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Some Results on Composition of n Entire Functions of Finite $[p, q]$ Order

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Abstract
The purpose of this paper is to study some growth properties of composition of n entire functions of finite $[p, q]$ order.

Keywords: order; iterated i-order; entire function; composition.

Introduction and Definitions

Let f and g be two transcendental entire functions. It is well known by a result of Clunie [1] that $\lim_{r\to\infty} \frac{T_{f\circ g}(r)}{T_f(r)} = \infty$ and $\lim_{r\to\infty} \frac{T_{f\circ g}(r)}{T_g(r)} = \infty$. Several authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] have studied on various properties of composition of entire functions with finite order and obtained many interesting results. In 2009, Tu et.al [12] investigated the growth of two composite entire functions of finite iterated order and proved several results connecting $T_{f\circ g}(r), T_f(r), T_g(r)$.
After this using the idea of generalised iteration of two entire functions Banerjee and Mandal [13, 14] extended the results of Jin tu et.al [12] for generalised iterated entire functions with finite iterated order. In 2019, Banerjee and Adhikary investigated the growth of composition of three entire functions as a generalisation of the results of Xu et. al [15]. In this paper, we investigate the growth of composition of n entire functions to generalise the results of Banerjee and Adhikary [16]. Here we use some basic definitions and standard notations of Nevanlinna theory [17]. First we recall some basic definitions [12, 15, 18].

Definition 1.1. Let $f(z)$ be an entire function. Then the order and lower order of $f(z)$ are respectively defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Definition 1.2. The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f) = \limsup_{r\to\infty} \frac{\log^{[i+1]} M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[i]} T_f(r)}{\log r},\ (i \in \mathbb{N}).$$

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Similarly, the iterated i lower order $\lambda_i(f)$ of an entire function f is defined by

$$\lambda_i(f) = \liminf_{r\to\infty} \frac{\log^{[i+1]} M_f(r)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[i]} T_f(r)}{\log r}, \ \ (i \in \mathbb{N}),$$

where

$$\log^{[1]}(r) = \log r, \ \ \log^{[i+1]}(r) = \log(\log^{[i]}(r)) \ \ (i \in \mathbb{N}), \ \ \text{for all sufficiently large } r.$$

Remark 1.1. We define $\exp^{[1]}(r) = e^r$, $exp^{[i+1]}(r) = \exp(\exp^{[i]} r)$ $(i \in \mathbb{N})$, $\exp^{[0]} r = r = \log^{[0]} r$ and $\exp^{[-1]} r = \log r$, for all $r \in (0, \infty)$.
Remark 1.2. Clearly $\rho_1(f) = \rho(f)$ and $\lambda_1(f) = \lambda(f)$.

Definition 1.3. If $f(z)$ is a transcendental entire function, the $[p, q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} T_f(r)}{\log^{[q]} r}.$$

Similarly, the lower $[p, q]$-order of $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r} = \liminf_{r\to\infty} \frac{\log^{[p]} T_f(r)}{\log^{[q]} r},$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Definition 1.4. A transcendental entire function $f(z)$ is said to have index-pair $[p, q]$, if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.

For an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the maximum term $\mu_f(r)$ of f on $|z| = r$ is defined by

$$\mu_f(r) = max_{n\geq 0}\{|a_n| r^n\}.$$

Since for $0 < r < R$ we have

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R),$$

an equivalent form for $\rho_{[p,q]}(f)$ and $\lambda_{[p,q]}(f)$ are

$$\rho_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} \mu_f(r)}{\log^{[q]} r}$$

and

$$\lambda_{[p,q]}(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} \mu_f(r)}{\log^{[q]} r},$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Let $f_1, f_2, \cdots, f_n$ be n entire functions having index-pair $[p_1, q_1]$, $[p_2, q_2]$, $\cdots$, $[p_n, q_n]$ respectively.
Set $A_1 = \rho_{[p_1,q_1]}(f_1), B_1 = \lambda_{[p_1,q_1]}(f_1), A_2 = \rho_{[p_2,q_2]}(f_2), B_2 = \lambda_{[p_2,q_2]}(f_2), \cdots, A_n = \rho_{[p_n,q_n]}(f_n), B_n = \lambda_{[p_n,q_n]}(f_n)$.

Lemmas

In this section we present two lemmas which will be needed in the sequel.

Lemma 1.1. *[19] Let f and g be two entire functions with $g(0) = 0$. Then for sufficiently large r,*

$$2\mu_f(4\mu_g(2r)) \geq \mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f(\frac{1}{8}\mu_g(\frac{1}{4}r)).$$

Lemma 1.2. *[9] Let f and g be two entire functions. Then for all large values of r*

$$T_{f\circ g}(r) \geq \frac{1}{3}\log M_f(\frac{1}{9}M_g(\frac{r}{4})).$$

Preliminary Theorems

The following preliminary theorems will be needed to prove our main results.

