ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2022) 2022:11

q-Analogue of n^{th} Order Opial's Integral Inequality and Its Extensions

Mohammed Muniru Iddrisu[1], Bashiru Abubakari[2], Kwara Nantomah[1]

[1]C.K. Tedam University of Technology and Applied Sciences, School of Mathematical Sciences, Department of Mathematics, P. O. Box 24, Navrongo, Ghana
[2]University for Development Studies, Faculty of Education, Department of Mathematics/ICT Education, P. O. Box TL 1350, Tamale, Ghana

Correspondence should be addressed to Mohammed Muniru Iddrisu: middrisu@cktutas.edu.gh

Email(s): middrisu@cktutas.edu.gh (M. Iddrisu), abashiru@uds.edu.gh (B. Abubakari), knantomah@cktutas.edu.gh (K. Nantomah)

Abstract
In this paper, we establish q-analogue of n^{th} Order Opial's integral inequality with some extensions. The fundamental theorem of q-calculus, the q-Cauchy's repeated integration formula, the q-Cauchy-Bunyakovsky-Schwarz's and q-Hölder's integral inequalities were employed to establish the results.

Keywords: q-analogue; opial's inequality; n^{th} order; extensions.

Introduction

The classical Opial's integral inequality was established in [16], which is an inequality involving integral of a function and its derivative defined as

$$\int_0^h |\psi(x)\psi'(x)|dx \leq \frac{h}{4}\int_0^h |\psi'(x)|^2 dx, \tag{1}$$

where $\psi \in C^1[0,h]$, such that $\psi(0) = \psi(h) = 0$, $\psi'(x) > 0$ and $x \in [0,h]$. The coefficient $h/4$ is the best constant possible.
This Opial's inequality, over time, experienced a lot of extensions and generalizations by many mathematicians in both classical and q-analogues, including [1], [5], [6], [7], [14], [17] & [19], among others. In Willet [19], an n^{th}- order generalization of the classical Opial's inequality is established as

$$\int_a^b \left|\psi(x)\psi^{(n)}(x)\right| dx \leq \frac{(b-a)^n}{2}\int_a^b \left|\psi^{(n)}(x)\right|^2 dx, \tag{2}$$

where $\psi \in C^{(n)}[a,b]$ with $\psi^{(i)}(a) = 0$ for $0 \leq i \leq n-1$, $(n \geq 1)$.
In [14], the authors established a q-analogue of a generalized Opial type inequality as

$$\int_a^b |D_q\psi(x)||\psi(x)|^p d_qx \leq (b-a)^p \int_a^b |D_q\psi(x)|^{p+1} d_qx, \tag{3}$$

where $\psi \in C^1[a,b]$ is a q-decreasing function with $\psi(bq^0) = 0$ and $p \geq 0$.
See also ([1]; [2]; [9]; [10]; [11]; [14] and [15]) for more q-analogues of the Opial's type inequalities.

Received: 2022-02-20

q-Calculus is a mathematical field of study which is analogous to the ordinary calculus. It is used to find q-derivatives and q-integrals of functions [13].

The Opial inequality plays essential role in establishing the existence and uniqueness of initial and boundary values problems for both ordinary and partial differential equations as well as in difference equations [2], [10]; [13] and [18].
Motivated by q-calculus our objective in this paper is to establish q-analogues of an n^{th}-order generalization of the Opial integral inequality (2) with some extensions.

Preliminaries

The basic concepts and terminologies of q-calculus that will be used to prove the main results are presented in this section.

Definition 1.1. *[13] For any arbitrary function ψ, the q-derivative is defined as*

$$D_q\psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x}, \quad x \neq 0. \tag{4}$$

Definition 1.2. *[13] For any positive real α, the q-Analogue of the Number α is*

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} = 1 + q + q^2 + \cdots + q^{\alpha-1}, \tag{5}$$

$0 < q < 1, \quad \alpha \in \mathbf{R}^+$.

Definition 1.3. *[13] The q-Derivative of sum or difference of ψ and ϕ is defined as*

$$D_q(\alpha(\psi(x) \pm \beta\phi(x)) = \alpha D_q\psi(x) \pm \beta D_q\phi(x), \tag{6}$$

where α and β are constants.

