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A Mathematical Model of Drug-Crime Dynamics

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Abstract
The incidence of Drug abuse and Crime are common phenomena in every society, especially within youthful populations. Several questions remains unanswered regarding Drug abuse and Crime. This study considered the dynamics of Drug abuse and Crime behaviour of an heterogeneous society which is divided into six-state mutual exclusive homogeneous classes; namely the Susceptible, Drug abuse, Crime, Drug abuse-Crime, Treatment/Rehabilitation and the Recovered. The developed model was analysed qualitatively with respective to the basic initiation number; $\mathcal{R}_0$ which is synonymous to the basic reproduction number in epidemiological modelling. Both the Drug only and Crime only models are observed to have asymptotically stable incidence-free equilibria when their respective basic initiation numbers are less than unity(ie $\mathcal{R}_0 < 1$). The full model also established stable equilibrium at $\mathcal{R}_0 < 1$. The sub-models and the full model exhibit forward bifurcation when their basic initiation numbers are at unity. The parameters that drug-abuse and crime initiation are influenced most were the rates of interactions, namely β_d and β_c.

Keywords: Mathematical model; local stability; basic reproduction number; drug and crime.

Introduction

Drug abuse and crime are inherent in every human society and it is common to find people involve in one or both. In Ghana, most drug users are often involved in social vices like prostitution and stealing, to get money to purchase their drugs [6]. [5] conducted a nationwide survey of senior high school students in Ghana and reported that at least 20 percent of students abuse drugs for various reasons like having good time with friends, alter their moods, relieve tension, feel good, and overcome boredom and problems. In fact, drug abuse is an age-old problem that has a big toll on society and health systems. Many people, particularly young people, engage in drug abuse for varied reasons, including physical, psychosocial, educational, political and moral gains. Drug abuse and crime remains a major global health and social problem and are now considered as a disease. It is not new that drug abuse and crime are world social menace, but what is new, in West Africa today is the wide variety of addictive drugs that are available for use by mostly young people and the increasing prevalence of drug use disorders and other social and physical problems associated with drug use and one of them is crime. There is a global burden of death, loss of property and disability all attributable to illicit drug use and crime; and this has remained a significant threat to public health for developed and developing nations. Deaths

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resulting from drug abuse and drug-related crimes have high impact on the communities [7]. The relationship between drug use and crime have been contradictory and complex [8] and this complex link can be explained as: drug use leads to crime, crime leads to drug use and this relationship might be mere coincidence or myriad of common causes. Some drug users commit crime in order to get money to continue to buy their drugs [9]. But the actual mechanisms by which new cases of drug abuse and crime is aggravated is poorly understood. The processes through which drug abuse and crime spread or control in a population is like epidemics spread or controlled. Epidemics spread through social interactions of individuals which could be physical contact or communication. Similarly, criminals within a community naturally influence others to join the criminal gang through verbal and non-verbal communication [10][11]. Hence, mathematical modelling of crime incidence lends itself to susceptible-infectious-recovery (SIR) disease model. To best of our knowledge, research on the mathematical model of Drug Abuse and Crime are scanty. Hence, this study sought to model Drug abuse and Crime epidemics to provide a rational basis for policy designers to control the spread of Crime and Drug abuse in society.

Model Formulation

We consider a heterogeneous population of size N that is divided into six homogeneous mutually-exclusive compartments consisting of the following:

i) Susceptibles, S; People who do not abuse drugs and are not engaged in crime.

ii) Drug abusers D; Individuals who abuse drugs but do not engage in crimes.

iii) Criminals, C; Individuals who engage in crime but do not abuse drugs;

iv) Drug abusive-Criminals, DC; Individuals who abuse drugs and engage in criminal behaviour.

v) Rehabilitated/Treated, T; Individuals undergoing rehabilitation in order to quit either crime or drug abuse or both.

vi) Recovered, R; Individuals who have successfully undergone rehabilitation.

So that the total population is given by $N = S + D + C + DC + T + R$. In order to construct the model under consideration, we assume that individuals engage in crime and drug as a result of initiation by those who are already involved in the practice. The population is assumed to increase via births and immigration at rate Λ and all recruits are assumed Susceptible. The Susceptible population is reduced as a result of initiation into Drug and Crime. Individuals (S, C) become drug abusers due to effective contact with drug abusers with force of initiation $\lambda_1 = \frac{\beta_d(D+\eta_1 DC)}{N}$. Also individuals($S$, D) join the criminals class through initiation by criminals/gangs with force of initiation $\lambda_2 = \frac{\beta_c(C+\eta_2 DC)}{N}$. Drug abusers and Criminals quite their activities at rates Q_d and Q_c respectively, while those involved in the both acts are assumed unable to quit unless through rehabilitation. Drug abusers and Criminals agree to undergo rehabilitation at rates τ_d and τ_c respectively, while those in the DC class join rehabilitation at rate τ_{dc}. We take ρ to be the rate of successful rehabilitation, so that individuals undergoing rehabilitation join the recovered class at rate ρ. We assume that μ, μ_d and μ_c are natural death rate, drug-use-induced death rate and the rate at which criminals die in the "line of duty" (ie. crime-induced death rate). Recovery from crime and drug-abuse is assumed to not last permanently so that the recovered individuals join the Susceptible class at rate r. With these considerations, the model under discussion is given

by the following set of differential equations.

