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On the K-g-Frames in Hilbert C^*-Modules

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Abstract
In this paper, we investigate the g-frame and Bessel g-sequence related to a linear bounded operator K in Hilbert C^*-module and we characterize the concept of a canonical K-dual Bessel sequence of a K-g-frame that generalizes the classical dual of a g-frame.

Keywords: g-frame; K-g-frame; Bessel g-sequence; K-dual Bessel g-sequence; Hilbert $\mathcal{A}$-modules.

Introduction

Frames were first introduced in 1952 by Duffin and Schaefer [4] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. The theory of frames has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert C^*-modules, for more see [7, 10–15].

The paper is organized as follows, in section 2 we briefly recall the definitions and basic properties of K-g-frames in Hilbert C^*-modules. In section 3, we characterize some result for K-dual Bessel g-sequence for given K-g-frames. In section 4, we use a family of linear operators to characterize atomic systems.

Preliminaries

Let I and J be two countable index sets. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, K-g-frames in Hilbert C^*-modules. For information about frames in Hilbert spaces, we refer to [1]. Our reference for C^*-algebras is [2, 3].

For a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive ($a \geq 0$) if $a = a^*$ and $sp(a) \subset \mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [8]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

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(i) $\langle x, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Example 1.2. [9] If $\{\mathcal{H}_k\}_{k\in\mathbf{N}}$ is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$.
On the $\mathcal{A}$-module $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$, of all sequences $x = (x_k)_{k\in\mathbb{N}} : x_k \in \mathcal{H}_k$, such that the series $\sum_{k\in\mathbb{N}}\langle x_k, x_k\rangle_{\mathcal{A}}$ is norm-convergent in the $\mathcal{C}^*$-algebra $\mathcal{A}$, we define the inner product by

$$\langle x, y\rangle := \sum_{k\in\mathbb{N}}\langle x_k, y_k\rangle_{\mathcal{A}}$$

for $x, y \in \oplus_{k\in\mathbb{N}}\mathcal{H}_k$.

Then $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module.

The direct sum of a countable number of copies of a Hilbert $\mathcal{C}^*$-module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \longrightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \longrightarrow \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 1.3. [16] Let $K \in End_{\mathcal{A}}^*(H)$, we call a sequence $\{\Lambda_i\}_{i\in I}$ a K-g-frame for H with respect to $\{H_i\}_{i\in I}$ if there are two positive constants A and B such that:

$$A\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle, \qquad f \in H$$

We call A and B the lower and upper frame bounds, respectively. If we only have the upper bound, then we call $\{\Lambda_i\}_{i\in I}$ a Bessel g-sequence.

Definition 1.4. [6]

Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$, a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called an atomic system for K, it the following conditions are satisfied :

1) $\{f_i\}_{i\in I}$ is a Bessel sequence.

2) There exists $c \geq 0$ such that for every $f \in \mathcal{H}$, there exists $a = \{a_i\}_{i\in I} \in l^2(I)$ such that $\|a\|_{l^2} \leq c\|f\|$ and $Kf = \sum_{i\in I} a_i f_i$.

Lemma 1.5. *[5] Let $U, V \in End_{\mathcal{A}}^*(\mathcal{H})$. The following statements are equivalent :*

1) $\mathcal{R}(U) \subset \mathcal{R}(V)$.

2) $UU^* \leq \lambda VV^*$ for some $\lambda \geq 0$.

3) There exists $Q \in End_{\mathcal{A}}^*(\mathcal{H})$ such that $U = VQ$.

Moreover, if 1), 2) et 3) are valid, then there exists a unique operator Q such that

1) $\|Q\|^2 = inf\{\mu : UU^* \leq \lambda VV^*\}$.

2) $\mathcal{N}(U) = \mathcal{N}(C)$.

3) $\mathcal{R}(C) \subset \overline{\mathcal{R}(V^*)}$.