Definition 1.4. *[13] The q-Derivative of product of ψ and ϕ is defined as*

$$\begin{aligned} D_q(\psi(x)\phi(x)) &= \psi(qx)D_q\phi(x) + \phi(x)D_q\psi(x) \\ &= \psi(x)D_q\phi(x) + \phi(qx)D_q\psi(x). \end{aligned} \tag{7}$$

Definition 1.5. *[13] The q-Derivative of a quotient of ψ and ϕ is defined as*

$$\begin{aligned} D_q\left(\frac{\psi(x)}{\phi(x)}\right) &= \frac{\phi(x)D_q\psi(x) - \psi(x)D_q\phi(x)}{\phi(qx)\phi(x)} \\ &= \frac{\phi(qx)D_q\psi(x) - \psi(qx)D_q g(x)}{\phi(qx)\phi(x)}, \quad \phi(qx)\phi(x) \neq 0. \end{aligned} \tag{8}$$

Definition 1.6. *[12] Let $\psi : C[0,b] \to \mathbf{R}(b > 0)$. Then, the Jackson's definite q-Integral on $[0,b]$ is defined as*

$$\int_0^b \psi(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j \psi(bq^j). \tag{9}$$

The q-integral on a generic $[a,b]$ is defined as

$$\int_a^b \psi(x) d_q x = \int_0^b \psi(x) d_q x - \int_0^a \psi(x) d_q x. \tag{10}$$

Definition 1.7. *[13] Let $\psi, \phi : [a,b]$ be q-Integrable functions, the q-Integration by Parts is defined as*

$$\int_a^b \psi(x) D_q \phi(x) d_q x = \psi(x)\phi(x)|_a^b - \int_a^b \phi(qx) D_q \psi(x) d_q x. \tag{11}$$

Definition 1.8. *[18] Let $\psi : [a,b] \to \mathbf{R}$ be a continuous function. Then, we have the following:*

$(i) D_q \int_a^x \psi(t) d_q t = \psi(x)$;

$(ii) \int_a^x D_q \psi(t) d_q t = \psi(x) - \psi(a), \quad for \quad x \in [a,b]$.

See also [1], [2], [8], [10], [11] & [14].

Definition 1.9. *[13] (The Fundamental Theorem of q-Calculus) If $\psi \in C[a,b]$ and Ψ is an antiderivative of ψ and defined on $x \in [a,b]$, then*

$$\Psi(x) = \int_a^x \psi(s) d_q s \tag{12}$$

and

$$\int_a^b \psi(x) d_q x = \Psi(b) - \Psi(a). \tag{13}$$

Definition 1.10. *(q-Cauchy's Formula) Let $\psi \in C^{(n)}[a,b]$ be such that $D_q^{(i)}\psi(a) = 0$, for $i = 0, 1, 2, ..., n-1$, $(n \geq 1)$ and $0 < q < 1$, we have*

$$\begin{aligned} \omega(x) &= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} D_q^{(n)} \psi(s) d_q s d_q x_1 \ldots d_q x_{n-1} \\ &= \frac{1}{[n-1]!} \int_a^x (x - qs)^{n-1} \psi(s) d_q s. \end{aligned} \tag{14}$$

See [3].

Definition 1.11. *[4] Let ψ and ϕ be continuous functions on $[a,b]$, $\alpha, \beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the q-Hölder's Integral Inequality is stated as*

$$\int_a^b |\psi(x)\phi(x)| d_q x \leq \left(\int_a^b |\psi(x)|^\alpha d_q x \right)^{\frac{1}{\alpha}} \left(\int_a^b |\phi(x)|^\beta d_q x \right)^{\frac{1}{\beta}}, \tag{15}$$

with equality when

$$|\phi(x)| = c|\psi(x)|^{p-1},$$

where c is a constant.

If $\alpha = \beta = 2$, the inequality becomes q-Cauchy-Bunyakovsky-Schwarz's Integral Inequality.

Results and Discussion

Lemma 1.1. *Let $\psi \in C^{(n)}[a,b]$ be such that $D_q \psi^{(i)} \in L_2[a,b]$, $0 \leq i \leq n$ and $0 < q < 1$, then*

$$\left(\int_a^b |D_q^{(n)} \psi(x)| d_q x \right)^2 \leq (b-a) \int_a^b |D_q^{(n)} \psi(x)|^2 d_q x. \tag{16}$$

Proof. Applying q-Cauchy-Bunyakovsky-Schwarz's inequality we have

$$\begin{aligned} \left(\int_a^b |D_q^{(n)} \psi(x)| d_q x \right)^2 &\leq \left[\left(\int_a^b 1^2 d_q x \right)^{\frac{1}{2}} \left(\int_a^b |D_q^{(n)} \psi(x)|^2 d_q x \right)^{\frac{1}{2}} \right]^2 \\ &= (b-a) \int_a^b |D_q^{(n)} \psi(x)|^2 d_q x. \end{aligned}$$

□

Theorem 1.1. *Let $\psi \in C^{(n)}[a,b]$ be such that $D_q^{(i)}\psi(a) = 0$, for $0 \le i \le n-1$, $(n \ge 1)$ and $0 < q < 1$, then*

$$\int_a^b |\psi(x)D_q^{(n)}\psi(x)|d_qx \le \frac{(b-a)^n}{2}\int_a^b |D_q^{(n)}\psi(x)|^2 d_qx. \tag{17}$$