$$\left.\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda + rR - (\lambda_1 + \lambda_2 + \mu)\, S \\
\frac{\mathrm{d}D}{\mathrm{d}t} &= \lambda_1 S - (\lambda_2 + Q_d + \tau_d + \mu_d + \mu)\, D \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \lambda_2 S - (\lambda_1 + Q_c + \tau_c + \mu_c + \mu)\, C \\
\frac{\mathrm{d}DC}{\mathrm{d}t} &= \lambda_1 C + \lambda_2 D - [\tau_{dc} + \mu + (\alpha\,\mu_c + (1-\alpha)\,\mu_d)]\, DC \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \tau_d D + \tau_c C + \tau_{dc} DC - (\rho + \mu)\, T \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= Q_d D + Q_c C + \rho\, T - (r + \mu)\, R
\end{aligned}\right\} \tag{1}$$

The expression $(\alpha\,\mu_c + (1-\alpha)\,\mu_d)$ represents the rate at which those involved in Drug-abuse and crime die from one drug-abuse or crime or both, where $0 \leq \alpha \leq 1$ models the weight associated with death in the course of crime.
The model (1) can be split into two sub-models, namely the Drug-only model and the Crime-only model which are given as follows

$$\overbrace{\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda + rR - (\lambda_1 + \mu)\, S \\
\frac{\mathrm{d}D}{\mathrm{d}t} &= \lambda_1 S - (Q_d + \tau_d + \mu_d + \mu)\, D \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \tau_d D - (\rho + \mu)\, T \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= Q_d D + \rho\, T - (r + \mu)\, R
\end{aligned}}^{\text{Drug-only model (DoM)}} \qquad \overbrace{\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda + rR - (\lambda_2 + \mu)\, S \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \lambda_2 S - (Q_c + \tau_c + \mu_c + \mu)\, C \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \tau_c C - (\rho + \mu)\, T \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= Q_c C + \rho\, T - (r + \mu)\, R
\end{aligned}}^{\text{Crime-only model (CoM)}} \tag{2}$$

In subsequent discussions, where necessary, the following substitutions are made for convenience

$$k_1 = Q_d + \tau_d + \mu_d + \mu,\ k_2 = Q_c + \tau_c + \mu_c + \mu,\ k_3 = \alpha\,\mu_c + \mu + \tau_{dc} + (1-\alpha)\,\mu_d,\ k_4 = \rho + \mu,\ k_5 = r + \mu.$$

In the next section, we discuss the qualitative properties of the Drug-Crime model (1).

Qualitative Properties of Model

Positivity and Boundedness of Solutions

There is the need to prove that all the state variables in the system (1) remain non-negative such that the solutions to system with positive initial conditions remain positive for all $t > 0$. Hence we give a Lemma:

Lemma 1. The following results hold concerning the model (1)

1. if the initial conditions are non-negative, so are all future solutions.
2. The set degined by $G = \left\{(S(t), D(t), C(t), DC(t), T(t), R(t)) \in \mathbb{R}^6_+ : N \leq \frac{\Lambda}{\mu}\right\}$ is a positively-invariant and attracting region of the model.

Proof. We note that the right-hand-side of all components of model (1) are continuous and also locally-Lipschitz at $t = 0$. Therefore for each non-negative initial condition, model (1) has a unique non-negative solution in $\mathbb{R}^6_+$ for all $t > 0$, concluding the first part of the Lemma.
Furthermore, it is easy to see that if $X = (S(t),\ D(t),\ C(t),\ DC(t),\ T(t),\ R(t)) \in \mathbb{R}^6_+$ and $X_i = 0$, then $\frac{\mathrm{d}X_i}{\mathrm{d}t} \geq 0$. Therefore, from **Theorem A.4** of [13], the region $\mathbb{R}^6_+$ is positively-invariant under the flow induced by model (1).
Now, adding all equations in model (1) gives

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \Lambda - \mu\, N - (D + (1-\alpha)DC))\,\mu_d - (C + \alpha DC)\,\mu_c \leq \Lambda - \mu\, N$$

so that we have $\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$. Since $N(t)$ is bounded then each component of X is also bounded and hence the feasible region of the model is given by G.

Concerning the invariance of G, we note that from $\frac{dN}{dt} \leq \Lambda - \mu N$ it is easy to see that $\frac{dN}{dt} < 0$ whenever $N > \frac{\Lambda}{\mu}$, showing that G is invariant. This concludes the proof of the second part of the Lemma. □

System (1) will therefore be analyzed in a suitable feasible region G of biological interest.The following Lemma applies to the region that system (1) is restricted to.