K-Dual Bessel g-Sequence for given K-g-Frames

Let $\Gamma = \{\Gamma_i\}_{i\in J}$ be a K-g-frames for $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)$. Define an operator

$$\begin{aligned} R_\Gamma : \mathcal{H} &\longrightarrow \oplus_{i\in\mathbb{N}}\mathcal{H}_i \\ x &\longrightarrow R_\Gamma x = \{\Gamma_i x\}_{i\in\mathbb{J}}. \end{aligned}$$

The operator R_Γ is called the analysis operator of the K-g-frames $\{\Gamma_i\}_{i\in\mathbb{J}}$.
The adjoint of the analysis operator R_Γ, is defined by

$$\begin{aligned} T_\Gamma : \oplus_{i\in\mathbb{N}}\mathcal{H}_i &\longrightarrow \mathcal{H} \\ \{x_i\}_{i\in\mathbb{J}} &\longrightarrow T_\Gamma(\{x_i\}_{i\in\mathbb{J}}) = \sum_{i\in\mathbb{J}} \Gamma_i^* x_i. \end{aligned}$$

The operator T_Γ is called the synthesis operator of the K-g-frames $\{\Gamma_i\}_{i\in\mathbb{J}}$.
By composing R_Γ and T_Γ, the frame operator S_Γ for the K-g-frames is given by,

$$\begin{aligned} S_\Gamma : \mathcal{H} &\longrightarrow \mathcal{H} \\ x &\longrightarrow S_\Gamma(x) = T_\Gamma R_\Gamma x = \sum_{i\in\mathbb{J}} \Gamma_i^*\Gamma_i x. \end{aligned}$$

Definition 1.6. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $\mathcal{H}$. A Bessel g-sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ is called a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$ if:

$$Kf = \sum_{i\in I} \Lambda_i^*\Gamma_i f, \qquad f \in \mathcal{H}.$$

Theorem 1.7. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $\mathcal{H}$ with optimal lower frame bound A, ($A = max\{\lambda \geq 0, \lambda\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle, f \in \mathcal{H}\}$.*

If $\Gamma = \{\Gamma_i\}_{i\in I}$ is a K-dual bessel sequence of $\{\Lambda_i\}_{i\in I}$, then $A \leq \|T_\Gamma\|^2$, where T_Γ denotes the synthesis operator of Γ.

Moreover, there exists a unique K-dual bessel sequence $\theta = \{\theta_i\}_{i\in I}$ of $\{\Lambda_i\}_{i\in I}$ such that $\|T_\theta\|^2 = A$ where T_θ denotes the synthesis operator of θ.

Proof. Suppose that $C \geq 0$ is a lower K-g-frame bound of $\{\Lambda_i\}_{i\in I}$, then for any $f \in H$, we have,

$$C\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle.$$

So,

$$C\langle K^*f, K^*f\rangle \leq \langle T_\Lambda f, T_\Lambda f\rangle.$$

This implies that :

$$\langle K^*f, K^*f\rangle \leq \frac{1}{C}\langle T_\Lambda f, T_\Lambda f\rangle$$

So,

$$\begin{aligned} A &= max\{\lambda \geq 0, \lambda\langle K^*f, K^*f\rangle \leq \langle T_\Lambda f, T_\Lambda f\rangle, \forall f \in \mathcal{H}\} \\ &= inf\{\mu \geq 0, \langle K^*f, K^*f\rangle \leq \mu\langle T_\Lambda f, T_\Lambda f\rangle, \forall f \in \mathcal{H}\} \end{aligned}$$

Since $\{\Gamma_i\}_{i\in I}$ is a K-dual Bessel g-sequence of $\{\Lambda_i\}_{i\in I}$.
For any $f \in \mathcal{H}$, we have :

$$Kf = \sum_{i\in I} \Lambda_i^*\Gamma_i f = T_\Lambda T_\Gamma^* f$$

So, $K = T_\Lambda T_\Gamma^*$

Thus : $KK^* = T_\Lambda T_\Gamma^* T_\Gamma T_\Lambda^* \leq \|T_\Gamma\|^2 T_\Lambda T_\Lambda^*$.

So for any $f \in \mathcal{H}$ we have,

$$\|K^*f\|^2 = \langle K^*f, K^*f\rangle = \langle KK^*f, f\rangle \leq \|T_\Gamma\|^2 \langle T_\Lambda T_\Lambda^* f, f\rangle = \|T_\Gamma\|^2 \|T_\Lambda f\|^2$$

So $\|T_\Lambda\|^2 \geq A$.