Theorem 1.3. *[20] Let $f_1, f_2, \cdots, f_n$ be n entire functions. Then for all large values of r*

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}\cdots\frac{1}{18}M_{f_n}(\frac{r}{2})\cdots))$$

and

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq M_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

Theorem 1.4. *[20] Let $f_1, f_2, \cdots, f_n$ be n entire functions such that $M_{f_i}(r) > \frac{2+\epsilon}{\epsilon}|f_i(0)|$ for $i = 2, 3, \ldots n$ and for any $\epsilon > 0$. Then for all large values of r*

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq (1+\epsilon)^{(n-1)}T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

Theorem 1.5. *[20] Let $f_1, f_2, \cdots, f_n$ be n entire functions. Then for all large values of r*

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\cdots\frac{1}{18}M_{f_n}(\frac{r}{8})\cdots)).$$

Theorem 1.6. *Let $f_1, f_2, \cdots, f_n$ be n entire functions. Then for all large values of r*

$$\mu_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq \mu_{f_1}(\frac{1}{9}\mu_{f_2}(\frac{1}{18}\mu_{f_3}\cdots\frac{1}{18}\mu_{f_n}(\frac{r}{2})\cdots))$$

and

$$\mu_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq \mu_{f_1}(\mu_{f_2}(\cdots\mu_{f_n}(r)\cdots)).$$

Proof. For two entire functions f_1 and f_2 we get from Lemma 1.1

$$\mu_{f_1 \circ f_2}(r) \geq \frac{1}{2}\mu_{f_1}(\frac{1}{8}\mu_{f_2}(\frac{r}{4})).$$

Now for three entire functions f_1, f_2 and f_3 as in the above we get

$$\mu_{f_1 \circ f_2 \circ f_3}(r) \geq \frac{1}{2}\mu_{f_1 \circ f_2}(\frac{1}{8}\mu_{f_3}(\frac{r}{4})) \geq \frac{1}{2}\mu_{f_1}(\frac{1}{8}\mu_{f_2}(\frac{1}{16}\mu_{f_3}(\frac{r}{4})).$$

Similarly for n-functions, we get

$$\mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \geq \frac{1}{2}\mu_{f_1}(\frac{1}{8}\mu_{f_2}(\frac{1}{16}\mu_{f_3}(\cdots \frac{1}{16}\mu_{f_n}(\frac{r}{4})\cdots))).$$

And using Lemma 1.1 successively, we get

$$\mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) \leq 2\mu_{f_1}(8\mu_{f_2}(\cdots 8\mu_{f_{n-1}}(4\mu_{f_n}(2^{n-1}r))\cdots)).$$

Main Theorems

In this section we present the main results of the paper.

Theorem 1.7. *Let $f_1, f_2, \cdots, f_n$ be n entire functions having index pair $[p_1, q_1], [p_2, q_2], \cdots, [p_n, q_n]$ respectively and $0 < B_1 \leq A_1 < \infty$, $\sum_{i=2}^{n-1} B_i > 0$. Now*
(i) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$ then $\rho_{[\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) = A_n$;
(ii) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$ then
$B_1 A_n \leq \rho_{[\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_1 A_n$;
(iii) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ then $B_1 \leq \rho_{[p_1, \sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_1$.

Proof. Clearly we have

$$A_1 = \limsup_{r\to\infty} \frac{\log^{[p_1]} T_{f_1}(r)}{\log^{[q_1]} r}, A_2 = \limsup_{r\to\infty} \frac{\log^{[p_2+1]} M_{f_2}(r)}{\log^{[q_2]} r}, \cdots, A_n = \limsup_{r\to\infty} \frac{\log^{[p_n+1]} M_{f_n}(r)}{\log^{[q_n]} r}. \tag{1.1}$$

So for arbitrary $\epsilon > 0$ and sufficiently large r,

$$\begin{aligned} T_{f_1}(r) \leq \exp^{[p_1]}\left\{(A_1+\epsilon)\log^{[q_1]}(r)\right\}, \quad M_{f_2}(r) \leq \exp^{[p_2+1]}\left\{(A_2+\epsilon)\log^{[q_2]}(r)\right\}, \cdots \\ \cdots, M_{f_n}(r) \leq \exp^{[p_n+1]}\left\{(A_n+\epsilon)\log^{[q_n]}(r)\right\}. \end{aligned} \tag{1.2}$$

From (1.2) and using Theorem 1.4, for sufficiently large values of r, we have

$$\begin{aligned} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) &\leq (1+\epsilon)^{(n-1)} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)) \\ &\leq (1+\epsilon)^{(n-1)} \exp^{[p_1]}\left\{(A_1+\epsilon)\log^{[q_1]}(M_{f_2}(\cdots M_{f_n}(r)\cdots))\right\} \\ &\leq (1+\epsilon)^{(n-1)} \exp^{[p_1]}\left\{(A_1+\epsilon)\log^{[q_1]}\left\{\exp^{[p_2+1]}((A_2+\epsilon)\log^{[q_2]} M_{f_3}(\cdots M_{f_n}(r)\cdots)\right\}\right\} \\ &\leq (1+\epsilon)^{(n-1)} exp^{[p_1]}[(A_1+\epsilon)\log^{[q_1]} \\ &\quad \left\{\exp^{[p_2+1]}\left((A_2+\epsilon)\log^{[q_2]}\left(\exp^{[p_3+1]}\left((A_3+\epsilon)\log^{[q_3]} M_{f_4}(\cdots M_{f_n}(r)\cdots)\right)\right)\right)\right\}] \\ &= (1+\epsilon)^{(n-1)} \exp^{[p_1]}[(A_1+\epsilon)\exp^{[p_2+1-q_1]}\left\{(A_2+\epsilon)\exp^{[p_3+1-q_2]}((A_3+\epsilon)\log^{[q_3]} M_{f_4}(\cdots M_{f_n}(r))\right\}] \\ &\vdots \\ &\leq (1+\epsilon)^{(n-1)} \exp^{[p_1]}[(A_1+\epsilon)\exp^{[\sum_{i=2}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i]}\left\{(A_n+(n-1)\epsilon)\log^{[q_n]}(r)\right\}]. \end{aligned} \tag{1.3}$$

Now
(i) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$ then from (1.3)

$$\limsup_{r\to\infty} \frac{\log^{[(\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i)]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[q_n]} r} \leq A_n,$$

since we always have $(\sum_{i=1}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) \geq q_n$.
So,

$$\rho_{[(\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i),q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_n. \tag{1.4}$$