Proof. Let $x \in [a,b]$, $D_q^{(n)}\psi(a) = 0$ and

$$\omega(x) = \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} |D_q^{(n)}\psi(s)|d_qsd_qx_1\dots d_qx_{n-1}, \tag{18}$$

so that
$D_q^{(n)}\omega(x) = |D_q^{(n)}\psi(x)|$, $\omega(x) \ge |\psi(x)|$ and $D_q^{(i)}\omega(x) \ge 0$, then we have

$$\int_a^b |\psi(x)D_q^{(n)}\psi(x)|d_qx \le \int_a^b \omega(x)D_q^{(n)}\omega(x)d_qx. \tag{19}$$

Since

$$D_q^{(i)}\omega(x) \le (x-a)D_q^{(i+1)}\omega(x), \quad x \in [a,b], \quad 0 \le i \le n-2,$$

it follows that

$$|\psi(x)| \le \omega(x) \le (x-a)D_q\omega(x) \le \dots \le (x-a)^{(n-1)}D_q^{(n-1)}\omega(x). \tag{20}$$

Applying (20) to (19) we have

$$\begin{aligned}
\int_a^b |\psi(x)D_q^{(n)}\psi(x)|d_qx &\le \int_a^b (x-a)D_q\omega(x)D_q^{(n)}\omega(x)d_qx \\
&\le \int_a^b (x-a)^2D_q^2\omega(x)D_q^{(n)}\omega(x)d_qx \\
&\vdots \\
&\le \int_a^b (x-a)^{(n-1)}D_q^{(n-1)}\omega(x)D_q^{(n)}\omega(x)d_qx \\
&= (b-a)^{(n-1)}\int_a^b D_q^{(n-1)}\omega(x)D_q^{(n)}\omega(x)d_qx \\
&= \frac{(b-a)^{(n-1)}\left[D_q^{(n)}\omega(b)\right]^2}{2} \\
&= \frac{(b-a)^{n-1}}{2}\left(\int_a^b |D_q^{(n)}\psi(x)|d_qx\right)^2.
\end{aligned} \tag{21}$$

Hence, by Lemma 1.1, we obtain

$$\int_a^b |\psi(x)D_q^{(n)}\psi(x)|d_qx \le \frac{(b-a)^{n-1}(b-a)}{2}\int_a^b |D_q^{(n)}\psi(x)|^2 d_qx, \tag{22}$$

which yields

$$\int_a^b |\psi(x)D_q^{(n)}\psi(x)|d_qx \le \frac{(b-a)^n}{2}\int_a^b |D_q^{(n)}\psi(x)|^2 d_qx.$$

□

Remark 1. *For $n = 1$, inequality (17) reduces to*

$$\int_a^b |\psi(x) D_q \psi(x)| d_q x \le \frac{(b-a)}{2} \int_a^b |D_q \psi(x)|^2 d_q x, \tag{23}$$

which is a q-analogue of the Opial's inequality established in [17].

Lemma 1.2. *Let $\psi \in C^{(n-1)}[a,b]$ be absolutely continuous such that $D_q^{(i)}\psi(a) = 0$, for $0 \le i \le n-1$, $(n \ge 1)$, $x \in [a,b]$ and $0 < q < 1$, then*

$$\int_a^x (x-qt)^{n-1} |D_q^{(n)} \psi(x)| d_q t = \frac{(x-qa)^{n-\frac{1}{2}}}{[2n-1]_q^{\frac{1}{2}}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}}. \tag{24}$$

Proof. By q-Cauchy-Bunyakovsky-Schwartz's Inequality we have

$$\begin{aligned}
\int_a^x (x-qs)^{n-1} |D_q^{(n)} \psi(s)| d_q s &= \left(\int_a^x (x-qs)^{2(n-1)} d_q t \right)^{\frac{1}{2}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}} \\
&= \left((1-q)(x-qa) \sum_{j=0}^{\infty} q^j ((x-qa)q^j)^{2(n-1)} \right)^{\frac{1}{2}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}} \\
&= \left((1-q) \sum_{j=0}^{\infty} q^{(2n-1)j} (x-qa)^{2n-1} \right)^{\frac{1}{2}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}} \\
&= \left(\frac{(1-q)}{1-q^{2n-1}} (x-qa)^{2n-1} \right)^{\frac{1}{2}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}} \\
&= \frac{(x-qa)^{n-\frac{1}{2}}}{[2n-1]_q^{\frac{1}{2}}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{\frac{1}{2}}.
\end{aligned}$$

□

Theorem 1.2. *Let $\psi \in C^{(n-1)}[a,b]$ be such that $D_q \psi^{(i)}(a) = 0$, for $0 \le i \le n-1$, $(n \ge 1)$. Also, let $D_q^{(n-1)}\psi(x)$ be absolutely continuous and $\int_a^b |D_q^n \psi(x)|^2 d_q x < \infty$, then*

$$\int_a^b |\psi(x) D_q^n \psi(x)| d_q x \le K(b-a)^n \int_a^b |D_q^n \psi(x)|^2 d_q x, \tag{25}$$

where $K = \frac{n}{n! \sqrt{2[2n]_q [2n-1]_q}}$.