Since $\{\Lambda_i\}_{i\in I}$ is a K-g-frame , we have that $Range(K) \subset Range(T_\Lambda)$ there exists a unique bounded operator $\Phi : \oplus_{i\in I}\mathcal{H}_i \to \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$ and

$$\|\Phi\|^2 = inf\{\mu : \|K^*f\|^2 \leq \mu\|T_\Lambda f\|^2, \forall f \in \mathcal{H}\} = A$$

Let $\Theta_i^* e_{ij} = \Phi(e_{ij}\delta_i)$, then it is easy to check that $\Theta = \{\Theta_i\}_{i\in I}$ is a Bessel g-sequence, since for any $f \in \mathcal{H}$ we have :

$$K^*f = \Phi T_\Lambda^* f = \sum_{i\in I} \Gamma_i^* \Lambda_i f$$

□

Theorem 1.8. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathcal{H}$ with a frame operator S_Λ. If $\{\Lambda_i\}_{i\in I}$ has a dual g-frame on $R(K)$ and $S_\Lambda(R(K)) \subset R(K)$, then it is a K-g-frame in $\mathcal{H}$.*

Proof. Assume that $\{\Gamma_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ on $R(K)$. Then for each $f \in \mathcal{H}$ can be expressed as $f = f_1 + f_2$, where $f_1 \in R(K)$ and $f_2 \in (R(K))^\perp$.

Then,

$$\begin{aligned}\sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle &= \sum_{i\in I}\langle \Lambda_i(f_1+f_2), \Lambda_i(f_1+f_2)\rangle \\ &= \sum_{i\in I}\langle \Lambda_i f_1, \Lambda_i f_1\rangle + \sum_{i\in I}\langle \Lambda_i f_2, \Lambda_i f_2\rangle + 2(\sum_{i\in I}\langle \Lambda_i^*\Lambda_i f_1, f_2\rangle)\end{aligned}$$

Note that,

$$\sum_{i\in I}\Lambda_i^*\Lambda_i f_1 = S_\Lambda f_1 \in S_\Lambda(R(K)) \subset R(K)$$

and so we have $\sum_{i\in I}\langle \Lambda_i^*\Lambda_i f_1, f_2\rangle = 0$.

Hence ,

$$\sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in I}\langle \Lambda_i f_1, \Lambda_i f_1\rangle + \sum_{i\in I}\langle \Lambda_i f_2, \Lambda_i f_2\rangle.$$

$\sum_{i\in I}\Gamma_i^*\Lambda_i f_1$ converge and so does $\sum_{i\in I}\pi_{R(K)}\Gamma_i^*\Lambda_i f$ where $\pi_{R(K)}$ is an orthogonal projection of $\mathcal{H}$ onto $R(K)$. Then for each $g \in R(K)$ we have,

$$\sum_{i\in I}\langle g, f_1\rangle = \sum_{i\in I}\langle \Lambda_i^*\Gamma_i g, f_1\rangle = \sum_{i\in I}\langle g, \Gamma_i^*\Lambda_i g\rangle = \langle g, \sum_{i\in I}\pi_{R(K)}\Gamma_i^*\Lambda_i g\rangle$$

It follows that,

$$f_1 = \sum_{i\in I}\pi_{R(K)}\Gamma_i^*\Lambda_i g.$$

Thus,

$$\begin{aligned}
\|K^*f\|^4 &= \|K^*(f_1+f_2)\|^4\\
&= \|K^*f_1\|^4\rangle\\
&= \|\langle K^*f_1, K^*f_1\rangle\|^2\\
&= \|\langle f_1, KK^*f_1\rangle\|^2\\
&= \|\langle \sum_{i\in I}\pi_{R(K)}\Gamma_i^*\Lambda_i f_1, KK^*f_1\rangle\|^2\\
&= \|\sum_{i\in I}\langle \Lambda_i f_1, \pi_{R(K)}\Gamma_i KK^*f_1\rangle\|^2\\
&\le (\sum_{i\in I}\|\Lambda_i f_1\|^2)(\sum_{i\in I}\|\Gamma\pi_{R(K)}KK^*f_1\|^2)\\
&\le D\|K\|^2\|K^*f\|^2(\sum_{i\in I}\|\Lambda_i f_1\|^2),
\end{aligned}$$

where D is the Bessel bound of $\{\Gamma_i\}_{i\in I}$.
Then we have,

$$D^{-1}\|K\|^{-2}\|K^*f\|^2 \le \sum_{i\in I}\|\Lambda_i f_1\|^2$$

Hence,

$$\begin{aligned}
\sum_{i\in I}\|\Lambda_i f\|^2 &= \sum_{i\in I}\|\Lambda_i f_1\|^2 + \sum_{i\in I}\|\Lambda_i f_2\|^2\\
&\ge \sum_{i\in I}\|\Lambda_i f_1\|^2\\
&\ge D^{-1}\|K\|^{-2}\|K^*f\|^2.
\end{aligned}$$

□

Atomic system in Hilbert C^*-module

We give a characterization of an atomic system with a sequence of linear operators.