(ii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$ then since $p_1 \geq q_n$ we have from (1.3)

$$\rho_{[p_1,q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_1 A_n.$$

(iii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ then $(\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)) \geq 1$ and we from (1.3)

$$\rho_{[p_1,(\sum_{i=1}^{n} q_i-\sum_{i=2}^{n} p_i-(n-1))]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_1,$$

as $p_1 \geq (\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1))$.
Since $A_n > 0$, there exists a sequence $\{r_n\} \to \infty$ such that for any ϵ $(0 < \epsilon < A_n)$ and sufficiently large r_n, we have

$$M_{f_n}(r_n) \geq \exp^{[p_n+1]}\left\{(A_n - \epsilon)\log^{[q_n]}(r_n)\right\}. \tag{1.5}$$

Now using Theorem 1.5 for sufficiently large r_n, we have from (1.5)

$$\begin{aligned}
T_{f_1\circ f_2\circ\cdots\circ f_n}(r_n) &\geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{8})\cdots))\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{(B_1-\epsilon)\log^{[q_1]}(\frac{1}{9}M_{f_2}(\frac{1}{9}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{8})\cdots))\right\}\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{(B_1-2\epsilon)\log^{[q_1]}\exp^{[p_2+1]}\left\{(B_2-\epsilon)\log^{[q_2]}(\frac{1}{9}M_{f_3}(\frac{1}{18}M_{f_4}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{8})\cdots))\right\}\right\} \qquad (1.6)\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{(B_1-2\epsilon)\log^{[q_1]}\exp^{[p_2+1]}\left\{(B_2-2\epsilon)\log^{[q_2]}\exp^{[p_3+1]}\left\{(B_3-2\epsilon)(\log^{[q_3]}((\frac{1}{18}M_{f_4}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{8})\cdots)))\right\}\right\}\right\}\\
&\vdots\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{(B_1-2\epsilon)\exp^{[\sum_{i=2}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i]}\left\{(A_n-(n-1)\epsilon)(\log^{[q_n]}(r_n)+O(1))\right\}\right\}. \qquad (1.7)
\end{aligned}$$

Now (i) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$ then from (1.7) we have

$$\limsup_{r\to\infty} \frac{\log^{[(\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i)]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[q_n]} r} \geq A_n$$

i.e.,

$$\rho_{[(\sum_{i=1}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i),q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \geq A_n. \tag{1.8}$$

So, from (1.4) and (1.8) we have

$$\rho_{[(\sum_{i=2}^{n} p_i+(n-1)-\sum_{i=1}^{n-1} q_i),q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) = A_n.$$

(ii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$ then from (1.7) we have

$$\rho_{[p_1,q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \geq B_1 A_n.$$

(iii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ then from (1.7) and since $(\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)) \geq 1$ we get

$$\rho_{[p_1,(\sum_{i=1}^{n} q_i-\sum_{i=2}^{n} p_i-(n-1))]}(f_1 \circ f_2 \circ \cdots \circ f_n) \geq B_1$$

This completes the proof.

Theorem 1.8. *Let $f_1, f_2, \cdots, f_n$ be n entire functions having index pair $[p_1, q_1], [p_2, q_2], \cdots, [p_n, q_n]$ respectively and $0 < A_1 < \infty$, $0 < \sum_{i=2}^{n-1} B_i < \infty$, $0 < B_n \leq A_n < \infty$. Now*
(i) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$ then $B_n \leq \rho_{[\sum_{i=1}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_n$;
(ii) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$ then
$A_1 B_n \leq \rho_{[p_1, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_1 A_n$;
(iii) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ then $\rho_{[p_1, \sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(f_1 \circ f_2 \circ \cdots \circ f_n) = A_n$.

Proof. Since $A_1 > 0$, there exists a sequence $\{R_n\} \to \infty$ such that for any ϵ $(0 < \epsilon < A_1)$, we have

$$M_{f_1}(R_n) \geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon)\log^{[q_1]}(R_n)\right\}.$$

Now $M_{f_n}(r)$ being an increasing continuous function, there exists a sequence $\{r_n\} \to \infty$ satisfying

$$R_n = \frac{1}{18} M_{f_n}(\frac{r_n}{2}) \geq \frac{1}{3}\exp^{[p_n+1]}\left\{(B_n - \epsilon)\log^{[q_n]}(\frac{r_n}{2})\right\}.$$

From Theorem 1.3, for sufficiently large $\{r_n\}$

$$\begin{aligned}
M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r_n) &\geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{2})\cdots)))\\
&\geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon)\log^{[q_1]} \frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{2})\cdots))\right\}\\
&\geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon)\log^{[q_1]}(\exp^{[p_2+1]}\left\{(B_2 - \epsilon)\log^{[q_2]} \frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{2})\cdots) + O(1)\right\})\right\}\\
&\geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon)\log^{[q_1]}(\exp^{[p_2+1]}\left\{(B_2 - \epsilon)\log^{[q_2]} \exp^{[p_3+1]}\left\{(B_3 - \epsilon)(\log^{[q_3]}(\frac{1}{18}M_{f_4}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{2})\cdots)) + O(1))\right\}\right\})\right\}\\
&\geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon) exp^{[p_2+1-q_1]}\left\{(B_2 - \epsilon)\exp^{[p_3+1-q_2]}\left\{(B_3 - \epsilon)(\log^{[q_3]}(\frac{1}{18}M_{f_4}(\cdots \frac{1}{18}M_{f_n}(\frac{r_n}{2})\cdots)) + O(1))\right\}\right\}\right\}\\
&\;\vdots\\
&\geq \exp^{[p_1+1]}\left\{(A_1 - \epsilon)\exp^{[\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i]}\left\{(B_n - (n-1)\epsilon)(\log^{[q_n]}(r_n) + O(1))\right\}\right\}.
\end{aligned} \tag{1.9}$$