Proof. Let $x \in [a,b]$. By applying the Cauchy's formula (14) we have

$$\omega(x) = \frac{1}{(n-1)!} \int_a^x (x-qs)^{n-1} D_q^n \psi(s) d_q s. \tag{26}$$

For $|\psi(x)| \le \omega(x)$, we have

$$|\psi(x) D_q^n \psi(x)| \le \frac{|D_q^n \psi(x)|}{(n-1)!} \int_a^x (x-qs)^{n-1} |D_q^n \psi(s)| d_q s, \tag{27}$$

Applying Lemma 1.2 yields

$$|\psi(x) D_q^n \psi(x)| \le \frac{|D_q^n \psi(x)|}{(n-1)!} \frac{(x-qa)^{n-1/2}}{[2n-1]_q^{1/2}} \left(\int_a^x |D_q^n \psi(s)|^2 d_q s \right)^{1/2}. \tag{28}$$

Integrating (28), we have

$$\int_a^b |\psi(x)D_q^n\psi(x)|d_qx \leq \frac{1}{(n-1)![2n-1]_q^{1/2}}\int_a^b (x-qa)^{n-1/2}|D_q^n\psi(x)|\left(\int_a^x |D_q^n\psi(s)|^2 d_qs\right)^{1/2} d_qx. \tag{29}$$

Applying q-Cauchy-Bunyakovsky-Schwartz's inequality to the right-hand of (29) yields

$$\begin{aligned}
&\int_a^b |\psi(x)D_q^n\psi(x)|d_qx \\
&\leq \frac{1}{(n-1)![2n-1]_q^{1/2}}\left(\int_a^b (x-qa)^{2n-1}d_qx\right)^{\frac{1}{2}}\left(\int_a^b |D_q^n\psi(x)|^2\left(\int_a^x |D_q^n\psi(s)|^2 d_qs\right)d_qx\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)(b-qa)\sum_{j=0}^{\infty} q^j((b-a)q^j)^{(2n-1)}\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{2}\int_a^b \frac{d}{d_qx}\left(\int_a^x |D_q^n\psi(s)|^2 d_qs\right)^2 d_qx\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)\sum_{j=0}^{\infty} q^{2nj}(b-qa)^{(2n)}\right)^{\frac{1}{2}}\sqrt{\frac{1}{2}}\int_a^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\sqrt{\frac{1}{2}}\left(\frac{(1-q)}{1-q^{2n}}(b-qa)^{(2n)}\right)^{\frac{1}{2}}\int_a^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\sqrt{\frac{1}{2}}\left(\frac{(b-qa)^{2n}}{[2n]_q}\right)^{\frac{1}{2}}\int_a^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{(b-a)^n}{(n-1)![2n-1]_q^{1/2}[2n]_q^{1/2}}\frac{1}{2^{1/2}}\int_a^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{n}{n!\sqrt{2[2n]_q[2n-1]_q}}(b-qa)^n\int_a^b |D_q^n\psi(x)|^2 d_qx.
\end{aligned} \tag{30}$$

Thus

$$\int_a^b |\psi(x)D_q^n\psi(x)|d_qx \leq K(b-qa)^n\int_a^b |D_q^n\psi(x)|^2 d_qx.$$

□

Remark 2. *For $n = 1$, inequality (25) reduces to*

$$\int_a^b |\psi(x)D_q\psi(x)|d_qx \leq \frac{(b-qa)}{\sqrt{2(1+q)}}\int_a^b |D_q\psi(x)|^2 d_qx, \tag{31}$$

which is a q-analogue of the Opial's inequality established in [17].

Remark 3. *Inequality (25) is sharper than (17) for $n \geq 2$ as $q \to 1$.*

Theorem 1.3. *If $\psi \in C^{(n)}[0,b]$ for $b > 0$, $D_q\psi, \ldots, D_q^{(n-1)}\psi$ being piecewise continuous and $D_q^{(n)}\psi$ is absolutely continuous with $\int_a^b |D_q^n\psi(x)|^2 d_qx < \infty$, $D_q^{(i)}\psi(0) = D_q^{(i)}\psi(b) = 0$ for $0 \leq i \leq n-1$, then*

$$\int_0^b |\psi(x)D_q^n\psi(x)|d_qx \leq K\left(\frac{qb}{2}\right)^n\int_0^b |D_q^n\psi(z)|^2 d_qx, \tag{32}$$

where $K = \frac{n}{n!\sqrt{2[2n]_q[2n-1]_q}}$.