Theorem 1.9. *Let $\{\Lambda_i\}_{i\in I}$ be a family of linear operator in $End_{\mathcal{A}}^*(\mathcal{H})$, then the following statements are equivalent :*

1) *$\{\Lambda_i\}_{i\in I}$ is an atomic system for K.*

2) *$\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$.*

3) *There exists a g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ such that $Kf = \sum_{i\in I}\Lambda_i^*\Gamma_i f$.*

Proof. it is easy. □

Theorem 1.10. *Let $K_1, K_2 \in End_{\mathcal{A}}^*(\mathcal{H})$, if $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1 and K_2, and α,β are a scalars. Then $\{\Lambda_i\}_{i\in I}$ is an atomic system for $\alpha K_1 + \beta K_2$ and K_1K_2.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1 and K_2. There exists a positives constants $A_n, B_n > 0$ $(n = 1, 2)$ such that

$$A_n\langle K_n^*f, K_n^*f\rangle \le \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \le B_n\langle f, f\rangle, \qquad f\in\mathcal{H} \tag{1.1}$$

Since

$$\begin{aligned}\|K_1^*f\|^2 &= \frac{1}{|\alpha|^2}\|\alpha K_1^*f\|^2 \\ &= \frac{1}{|\alpha|^2}\|(\alpha K_1^*+\beta K_2^*)f-\beta K_2^*f\|^2 \\ &\geq \frac{1}{|\alpha|^2}\|(\alpha K_1^*+\beta K_2^*)f\|^2-\frac{1}{|\alpha|^2}\|\beta K_2^*f\|^2.\end{aligned}$$

Then,

$$\begin{aligned}\|(\alpha K_1^*+\beta K_2^*)f\|^2 &\leq |\alpha|^2\|K_1^*f\|^2+|\beta|^2\|K_2^*f\|^2 \\ &\leq \frac{1}{2}(|\alpha|^2\|K_1^*f\|^2+|\beta|^2\|K_2^*f\|^2+\frac{A_1}{A_2}|\beta|^2\|K_1^*f\|^2+\frac{A_2}{A_1}|\alpha|^2\|K_2^*f\|^2) \\ &= \frac{A_2|\alpha|^2+A_1|\beta|^2}{2A_1A_2}(A_1\|K_1^*f\|^2+A_2\|K_2^*f\|^2)\end{aligned}$$

Hence,

$$\sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle \geq \frac{1}{2}(A_1\|K_1^*f\|^2+A_2\|K_2^*f\|^2) \geq \frac{A_1A_2}{A_2|\alpha|^2+A_1|\beta|^2}(\|(\alpha K_1^*+\beta K_2^*)f\|^2).$$

From inequalities (1.1), we get :

$$\sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle \leq \frac{B_1+B_2}{2}\|f\|^2, \qquad f\in H.$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is an $(\alpha K_1+\beta K_2)$-g-frame.

By theorem (1.9) $\{\Lambda_i\}_{i\in I}$ is an atomic system of $(\alpha K_1+\beta K_2)$.
Now for each $f\in\mathcal{H}$, we have

$$\begin{aligned}\|(K_1K_2)^*f\|^2 &= \|K_2^*K_1^*f\|^2 \\ &\leq \|K_2^*\|^2\|K_1^*f\|^2\end{aligned}$$

Hence $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1, we have

$$\frac{A_1}{\|K_2^*\|^2}\|(K_1K_2)^*f\|^2 \leq A_1\|K_1^*f\|^2 \leq \sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle \leq B_1\|f\|^2,\ f\in H.$$

By theorem 1.9 we conclude that $\{\Lambda_i\}_{i\in I}$ is an atomic system for K_1K_2. □