Now,
(i) if $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$, then from (1.9)

$$\rho_{[(\sum_{i=1}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i), q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \geq B_n. \tag{1.10}$$

Also from Theorem 1.3, we have for large values of r

$$\begin{aligned}
M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r) &\leq M_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots))\\
&\leq \exp^{[p_1+1]}\left\{(A_1 + \epsilon)\log^{[q_1]} M_{f_2}(M_{f_3}(\cdots M_{f_n}(r)\cdots))\right\}\\
&\leq \exp^{[p_1+1]}\left\{(A_1 + \epsilon)\log^{[q_1]}(\exp^{[p_2+1]}\left\{(A_2 + \epsilon)\log^{[q_2]} M_{f_3}(M_{f_4}(\cdots M_{f_n}(r)\cdots))\right\})\right\}\\
&\leq \exp^{[p_1+1]}\left\{(A_1 + \epsilon)\log^{[q_1]}\left\{\exp^{[p_2+1]}\left\{(A_2 + \epsilon)\log^{[q_2]}(\exp^{[p_3+1]}\left\{(A_3 + \epsilon)\log^{[q_3]} M_{f_4}(M_{f_5}(\cdots M_{f_n}(r)\cdots))\right\})\right\}\right\}\right\}\\
&\leq \exp^{[p_1+1]}\left\{(A_1 + \epsilon)(\exp^{[p_2+1-q_1]}\left\{(A_2 + \epsilon)\exp^{[p_3+1-q_2]}\left\{(A_3 + \epsilon)\log^{[q_3]} M_{f_4}(M_{f_5}(\cdots M_{f_n}(r)\cdots))\right\}\right\}\right\})\\
&\;\vdots\\
&\leq \exp^{[p_1+1]}\left\{(A_1 + \epsilon)\exp^{[\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i]}\left\{(A_n + (n-1)\epsilon)\log^{[q_n]}(r)\right\}\right\}.
\end{aligned} \tag{1.11}$$

So for $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$, we have from (1.11)

$$\rho_{[\sum_{i=1}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \leq A_n. \tag{1.12}$$

Hence (i) follows from the (1.10) and (1.12).
(ii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$, then from (1.9)

$$\rho_{[p_1, q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \geq A_1 B_n;$$

and from (1.11)

$$\rho_{[p_1,q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \le A_1 A_n.$$

So,

$$A_1 B_n \le \rho_{[p_1,q_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \le A_1 A_n.$$

This completes the proof of (ii).
(iii) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$, then from (1.9)

$$\rho_{[p_1,\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(f \circ g \circ h) \ge A_1.$$

and from (1.11)

$$\rho_{[p_1,\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(f_1 \circ f_2 \circ \cdots \circ f_n) \le A_1.$$

Consequently,

$$\rho_{[p_1,\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(f_1 \circ f_2 \circ \cdots \circ f_n) = A_1.$$

This completes the proof of (iii).

Theorem 1.9. *Let $f_1, f_2, \cdots, f_n$ be n entire functions having index pair $[p_1, q_1], [p_2, q_2], \cdots, [p_n, q_n]$ respectively.*
(1) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$
and (i) $q_1 > q_n$ and $B_n > 0$, then

$$\lim_{r\to\infty} \frac{\log^{[\sum_{i=1}^{n} p_i + n - \sum_{i=1}^{n-1} q_i]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = \infty;$$

(ii) $q_1 = q_n$ and $0 < A_1, B_1, B_n, A_n < \infty$, then

$$\frac{B_n}{A_1} \le \lim_{r\to\infty} \frac{\log^{[\sum_{i=1}^{n} p_i + n - \sum_{i=1}^{n-1} q_i]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} \le \frac{A_n}{B_1};$$

(iii) $q_1 < q_n$ and $B_1 > 0$, then

$$\lim_{r\to\infty} \frac{\log^{[\sum_{i=1}^{n} p_i + n - \sum_{i=1}^{n-1} q_i]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = 0.$$

(2) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$
and (i) $q_1 > q_n$ and $B_1, B_n > 0$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = \infty;$$

(ii) $q_1 = q_n$ and $0 < A_1, B_1, B_n, A_n < \infty$, then

$$\frac{B_n B_1}{A_1} \le \lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} \le \frac{A_1 A_n}{B_1};$$

(iii) $q_1 < q_n$ and $0 < A_1, B_1, B_n, A_n < \infty$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = 0.$$

(3) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ and $0 < A_1, B_1 < \infty$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = \infty.$$