Proof. Let $x \in [0, b]$. By applying the Cauchy's formula (14) we have

$$|\psi(x)D_q^n\psi(x)| = \frac{|D_q^n\psi(x)|}{(n-1)!}\int_0^x (x-qs)^{n-1}|D_q^n\psi(s)|d_qs. \tag{33}$$

Applying Lemma 1.2 we obtain

$$|\psi(x)D_q^n\psi(x)| \leq \frac{|D_q^n\psi(x)|}{(n-1)!}\frac{(x-q(0))^{n-1/2}}{[2n-1]_q^{1/2}}\left(\int_0^x |D_q^n\psi(s)|^2d_qs\right)^{1/2}. \tag{34}$$

Let $\left[0, \frac{qb}{2}\right]$ and $\left[\frac{qb}{2}, b\right]$ be subintervals of $[0, b]$.
From (34) we have

$$\int_0^{\frac{qb}{2}} |\psi(x)D_q^n\psi(x)|d_qx \leq$$
$$\frac{1}{(n-1)![2n-1]_q^{1/2}}\int_0^{\frac{qb}{2}} (x)^{n-1/2}|D_q^n\psi(x)|\left(\int_0^x |D_q^n\psi(s)|^2d_qs\right)^{1/2} d_qx. \tag{35}$$

Applying q-analogue of Cauchy-Bunyakovsky-Schwarz's inequality to the right-side of (35) we obtain

$$\int_0^{\frac{qb}{2}} |\psi(x)D_q^n\psi(x)|d_qx \leq$$
$$\frac{1}{(n-1)![2n-1]_q^{1/2}}\left(\int_0^{\frac{qb}{2}} (x)^{2n-1}d_qx\right)^{\frac{1}{2}}\left(\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2\left(\int_0^x |D_q^n\psi(s)|^2d_qs\right)d_qx\right)^{\frac{1}{2}}$$
$$= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)\left(\frac{qb}{2}\right)\sum_{j=0}^{\infty} q^j\left(\left(\frac{qb}{2}\right)q^j\right)^{(2n-1)}\right)^{\frac{1}{2}}$$
$$\left(\frac{1}{2}\int_0^{\frac{qb}{2}} D_q\int_0^x |D_q^n\psi(s)|^2d_qs\left(\int_0^x |D_q^n\psi(s)|^2d_qs\right)d_qx\right)^{\frac{1}{2}}$$
$$= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)\sum_{j=0}^{\infty} q^{2nj}\left(\frac{qb}{2}\right)^{(2n)}\right)^{\frac{1}{2}}$$
$$\left(\frac{1}{2}\int_0^{\frac{qb}{2}} D_q\left(\int_0^x |D_q^n\psi(s)|^2d_qs\right)^2 d_qx\right)^{\frac{1}{2}}$$
$$= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left(\frac{(1-q)}{1-q^{2n}}\left(\frac{(qb)}{2}\right)^{(2n)}\right)^{\frac{1}{2}}\sqrt{\frac{1}{2}}\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2d_qx$$
$$= \frac{1}{(n-1)![2n-1]_q^{1/2}}\sqrt{\frac{1}{2}}\left(\frac{\left(\frac{qb}{2}\right)^{2n}}{[2n]_q}\right)^{\frac{1}{2}}\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2d_qx$$
$$= \frac{\left(\frac{qb}{2}\right)^n}{(n-1)![2n-1]_q^{1/2}[2n]_q^{1/2}}\frac{1}{2^{1/2}}\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2d_qx$$
$$= \frac{n\left(\frac{qb}{2}\right)^n}{n!\sqrt{2[2n]_q[2n-1]_q}}\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2d_qx. \tag{36}$$

Now from (14) we have

$$|\omega(x)D_q^n\psi(x)| = \frac{-|D_q^n\psi(x)|}{(n-1)!}\int_x^b (qs-x)^{n-1}|D_q^n\psi(s)|d_qs. \tag{37}$$

Applying Lemma 1.2 and $|\psi(x)| \leq \omega(x)$, (37) becomes

$$|\psi(x)D_q^n\psi(x)| \leq \frac{-|D_q^n\psi(x)|}{(n-1)!}\frac{(qb-x)^{n-1/2}}{[2n-1]_q^{1/2}}\left(\int_x^b |D_q^n\psi(s)|^2 d_qs\right)^{1/2}. \tag{38}$$