Theorem 1.11. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two atomic systems for K, and let the corrsponding synthesis operators be T_Λ and T_Γ respectively. If $T_\Lambda T_\Gamma^*=0$ and U or V is surjective satisfying $UK^*=K^*U$ or $VK^*=K^*V$, then $\{\Lambda_iU+\Gamma_iV\}_{i\in I}$ is an atomic system for K.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are two atomic systems for K.
By theorem 1.9, $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are two K-g-frames for $\mathcal{H}$, and so there exist $B_1\geq A_1>0$ and $B_2\geq A_2>0$ such that :

$$A_1\langle K^*f,K^*f\rangle \leq \sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle \leq B_1\langle f,f\rangle,$$

and

$$A_2\langle K^*f,K^*f\rangle \leq \sum_{i\in I}\langle\Gamma_i f,\Gamma_i f\rangle \leq B_1\langle f,f\rangle.$$

Since $T_\Lambda T_\Gamma^* = 0$, for any $f \in \mathcal{H}$, we have:

$$\sum_{i\in I} \Lambda_i^* \Gamma_i f = \sum_{i\in I} \Gamma_i^* \Lambda_i f = 0$$

Therefore, for any $f \in \mathcal{H}$, we have :

$$\begin{aligned}\sum_{i\in I} \|(\Lambda_i U + \Gamma_i V)f\|^2 &= \sum_{i\in I} \langle \Lambda_i U + \Gamma_i V f, \Lambda_i U + \Gamma_i V f\rangle \\ &= \sum_{i\in I} \|\Lambda_i U f\|^2 + \sum_{i\in I} \|\Gamma_i V f\|^2 + 2\sum_{i\in I} \langle \Lambda_i^* \Gamma_i f, U f\rangle \\ &= \sum_{i\in I} \|\Lambda_i U f\|^2 + \sum_{i\in I} \|\Gamma_i V f\|^2 \\ &\le B_1 \|Uf\|^2 + B_2 \|Vf\|^2 \\ &\le (B_1\|U\|^2 + B_2\|V\|^2)\|f\|^2\end{aligned}$$

Without loss of generality, assume that U is surjective; then there exists $C > 0$ such that $\langle Uf, Uf\rangle \ge C\langle f, f\rangle$ for any $f \in H$.
Since $UK^* = K^*U$, we have :

$$\begin{aligned}\sum_{i\in I} \|(\Lambda_i U + \Gamma_i V)f\|^2 &= \sum_{i\in I} \|\Lambda_i U f\|^2 + \sum_{i\in I} \|\Gamma_i V f\|^2 \\ &\ge \sum_{i\in I} \|\Lambda_i U f\|^2 \\ &\ge A_1 \|K^* U f\|^2 \\ &= A_1 \|U K^* f\|^2 \\ &\ge A_1 C \|K^* f\|^2.\end{aligned}$$

So, $\{\Lambda_i U + \Gamma_i V\}_{i\in I}$ is a K-g-frame and thus an atomic system for K by theorem 1.9. □

Let B=0, and we get the following corollary.

Corollary 1.12. *Suppose that $K \in End_A^*(\mathcal{H})$ and $\{\Lambda_i\}_{i\in I}$ is an atomic system for K. If U is surjective and $UK^* = K^*U$, then $\{\Lambda_i U\}_{i\in I}$ is an atomic system for K.*

Let $U = V = Id_{\mathcal{H}}$, then we obtain the following corollory for a K-g-frame.

Corollary 1.13. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two parseval K-g-frames for $\mathcal{H}$, with synthesis operator T_Λ and T_Γ respectively.*
If $T_\Lambda T_\Gamma^* = 0$ then $\{\Lambda_i + \Gamma_i\}_{i\in I}$ is a 2-tight K-g-frame for $\mathcal{H}$.

Theorem 1.14. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two atomic system for K and let the corresponding synthesis operators be T_1 and T_2 respectively.*
If $T_\Lambda T_\Gamma^* = 0$ and $U_i \in End_A^*(H)$ satisfies $R(T_i) \subset R(U_1^* T_i)$ for $i = 1, 2$; then $\{\Lambda_i U_1 + \Gamma_i U_2\}_{i\in I}$ is an atomic system for K.