Proof. Using Theorem 1.3 and for $\epsilon > 0$ there exists a positive number r_0 such that for all $r > r_0$

$$\begin{aligned} M_{f_1\circ f_2\circ\cdots\circ f_n}(r) &\geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}\cdots\frac{1}{18}M_{f_n}(\frac{r}{2})\cdots)) \\ &\geq \exp^{[p_1+1]}\left\{(B_1-\epsilon)\log^{[q_1]}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}\cdots\frac{1}{18}M_{f_n}(\frac{r}{2})\cdots))\right\} \\ &\geq \exp^{[p_1+1]}\left\{(B_1-\epsilon)\log^{[q_1]}(\exp^{[p_2+1]}\left\{(B_2-\epsilon)\log^{[q_2]}(\frac{1}{18}M_{f_3}(\frac{1}{18}M_{f_4}\cdots\frac{1}{18}M_{f_n}(\frac{r}{2})\cdots))\right\})\right\}+O(1) \\ &\vdots \\ &\geq \exp^{[p_1+1]}\left\{(B_l-\epsilon)\exp^{[(\sum_{i=2}^{n}p_i+(n-1)-\sum_{i=1}^{n-1}q_i)]}\left\{(B_n-(n-1)\epsilon)\log^{[q_n]}(r)\right\}\right\}+O(1) \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} M_{f_1\circ f_2\circ\cdots\circ f_n}(r) &\leq M_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)) \\ &\leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)\log^{[q_1]}M_{f_2}(M_{f_3}(\cdots M_{f_n}(r)\cdots))\right\} \\ &\leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)\log^{[q_1]}(\exp^{[p_2+1]}\left\{(A_2+\epsilon)\log^{[q_2]}M_{f_3}(M_{f_4}(\cdots M_{f_n}(r)\cdots))\right\})\right\} \\ &\leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)\log^{[q_1]}\left\{\exp^{[p_2+1]}\left\{(A_2+\epsilon)\log^{[q_2]}(\exp^{[p_3+1]}\left\{(A_3+\epsilon)\log^{[q_3]}M_{f_4}(M_{f_5}(\cdots M_{f_n}(r)\cdots))\right\})\right\}\right\}\right\} \\ &\leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)(\exp^{[p_2+1-q_1]}\left\{(A_2+\epsilon)\exp^{[p_3+1-q_2]}\left\{(A_3+\epsilon)\log^{[q_3]}M_{f_4}(M_{f_5}(\cdots M_{f_n}(r)\cdots))\right\}\right\}\right\}) \\ &\vdots \\ &\leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)\exp^{[\sum_{i=2}^{n}p_i+(n-1)-\sum_{i=1}^{n-1}q_i]}\left\{(A_n+(n-1)\epsilon)\log^{[q_n]}(r)\right\}\right\}. \end{aligned} \tag{1.14}$$

Also for all large values of r,

$$\exp^{[p_1+1]}\left\{(B_l-\epsilon)\log^{[q_1]}r\right\} \leq M_{f_1}(r) \leq \exp^{[p_1+1]}\left\{(A_1+\epsilon)\log^{[q_1]}r\right\} \tag{1.15}$$

Now we consider the following three cases.

Case I. $(\sum_{i=2}^{n}p_i+(n-1)-\sum_{i=1}^{n-1}q_i) > 0$. Then from (1.13), (1.14) and (1.15)

$$\frac{(A_n+(n-1)\epsilon)\log^{[q_n]}(r)}{(B_1-\epsilon)\log^{[q_1]}r} \geq \frac{\log^{[\sum_{i=1}^{n}p_i+n-\sum_{i=1}^{n-1}q_i]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} \geq \frac{(B_n-(n-1)\epsilon)\log^{[q_n]}(r)+O(1)}{(A_1+\epsilon)\log^{[q_1]}r}. \tag{1.16}$$

(i) When $q_1 > q_n$. Since $B_n > 0$, from (1.16)

$$\lim_{r\to\infty}\frac{\log^{[\sum_{i=1}^{n}p_i+n-\sum_{i=1}^{n-1}q_i]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} = \infty.$$

(ii) When $q_1 = q_n$. Since $0 < A_1, B_1, B_n, A_n < \infty$, from (1.16)

$$\frac{B_n}{A_1} \leq \lim_{r\to\infty}\frac{\log^{[\sum_{i=1}^{n}p_i+n-\sum_{i=1}^{n-1}q_i]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} \leq \frac{A_n}{B_1}.$$

(iii) When $q_1 < q_n$. Since $B_1 > 0$, from (1.16)

$$\lim_{r\to\infty}\frac{\log^{[\sum_{i=1}^{n}p_i+n-\sum_{i=1}^{n-1}q_i]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} = 0.$$

Case II. $(\sum_{i=2}^{n}p_i+(n-1)-\sum_{i=1}^{n-1}q_i) = 0$. Then from (1.13), (1.14) and (1.15)

$$\frac{(A_1+\epsilon)(A_n+(n-1)\epsilon)\log^{[q_n]}(r)}{(B_1-\epsilon)\log^{[q_1]}r} \geq \frac{\log^{[\sum_{i=1}^{n}p_i+n-\sum_{i=1}^{n-1}q_i]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} \geq \frac{(B_1-\epsilon)(B_n-(n-1)\epsilon)\log^{[q_n]}(r)+O(1)}{(A_1+\epsilon)\log^{[q_1]}r}. \tag{1.17}$$

(i) When $q_1 > q_n$. Since $B_1, B_n > 0$, from (1.17)

$$\lim_{r\to\infty}\frac{\log^{[p_1+1]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1+1]}M_{f_1}(r)} = \infty.$$

(ii) When $q_1 = q_n$. Since $0 < A_1, B_1, B_n, A_n < \infty$, from (1.17)

$$\frac{B_1 B_n}{A_1} \leq \lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} \leq \frac{A_1 A_n}{B_1}.$$

(iii) When $q_1 < q_n$. Since $0 < A_1, B_1, B_n, A_n < \infty$, from (1.17)

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} = 0.$$

Case III. $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$. Since $p_n \geq q_n \geq 1$, we have from (1.13) and (1.15) and $0 < A_1, B_1 < \infty$, we have

$$\lim_{r\to\infty} \frac{\log^{[p_1]} M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} M_{f_1}(r)} \geq \frac{(B_1 - \epsilon) \log^{[\sum_{i=1}^{n} q_i - \sum_{i=2}^{n} p_i - (n-1)]}(r)}{(A_1 + \epsilon) \log^{[q_1]} r} \to \infty. \tag{1.18}$$