Thus

$$\begin{aligned}
&\int_{\frac{qb}{2}}^b |\psi(x)D_q^n\psi(x)|d_qx \\
&\quad\leq \frac{-1}{(n-1)![2n-1]_q^{1/2}}\int_{\frac{qb}{2}}^b (qb-x)^{n-1/2}|D_q^n\psi(x)|\left(\int_x^b |D_q^n\psi(s)|^2 d_qs\right)^{1/2} d_qx.
\end{aligned} \tag{39}$$

By the q-analogue of Cauchy-Bunyakovsky-Schwarz's inequality, we obtain

$$\begin{aligned}
&\int_{\frac{qb}{2}}^b |\psi(x)D_q^n\psi(x)|d_qx \\
&\leq \frac{-1}{(n-1)![2n-1]_q^{1/2}}\left(\int_{\frac{qb}{2}}^b (qb-x)^{2n-1}d_qx\right)^{\frac{1}{2}}\left(\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2\left(\int_x^b |D_q^n\psi(s)|^2 d_qs\right)d_qx\right)^{\frac{1}{2}} \\
&= \frac{-1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)\left(qb-\frac{qb}{2}\right)\sum_{j=0}^{\infty} q^j\left(\left(qb-\frac{qb}{2}\right)q^j\right)^{(2n-1)}\right)^{\frac{1}{2}} \\
&\times\left(\int_{\frac{qb}{2}}^b D_q\int_x^b |D_q^n\psi(s)|^2 d_qs\left(\int_x^b |D_q^n\psi(s)|^2 d_qs\right)d_qx\right)^{\frac{1}{2}} \\
&= \frac{-1}{(n-1)![2n-1]_q^{1/2}}\left((1-q)\sum_{j=0}^{\infty} q^{2nj}\left(qb-\frac{qb}{2}\right)^{(2n)}\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{2}\int_{\frac{qb}{2}}^b D_q\left(\int_x^b |D_q^n\psi(s)|^2 d_qs\right)^2 d_qx\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\left(\frac{(1-q)}{1-q^{2n}}\left(\frac{qb}{2}\right)^{(2n)}\right)^{\frac{1}{2}}\sqrt{\frac{1}{2}}\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{1}{(n-1)![2n-1]_q^{1/2}}\sqrt{\frac{1}{2}}\left(\frac{\left(\frac{qb}{2}\right)^{2n}}{[2n]_q}\right)^{\frac{1}{2}}\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{\left(\frac{qb}{2}\right)^n}{(n-1)![2n-1]_q^{1/2}[2n]_q^{1/2}}\frac{1}{2^{1/2}}\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2 d_qx \\
&= \frac{n\left(\frac{qb}{2}\right)^n}{n!\sqrt{2[2n]_q[2n-1]_q}}\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2 d_qx.
\end{aligned} \tag{40}$$

Adding (36) and (40) we have

$$\int_0^b |\psi(x)D_q^n\psi(x)|d_qx \leq \frac{n\left(\frac{qb}{2}\right)^n}{n!\sqrt{2[2n]_q[2n-1]_q}}\int_0^{\frac{qb}{2}} |D_q^n\psi(x)|^2 d_qx + \frac{n\left(\frac{qb}{2}\right)^n}{n!\sqrt{2[2n]_q[2n-1]_q}}\int_{\frac{qb}{2}}^b |D_q^n\psi(x)|^2 d_qx. \tag{41}$$

This yields

$$\int_0^b |\psi(x)D_q^n\psi(x)|d_qx \leq \frac{n}{n!\sqrt{2[2n]_q[2n-1]_q}}\left(\frac{qb}{2}\right)^n \int_0^b |D_q^n\psi(x)|^2 d_qx. \tag{42}$$

Hence

$$\int_0^b |\psi(x)D_q^n\psi(x)|d_qx \leq K\left(\frac{qb}{2}\right)^n \int_0^b |D_q^n\psi(x)|^2 d_qx.$$

This completes the proof. □

Remark 4. *For $n = 1$, inequality (32) reduces to*

$$\int_0^b |\psi(x)D_q\psi(x)|d_qx \leq \frac{(qb)}{2\sqrt{2(1+q)}}\int_0^b |D_q\psi(x)|^2 d_qx, \tag{43}$$

which gives the q-analogue of the Opial's inequality established in [16].