Proof. Since $T_1 T_2^* = 0$, we have :

$$\begin{aligned}\sum_{i\in I} \langle (\Lambda_i U_1 + \Gamma_i U_2)f, (\Lambda_i U_1 + \Gamma_i U_2)f\rangle &= \sum_{i\in I} \langle \Lambda_i U_1 f, \Lambda_i U_1 f\rangle + \sum_{i\in I} \langle \Gamma_i U_2 f, \Gamma_i U_2 f\rangle \\ &= \langle T_1^* U_1 f, T_1^* U_1 f\rangle + \langle T_2^* U_2 f, T_2^* U_2 f\rangle \\ &= \langle (U_1^* T_1)^* f, (U_1^* T_1)^* f\rangle + \langle (U_2^* T_2)^* f, (U_2^* T_2)^* f\rangle\end{aligned}$$

Since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are atomic systems for K, then by theorem 1.9, $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are K-g-frames for $\mathscr{H}$.
Thus we have that $R(K)\subset R(T_i)\subset R(U_i^*T_i)$, then for each $i=1,2$, there exists $\lambda_i>0$ such that,

$$KK^*\leq \lambda_i(U_i^*T_i)(U_i^*T_i)^*$$

Then,

$$\begin{aligned}\sum_{i\in I}\langle(\Lambda_iU_1+\Gamma_iU_2)f,(\Lambda_iU_1+\Gamma_iU_2)f\rangle &= \sum_{i\in I}\langle(U_1^*T_i)^*f,(U_1^*T_i)^*f\rangle+\sum_{i\in I}\langle(U_2^*T_i)^*f,(U_2^*T_i)^*f\rangle\\ &\geq (\frac{1}{\lambda_1}+\frac{1}{\lambda_2})\sum_{i\in I}\langle K^*f,K^*f\rangle\end{aligned}$$

Hence $\{\Lambda_iU+\Gamma_iV\}_{i\in I}$ is a K-g-frame and thus an atomic system for K by theorem (1.9). □

Before the following result, we need a simple lemma.

Lemma 1.15. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathscr{H}$ with a frame operator S_Λ. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if there exists $\lambda>0$ such that $S_\Lambda\geq\lambda KK^*$.*

Proof. Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame with frame bounds, A,B and with a frame operator S_Λ if and only if

$$A\langle K^*f,K^*f\rangle\leq\sum_{i\in I}\langle\Lambda_if,\Lambda_if\rangle\leq A\langle f,f\rangle,\qquad f\in H$$

That is,

$$\langle AKK^*f,f\rangle\leq\langle S_\Lambda f,f\rangle\leq\langle Bf,f\rangle,\qquad f\in\mathscr{H}.$$

So the conclusion holds. □

Theorem 1.16. *Let $\{\Lambda_i\}_{i\in I}$ be an atomic system for K, and let S_Λ be the frame operator of $\{\Lambda_i\}_{i\in I}$. Let U be a positive operator. Then $\{\Lambda_i+\Lambda_iU\}_{i\in I}$ is an atomic system for K.*
Moreover, for any integer number n, $\{\Lambda_i+\Lambda_iU^n\}_{i\in I}$ is an atomic system for K.

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an atomic system for K, then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H.
Then there exists $\lambda>0$ such that

$$S_\Lambda\geq\lambda KK^*.$$

The frame operator for $\{\Lambda_i+\Lambda_iU\}_{i\in I}$ is $(I_{\mathscr{H}}+U)^*S_\Lambda(I_{\mathscr{H}}+U)$.
In fact, for each $f\in\mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in I}(\Lambda_i+\Lambda_iU)^*(\Lambda_i+\Lambda_iU)f &= (I_{\mathscr{H}}+U)^*\sum_{i\in I}\Lambda_i^*\Lambda_i(I_{\mathscr{H}}+U)f\\ &= (I_{\mathscr{H}}+U)^*S_\Lambda(I_{\mathscr{H}}+U)f\end{aligned}$$

So,

$$(I_H+U)^*S_\Lambda(I_H+U)\geq S_\Lambda\geq\lambda KK^*.$$

And again, we can conclude that $\{\Lambda_i+\Lambda_iU\}_{i\in I}$ is a K-g-frame and an atomic system for K by theorem 3.3 for any natural number n, the frame operator fo $\{\Lambda_i+\Lambda_iU^*\}_{i\in I}$ is $(I_{\mathscr{H}}+U^n)^*S_\Lambda(I_{\mathscr{H}}+U^n)$.
Similarly, $\{\Lambda_i+\Lambda_iU^n\}_{i\in I}$ is an atomic system for K □

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