In terms of $\mu(r, f)$ instead of $M(r, f)$ Theorem 1.9 can be stated as follows:

Theorem 1.10. *Let $f_1, f_2, \cdots, f_n$ be n entire functions having index pair $[p_1, q_1], [p_2, q_2], \cdots, [p_n, q_n]$ respectively. Then*
(1) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) > 0$
and (i) $q_1 > q_n$ and $B_n > 0$, then

$$\lim_{r\to\infty} \frac{\log^{[\sum_{i=1}^{n} p_i + n - \sum_{i=1}^{n-1} q_i]} \mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} \mu_{f_1}(r)} = \infty;$$

(ii) $q_1 = q_n$ and $0 < A_1, B_n < \infty$, then

$$\lim_{r\to\infty} \frac{\log^{[\sum_{i=1}^{n} p_i + n - \sum_{i=1}^{n-1} q_i]} \mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} \mu_{f_1}(r)} \geq \frac{B_n}{A_1};$$

(2) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) = 0$
and (i) $q_1 > q_n$ and $B_1, B_n > 0$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} \mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} \mu_{f_1}(r)} = \infty;$$

(ii) $q_1 = q_n$ and $0 < A_1, B_1, B_n, A_n < \infty$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} \mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} \mu_{f_1}(r)} \leq \frac{B_1 B_n}{A_1};$$

(3) If $(\sum_{i=2}^{n} p_i + (n-1) - \sum_{i=1}^{n-1} q_i) < 0$ and $0 < A_1, B_1 < \infty$, then

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} \mu_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1+1]} \mu_{f_1}(r)} = \infty.$$

Proof. Using Theorem 1.6 and applying the same arguments as in Theorem (1.9), we can easily prove the Theorem 1.10.

Theorem 1.11. *Let $f_1, f_2, \cdots, f_n$ be n entire functions and $f_2, f_3, \cdots, f_n$ have index pair $[p_2, q_2], [p_3, q_3], \cdots \cdots, [p_n, q_n]$ respectively satisfying $B_2, B_3, \cdots, B_n > 0$. If $(f_1 \circ f_2 \circ \cdots \circ f_n)$ have index-pair $[p_1, q_1]$ and $0 < \lambda_{[p_1,q_1]}(f_1 \circ f_2 \circ \cdots \circ f_n) = \chi < \infty$, then $\lambda_{[p_1,q_1]}(f_1 \circ f_2 \circ \cdots \circ f_{n-1}) = 0$, $\lambda_{[p_1,q_1]}(f_1 \circ f_2 \circ \cdots \circ f_{n-2}) = 0, \cdots, \lambda_{[p_1,q_1]}(f_1) = 0$.*

Proof. By definition there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n and using Lemma 1.2

$$\frac{1}{3}\log M_{f_1\circ f_2\circ f_3\circ\cdots\circ f_{n-1}}\left(\frac{1}{9}M_{f_n}\left(\frac{r_n}{4}\right)\right) \le T_{f_1\circ f_2\circ\cdots\circ f_n}(r_n) \le \exp^{[p_1]}\left\{(\chi+\epsilon)\log^{[q_1]} r_n\right\}. \tag{1.19}$$

Since $\lambda_{[p_n,q_n]}(f_n) = B_n > 0$ so for given ϵ $(0 < \epsilon < B_n)$ and sufficiently large r_n

$$\begin{aligned}\frac{1}{9}M_{f_n}\left(\frac{r_n}{4}\right) &\ge \exp^{[p_n+1]}\left\{\left(B_n - \frac{\epsilon}{2}\right)\log^{[q_n]}\frac{r_n}{4}\right\}\\ &\ge \exp^{[p_n+1]}\left\{(B_n-\epsilon)\log^{[q_n]} r_n\right\}.\end{aligned}$$

Set $R_n = \frac{1}{9}M_{f_n}(\frac{r_n}{4})$. Then

$$r_n \le \exp^{[q_n]}\left\{\frac{1}{(B_n-\epsilon)}\log^{[p_n+1]} R_n\right\}. \tag{1.20}$$

Now from (1.19) and (1.20),

$$\log M_{f_1\circ f_2\circ\cdots\circ f_{n-1}}(R_n) \le 3\exp^{[p_1]}\left\{(\chi+\epsilon)\log^{[q_1]}\left(\exp^{[q_n]}\left\{\frac{1}{(B_n-\epsilon)}\log^{[p_n+1]} R_n\right\}\right)\right\}.$$

Since $(p_n + 1 - q_n) > 0$, we have

$$\frac{\log^{[p_1+1]} M_{f_1\circ f_2\circ\cdots\circ f_{n-1}}(R_n)}{\log^{[q_1]} R_n} \le \frac{(\chi+\epsilon)\exp^{[q_n-q_1]}\left\{\frac{1}{(B_n-\epsilon)}\log^{[p_n+1]} R_n\right\}}{\log^{[q_1]} R_n} \to 0.$$

Consequently,

$$\lambda_{[p_1,q_1]}(f_1 \circ f_2 \circ \cdots \circ f_{n-1}) = 0.$$

Again for sufficiently large r_n,

$$\frac{1}{3}\log M_{f_1\circ f_2\circ f_3\circ\cdots\circ f_{n-2}}\left(\frac{1}{9}M_{f_{n-1}}\left(\frac{r_n}{4}\right)\right) \le T_{f_1\circ f_2\circ\cdots\circ f_{n-1}}(r_n) \le \exp^{[p_1]}\left\{\epsilon\log^{[q_1]} r_n\right\}. \tag{1.21}$$

