uncertainty including Renyi, Shannon and s-entropies are derived.

TABLE 3. Shannon and s-entropies of the sub-models of the FP distribution

Family name	Shannon entropy	s-entropy
$P(I)$	$log\left(\frac{\Gamma(\alpha+1)}{\theta}\right)-\Gamma(\alpha)-1+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}*\left[1-\theta\frac{[\Gamma(\alpha+1)]^s\Gamma(s+\alpha s-1)}{\Gamma(\alpha)^s\Gamma(s+\alpha s)}\right]$
$P(II)$	$log\left(\frac{\Gamma(\alpha+1)}{\theta}\right)-\Gamma(\alpha)-1+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}*\left[1-\theta\frac{[\Gamma(\alpha+1)]^s\Gamma(s+\alpha s-\gamma)}{\Gamma(\alpha)^s\Gamma(s+\alpha s)}\right]$
$P(III)$	$log\left(\frac{\Gamma(2)}{\theta\gamma}\right)-2+2(\psi(1)-\psi(2))$	$\frac{1}{s-1}*\left[1-\theta\gamma^{1-s}\frac{\Gamma(s-\gamma s+\gamma)\Gamma(\gamma s+s-\gamma)}{\Gamma(2s)}\right]$
$P(IV)$	$log\left(\frac{\Gamma(\alpha+1)}{\theta\gamma}\right)-\Gamma(\alpha)-1-(1-\gamma)[\psi(1)-\psi(\alpha)]+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}*\left[1-\frac{\theta\gamma^{1-s}[\alpha]^s\Gamma(s-\gamma s+\gamma)\Gamma(\gamma s+\alpha s-\gamma)}{\Gamma(s+\alpha s)}\right]$

TABLE 4. Shannon and s-entropies of the WFP Distribution

Family name	Shannon entropy	s-entropy
$WP(I)$	$logB(\alpha,1)+log(\theta)-\Gamma(\alpha)-1+log\Gamma(k+1)+log\Gamma(\alpha-k)-k[\psi(1)-\psi(\alpha)]+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}\left[1-\theta\frac{[\Gamma(\alpha+1)]^s\Gamma(ks+1)\Gamma(s\alpha-ks+s-1)}{\Gamma(k+1)^s\Gamma(\alpha-k)^s\Gamma(s+s\alpha)}\right]$
$WP(II)$	$logB(\alpha,1)+log(\theta)-\Gamma(\alpha)-1+log\Gamma(k+1)+log\Gamma(\alpha-k)-k[\psi(1)-\psi(\alpha)]+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}\left[1-\theta\frac{[\Gamma(\alpha+1)]^s\Gamma(ks+1)\Gamma(s\alpha-ks+s-1)}{\Gamma(k+1)^s\Gamma(\alpha-k)^s\Gamma(s+s\alpha)}\right]$
$WP(III)$	$logB(1,1)+log(\theta\gamma)-2+log\Gamma(k\gamma+1)+log\Gamma(1-k\gamma)-(k\gamma+1-\gamma)[\psi(1)-\psi(1)]+2(\psi(1)-\psi(2))$	$\frac{1}{s-1}\left[1-\frac{\theta\gamma^{1-s}\Gamma(ks\gamma+\gamma)\Gamma(s-ks\gamma+s-\gamma)}{\Gamma(k\gamma+1)^s\Gamma(1-k\gamma)^s\Gamma(2s)}\right]$
$WP(IV)$	$logB(\alpha,1)+log(\theta\gamma)-\Gamma(\alpha)-1+log\Gamma(k\gamma+1)+log\Gamma(\alpha-k\gamma)-(k\gamma+1-\gamma)[\psi(1)-\psi(\alpha)]+(\alpha+1)(\psi(\alpha)-\psi(\alpha+1))$	$\frac{1}{s-1}\left[1-\frac{[\Gamma(\alpha+1)]^s\Gamma(ks\gamma+\gamma)\Gamma(s\alpha-ks\gamma+s-\gamma)}{\theta\gamma^{s-1}\Gamma(k\gamma+1)^s\Gamma(\alpha-k\gamma)^s\Gamma(s+s\alpha)}\right]$

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