Remark 5. *The inequality (32) is also sharper than (25) and (17) for $n \geq 2$ as $q \to 1$.*

Theorem 1.4. *Let $r, s > 0$ satisfying $\beta = s + r > 1$. Also, let $\psi \in C^{(n)}[a, b]$ such that $D_q^{(i)}\psi(a) = 0$, $0 \leq i \leq n-1$ $(n \geq 1)$, $D_q^{(n-1)}\psi$ is absolutely continuous with $\int_a^b |D_q^n\psi(x)|^\beta d_qx < \infty$, then*

$$\int_a^b |\psi(x)|^s |D_q^n\psi(x)|^r d_qx \leq M(b-a)^{ns}\int_a^b |D_q^n\psi(x)|^\beta d_qx, \tag{44}$$

where

$$M = \phi r^{r\phi}\left(\frac{n[\beta n]_q^\phi}{\left[\frac{\beta n-1}{\beta-1}\right]_q^{1-\phi}}\right)^s (n!)^{-s}, \quad \phi = \beta^{-1}. \tag{45}$$

Proof. Let $x \in [a, b]$. From (14) we have

$$\omega(x) = \frac{1}{(n-1)!}\int_a^x (x-qs)^{n-1}D_q^n\psi(s)d_qs. \tag{46}$$

Applying Hölder's inequality with $\alpha = \frac{s+r}{s+r-1}$ and $\beta = s + r$ to (46) we obtain

$$
\begin{aligned}
\omega(x) &= \frac{1}{(n-1)!}\left(\int_a^x (x-qs)^{\alpha(n-1)} d_qs\right)^{\frac{1}{\alpha}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}} \\
&= \frac{1}{(n-1)!}\left((1-q)(x-qa)\sum_{j=0}^{\infty} q^j\left((x-qa)q^j\right)^{\alpha(n-1)}\right)^{\frac{1}{\alpha}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}} \\
&= \frac{1}{(n-1)!}\left((1-q)\sum_{j=0}^{\infty} q^{(\alpha(n-1)+1)j}(x-qa)^{\alpha(n-1)+1}\right)^{\frac{1}{\alpha}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}} \\
&= \frac{1}{(n-1)!}\left(\frac{(1-q)}{1-q^{(\alpha(n-1)+1)}}(x-qa)^{(\alpha(n-1)+1)}\right)^{\frac{1}{\alpha}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}} \\
&= \frac{1}{(n-1)!}\left(\frac{1-q}{1-q^{(\alpha(n-1)+1)}}\right)^{\frac{1}{\alpha}}(x-qa)^{(n-1)+\alpha^{-1}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}} \\
&= \frac{n}{n!\,[\alpha(n-1)+1]_q^{\frac{1}{\alpha}}}(x-qa)^{(n-1)+\alpha^{-1}}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{1}{\beta}}.
\end{aligned} \tag{47}
$$

Letting $A = \frac{n}{n![\alpha(n-1)+1]_q^{\frac{1}{\alpha}}}$ and for $|\psi(x)| \leq \omega(x)$ we have

$$
\begin{aligned}
&\int_a^b |\psi(x)|^s|D_q^n\psi(x)|^r d_qx \\
&\qquad \leq A^s\int_a^b (x-a)^{s((n-1)+\alpha^{-1})}|D_q^n\psi(x)|^r\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{s}{\beta}} d_qx.
\end{aligned} \tag{48}
$$

Applying q-Hölder's inequality with indices $\frac{\beta}{s}$ and $\frac{\beta}{r}$ to the right-side of (48) we obtain

$$
\begin{aligned}
&\int_a^b |\psi(x)|^s|D_q^n\psi(x)|^r d_qx \leq \\
&A^s\left(\int_a^b (x-qa)^{\beta((n-1)+\alpha^{-1})} d_qx\right)^{\frac{s}{\beta}}\left(\int_a^b |D_q^n\psi(x)|^\beta\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{s/r} d_qx\right)^{\frac{r}{\beta}} \\
&= A^s\left((1-q)(b-a)\sum_{j=0}^{\infty} q^j\left((b-a)q^j\right)^{\beta((n-1)+\alpha^{-1})}\right)^{\frac{s}{\beta}} \\
&\times\left(\frac{r}{\beta}\int_a^b \frac{d_q}{d_qx}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{s}{r}+1} d_qx\right)^{\frac{r}{\beta}} \\
&= A^s\left((1-q)\sum_{j=0}^{\infty} q^{(\beta((n-1)+\alpha^{-1})+1)j}(b-a)^{(\beta((n-1)+\alpha^{-1})+1)}\right)^{\frac{s}{\beta}} \\
&\times\left(\frac{r}{\beta}\int_a^b \frac{d_q}{d_qx}\left(\int_a^x |D_q^n\psi(s)|^\beta d_qs\right)^{\frac{\beta}{r}} d_qx\right)^{\frac{r}{\beta}}
\end{aligned}
$$

$$= A^s \left(\frac{(1-q)}{1-q^{\beta n}} (b-a)^{\beta n} \right)^{\frac{s}{\beta}} \phi r^{r\phi} \int_a^b |D_q^n \psi(x)|^\beta d_q x \tag{49}$$