Since $\lambda_{[p_{n-1},q_{n-1}]}(f_{n-1}) = B_{n-1} > 0$ so for given ϵ $(0 < \epsilon < B_{n-1})$ and sufficiently large r_n, we have

$$\begin{aligned}\frac{1}{9}M_{f_{n-1}}\left(\frac{r_n}{4}\right) &\ge \exp^{[p_{n-1}+1]}\left\{\left(B_{n-1} - \frac{\epsilon}{2}\right)\log^{[q_{n-1}]}\frac{r_n}{4}\right\}\\ &\ge \exp^{[p_{n-1}+1]}\left\{(B_{n-1}-\epsilon)\log^{[q_{n-1}]} r_n\right\}.\end{aligned}$$

Put $R'_n = \frac{1}{9}M_{f_{n-1}}(\frac{r_n}{4})$. Then

$$r_n \le \exp^{[q_{n-1}]}\left\{\frac{1}{(B_{n-1}-\epsilon)}\log^{[p_{n-1}+1]} R'_n\right\}. \tag{1.22}$$

From (1.21) and (1.22), we get

$$\log M_{f_1\circ f_2\circ\cdots\circ f_{n-2}}(R'_n) \le 3\exp^{[p_1]}\left\{\epsilon\log^{[q_1]}\left(\exp^{[q_{n-1}]}\left\{\frac{1}{(B_{n-1}-\epsilon)}\log^{[p_{n-1}+1]} R'_n\right\}\right)\right\}.$$

Since $(p_{n-1} + 1 - q_{n-1}) > 0$, we have

$$\frac{\log^{[p_1+1]} M_{f_1\circ f_2\circ\cdots\circ f_{n-2}}(R'_n)}{\log^{[q_1]} R'_n} \le \frac{\epsilon\exp^{[q_{n-1}-q_1]}\left\{\frac{1}{(B_{n-1}-\epsilon)}\log^{[p_{n-1}+1]} R'_n\right\}}{\log^{[q_1]} R'_n} \to 0.$$

So,

$$\lambda_{[p_1,q_1]}(f_1 \circ f_2 \circ \cdots \circ f_{n-2}) = 0.$$

Finally for sufficiently large r_n,

$$\frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\frac{r_n}{4})) \le T_{f_1\circ f_2}(r_n) \le \exp^{[p_1]}\left\{\epsilon \log^{[q_1]} r_n\right\}. \tag{1.23}$$

Since $\lambda_{[p_2,q_2]}(f_2) = B_2 > 0$ so for given ϵ $(0 < \epsilon < B_2)$ and sufficiently large r_n,

$$\begin{aligned}\frac{1}{9}M_{f_2}(\frac{r_n}{4}) &\ge \exp^{[p_2+1]}\left\{(B_2 - \frac{\epsilon}{2})\log^{[q_2]}\frac{r_n}{4}\right\}\\ &\ge \exp^{[p_2+1]}\left\{(B_2-\epsilon)\log^{[q_2]} r_n\right\}.\end{aligned}$$

Set $R_n'' = \frac{1}{9}M_{f_2}(\frac{r_n}{4})$. Then

$$r_n \le \exp^{[q_2]}\left\{\frac{1}{(B_2-\epsilon)}\log^{[p_2+1]} R_n''\right\}. \tag{1.24}$$

Now from (1.23) and (1.24), we get

$$\log M_{f_1}(R_n'') \le 3\exp^{[p_1]}\left\{\epsilon \log^{[q_1]}(\exp^{[q_2]}\left\{\frac{1}{(B_2-\epsilon)}\log^{[p_2+1]} R_n''\right\})\right\}.$$

Since $(p_2 + 1 - q_2) > 0$, we have

$$\frac{\log^{[p_1+1]} M_{f_1}(R_n'')}{\log^{[q_1]} R_n''} \le \frac{\epsilon \exp^{[q_2-q_1]}\left\{\frac{1}{(B_2-\epsilon)}\log^{[p_2+1]} R_n''\right\}}{\log^{[q_1]} R_n''} \to 0.$$

Thus

$$\lambda_{[p_1,q_1]}(f_1) = 0.$$

This completes the proof.

Theorem 1.12. *Let $f_1, f_2, \cdots, f_n$ be n entire functions and p, q be two positive integers satisfying $p \ge q \ge 1$. If $\lambda_{[p,q]}(f_1 \circ f_2 \circ \cdots \circ f_n) = \tau < \lambda_{[p,q]}(f_n) = \sigma < \infty$, then $\lambda_{[p,q]}(f_1 \circ f_2 \circ \cdots \circ f_{n-1}) = 0$, $\lambda_{[p,q]}(f_1 \circ f_2 \circ \cdots \circ f_{n-2}) = 0, \cdots, \lambda_{[p,q]}(f_1) = 0$.*

Proof. By definition there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n and using Lemma 1.2

$$\frac{1}{3}\log M_{f_1\circ f_2\circ f_3\circ\cdots\circ f_{n-1}}(\frac{1}{9}M_{f_n}(\frac{r_n}{4})) \le T_{f_1\circ f_2\circ\cdots\circ f_n}(r_n) \le \exp^{[p]}\left\{(\tau+\epsilon)\log^{[q]} r_n\right\}. \tag{1.25}$$

For given ϵ $(0 < \epsilon < \sigma - \tau)$ and sufficiently large r_n, we have

$$\begin{aligned}\frac{1}{9}M_{f_n}(\frac{r_n}{4}) &\ge \exp^{[p+1]}\left\{(\sigma - \frac{\epsilon}{2})\log^{[q]}\frac{r_n}{4}\right\}\\ &\ge \exp^{[p+1]}\left\{(\sigma-\epsilon)\log^{[q]} r_n\right\}.\end{aligned}$$

Set $R_n = \frac{1}{9}M_{f_n}(\frac{r_n}{4})$. Then

$$\log^{[q]} r_n \le \frac{1}{(\sigma-\epsilon)}\log^{[p+1]} R_n. \tag{1.26}$$

Now from (1.25) and (1.26),

$$\log M_{f_1\circ f_2\circ\cdots\circ f_{n-1}}(R_n) \le 3\exp^{[p]}\left\{\frac{\tau+\epsilon}{\sigma-\epsilon}\log^{[p+1]} R_n\right\}.$$

Since $p \geq q \geq 1$, so we have

$$\frac{\log^{[p+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_{n-1}}(R_n)}{\log^{[q]} R_n} \leq \frac{\frac{\tau+\epsilon}{\sigma-\epsilon} \log^{[p+1]} R_n}{\log^{[q]} R_n} \to 0.$$

Consequently,

$$\lambda_{[p,q]}(f_1 \circ f_2 \circ \cdots \circ f_{n-1}) = 0.$$

Again for sufficiently large r_n,

$$\frac{1}{3} \log M_{f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_{n-2}}(\frac{1}{9} M_{f_{n-1}}(\frac{r_n}{4})) \leq T_{f_1 \circ f_2 \circ \cdots \circ f_{n-1}}(r_n) \leq \exp^{[p]} \left\{ \epsilon \log^{[q]} r_n \right\}. \tag{1.27}$$

Since $\lambda_{[p,q]}(f_{n-1}) = B_{n-1} > 0$ so for given ϵ $(0 < \epsilon < B_{n-1})$ and sufficiently large r_n,

$$\begin{aligned} \frac{1}{9} M_{f_{n-1}}(\frac{r_n}{4}) &\geq \exp^{[p+1]} \left\{ (B_{n-1} - \frac{\epsilon}{2}) \log^{[q]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p+1]} \left\{ (B_{n-1} - \epsilon) \log^{[q]} r_n \right\}. \end{aligned}$$

Set $R_n' = \frac{1}{9} M_{f_{n-1}}(\frac{r_n}{4})$. Then

$$r_n \leq \exp^{[q]} \left\{ \frac{1}{(B_{n-1} - \epsilon)} \log^{[p+1]} R_n' \right\}. \tag{1.28}$$

From (1.27) and (1.28),

$$\begin{aligned} \log M_{f_1 \circ f_2 \circ \cdots \circ f_{n-2}}(R_n') &\leq 3 \exp^{[p]} \left\{ \epsilon \log^{[q]} (\exp^{[q]} \left\{ \frac{1}{(B_{n-1} - \epsilon)} \log^{[p+1]} R_n' \right\}) \right\} \\ &\leq 3 \exp^{[p]} \left\{ \epsilon \left\{ \frac{1}{(B_{n-1} - \epsilon)} \log^{[p+1]} R_n' \right\} \right\}. \end{aligned}$$

So

$$\frac{\log^{[p+1]} M_{f_1 \circ f_2 \circ \cdots \circ f_{n-2}}(R_n')}{\log^{[q]} R_n'} \leq \frac{\left\{ \frac{\epsilon}{(B_{n-1} - \epsilon)} \log^{[p+1]} R_n' \right\}}{\log^{[q]} R_n'} \to 0.$$

Thus

$$\lambda_{[p,q]}(f_1 \circ f_2 \circ \cdots \circ f_{n-2}) = 0.$$

Finally for sufficiently large r_n,

$$\frac{1}{3} \log M_{f_1}(\frac{1}{9} M_{f_2}(\frac{r_n}{4})) \leq T_{f_1 \circ f_2}(r_n) \leq \exp^{[p]} \left\{ \epsilon \log^{[q]} r_n \right\}. \tag{1.29}$$

Since $\lambda_{[p,q]}(f_2) = B_2 > 0$ so for given ϵ $(0 < \epsilon < B_2)$ and sufficiently large r_n,

$$\begin{aligned} \frac{1}{9} M_{f_2}(\frac{r_n}{4}) &\geq \exp^{[p+1]} \left\{ (B_2 - \frac{\epsilon}{2}) \log^{[q]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p+1]} \left\{ (B_2 - \epsilon) \log^{[q]} r_n \right\}. \end{aligned}$$

Set $R_n'' = \frac{1}{9} M_{f_2}(\frac{r_n}{4})$. Then

$$r_n \leq \exp^{[q]} \left\{ \frac{1}{(B_2 - \epsilon)} \log^{[p+1]} R_n'' \right\}. \tag{1.30}$$

From (1.29) and (1.30),

$$\begin{aligned} \log M_{f_1}(R_n'') &\leq 3 \exp^{[p]} \left\{ \epsilon \log^{[q]} (\exp^{[q]} \left\{ \frac{1}{(B_2 - \epsilon)} \log^{[p+1]} R_n'' \right\}) \right\} \\ &\leq 3 \exp^{[p]} \left\{ \epsilon \left\{ \frac{1}{(B_2 - \epsilon)} \log^{[p+1]} R_n'' \right\} \right\}. \end{aligned}$$

So

$$\frac{\log^{[p+1]} M_{f_1}(R_n'')}{\log^{[q]} R_n''} \leq \frac{\left\{\frac{\epsilon}{(B_2-\epsilon)} \log^{[p+1]} R_n''\right\}}{\log^{[q]} R_n''} \to 0.$$

Thus

$$\lambda_{[p,q]}(f_1) = 0.$$

This completes the proof.

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