$$= A^s \left(\frac{(1-q)}{1-q^{\beta n}} \right)^{\frac{s}{\beta}} (b-a)^{ns} \phi r^{r\phi} \int_a^b |D_q^n \psi(x)|^\beta d_q x$$

$$= \phi r^{r\phi} A^s [\beta n]_q^{s\phi} (b-a)^{ns} \int_a^b |D_q^n f(x)|^\beta d_q x$$

$$= \phi r^{r\phi} \left(\frac{n\,[\beta n]_q^{\phi}}{\left[\frac{\beta n-1}{\beta-1}\right]_q^{1-\phi}} \right)^s (n!)^{-s} (b-a)^{ns} \int_a^b |D_q^n \psi(x)|^\beta d_q x.$$

Thus

$$\int_a^b |\psi(x)|^s |D_q^n \psi(x)|^r d_q x \leq M(b-a)^{ns} \int_a^b |D_q^n \psi(x)|^\beta d_q x. \tag{50}$$

□

Remark 6. *For $n = 1$, and $s = r = 1$ (50) reduces to*

$$\int_a^b |\psi(x) D_q \psi(x)| d_q x \leq \sqrt{\frac{1+q}{2}} (b-a) \int_a^b |D_q \psi(x)|^2 d_q x. \tag{51}$$

Conclusion

This work presented Opial's integral inequality and its n^{th} order as well as q-Calculus. The q-analogue of the n^{th} Order of the Opial's integral inequality was established and further deductions were made. The fundamental theorem of q-calculus, q-Cauchy's repeated integration formula, q-Hölder's integral inequality together with some other mathematical techniques were employed to establish the results.

Acknowledgments

The authors would like to thank the anonymous referees for their comments that helped to improve this article.

References

[1] B. Abubakari, M. M. Iddrisu, K. Nantomah, On generalized Opial's integral inequalities in q-calculus, J. Adv. Math. Comput. Sci. 35 (2020), 106–114.

[2] N. Alp, C. C. Bilisik, M. Z. Sarikaya, On q-Opial type inequality for quantum integral, Filomat. 33 (2018) 4175–4184.

[3] W. A. Al-Salam, q-Analogues of Cauchy's formulas, Proc. Amer. Math. Soc. 17 (1966), 616 – 621.

[4] G. A. Anastassiou, Intelligent mathematics: computational analysis, Heidelberg: Springer. Vol. 51 (2011).

[5] P. R. Beesack, On an integral inequality of Z. Opial, Trans. Amer. Math. Soc. 104 (1962), 470–475.

[6] J. Calvert, Some generalizations of Opial's inequality, Proc. Amer. Math. Soc. 18 (1967), 72–75.

[7] K. M. Das, An inequality similar to Opial's inequality, Proc. Amer. Math. Soc. 22 (1969), 258–261.

[8] T. Ernest, The history of q-calculus and a new method, Department of Mathematics, Uppsala University, Box 480, SE - 751, 06 Uppsala Sweden, (2001).

[9] H. Gauchman, Integral inequalities in q-calculus, J. Comput. Math. Appl. 47 (2004), 281-300.

[10] E. Gov, O. Tasbozan, Some quantum estimates of Opial inequality and some of its generalizations, New Trends Math. Sci. 6 (2018), 76–84.

[11] M. M. Iddrisu, q-Steffensen's Inequality for Convex Functions, Intl. J. Math. Appl. 6 (2018), 157–162.

[12] F. H. Jackson, On a definite q-integrals, Quart. J. Pure Appl. Math. 41 (1910), 193–202.

[13] V. Kac, P. Cheung, Quantum calculus, Universitext, Springer-Verlag, New York, (2002).

[14] T. Z. Mirkovic, S. B. Trickovic, M. S. Stankovic, Opial inequality in q-calculus, J. Inequal. Appl. 2018 (2018), 347, 8 pages.

[15] K. Nantomah, Generalized Holder's and Minkowski's inequalities for jackson's q-integral and some applications to the incomplete q-gamma function, Abstr. Appl. Anal. 2017 (2017), Article ID 9796873, 6 pages.

[16] Z. Opial, Sur une inegaliti, J. Ann. Polon. Math. 8 (1960), 29–32.

[17] M. Z. Sarikaya, On the generalization of opial type inequality for convex function, Konuralp J. Math. 7 (2018), 456–461.

[18] J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, Adv. Differ. Equ. 2013 (2013), 282, 19 pages.

[19] D. Willet, The existence-uniqueness theorem for an nth order linear ordinary differential equation. Amer. Math. Monthly, 75 (1968), 174-178.