Equilibrium Points of Model

Let $\mathscr{E}^* = (S^*, D^*, C^*, DC^*, T^*, R^*)$ be a typical equilibrium point. Then solving system (1) gives

$$\left.\begin{aligned}
S^* &= \frac{\Lambda\left(\lambda_1^*+k_2\right)\left(\lambda_2^*+k_1\right)k_3k_4k_5}{\Gamma_1},\\
D^* &= \frac{\lambda_1^* S^*}{\lambda_2^*+k_1},\\
C^* &= \frac{\lambda_2^* S^*}{\lambda_1^*+k_2},\\
DC^* &= \left[\frac{\lambda_1^*\lambda_2^*\left(\lambda_2^*+k_1+\lambda_1^*+k_2\right)}{\left(\lambda_1^*+k_2\right)\left(\lambda_2^*+k_1\right)k_3}\right]S^*,\\
T^* &= \left[\frac{\lambda_2^*\left(k_3\tau_c+\tau_{dc}\lambda_1^*\right)}{\left(\lambda_1^*+k_2\right)k_3k_4}+\frac{\lambda_1^*\left(k_3\tau_d+\lambda_2\tau_{dc}\right)}{\left(\lambda_2^*+k_1\right)k_3k_4}\right]S^*,\\
R^* &= \left[\frac{\lambda_2^*\left(\rho\,k_3\tau_c+\rho\,\tau_{dc}\lambda_1^*+Q_ck_3k_4\right)}{\left(\lambda_1^*+k_2\right)k_3k_4k_5}+\frac{\lambda_1^*\left(\rho\,k_3\tau_d+\tau_{dc}\rho\,\lambda_2^*+Q_dk_3k_4\right)}{\left(\lambda_2^*+k_1\right)k_3k_4k_5}\right]S^*,\\
\Gamma_1 &= \left(\lambda_1^*+\lambda_2^*+\mu\right)\left(\lambda_1^*+k_2\right)\left(\lambda_2^*+k_1\right)k_3k_4k_5\\
&\quad -r\left[\left(\tau_{dc}\rho\,\lambda_2^*+k_3\left(\rho\,\tau_d+Q_dk_4\right)\right)\lambda_1^*\left(\lambda_1^*+k_2\right)\right.\\
&\quad \left.+\left(k_3\left(\rho\,\tau_c+Q_ck_4\right)+\rho\,\tau_{dc}\lambda_1^*\right)\lambda_2^*\left(\lambda_2^*+k_1\right)\right]
\end{aligned}\right\} \tag{3}$$

where λ_1^* and λ_2^* satisfy the following set of non-linear equations:

$$\left.\begin{aligned}
\sum_{i,\,k=0}^{2}\Phi_{ik}\left(\lambda_2^*\right)^k\left(\lambda_1^*\right)^i &= 0;\\
\sum_{i,\,k=0}^{2}\Psi_{ik}\left(\lambda_1^*\right)^k\left(\lambda_2^*\right)^i &= 0.
\end{aligned}\right\} \tag{4}$$

where

$$\begin{aligned}
\Phi_{22} &= 0;\\
\Phi_{21} &= \left(\left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)k_4k_5+r\rho\,\tau_{dc}-k_3k_4k_5\right);\\
\Phi_{20} &= k_3\mu_dk_4k_5+rk_3\left(\rho\,\tau_d+Q_dk_4\right)-k_1k_3k_4k_5\\
\Phi_{12} &= \left(\left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)k_4k_5+r\rho\,\tau_{dc}-k_3k_4k_5\right);\\
\Phi_{11} &= \left(\beta_d\mu\,\eta_1+\left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)\left(k_1+k_2\right)\right)k_4k_5\\
&\quad +r\rho\,\tau_{dc}\left(k_1+k_2\right)-\left(\mu+k_2+k_1\right)k_3k_4k_5;\\
\Phi_{10} &= \left(k_3\beta_d\mu+\mu_dk_2k_3\right)k_4k_5+rk_3\left(\rho\,\tau_d+Q_dk_4\right)k_2\\
&\quad -\left(\mu+k_2\right)k_1k_3k_4k_5\\
\Phi_{02} &= \left(\beta_d\mu\,\eta_1+k_3\mu_c-k_2k_3\right)k_4k_5+rk_3\left(\rho\,\tau_c+Q_ck_4\right);\\
\Phi_{01} &= \left(\beta_d\mu\,\eta_1\left(k_1+k_2\right)+k_3\mu_ck_1\right)k_4k_5\\
&\quad +rk_3\left(\rho\,\tau_c+Q_ck_4\right)k_1-\left(\mu\,k_2+k_2k_1\right)k_3k_4k_5;\\
\Phi_{00} &= \mu\,k_1k_3k_4k_5\left(\beta_d-k_2\right)
\end{aligned}$$

$$\begin{aligned}
\Psi_{22} &= 0;\\
\Psi_{21} &= \left(\left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)k_4k_5+r\rho\,\tau_{dc}-k_3k_4k_5;\right.\\
\Psi_{10} &= k_3\mu_ck_4k_5+rk_3\left(\rho\,\tau_c+Q_ck_4\right)-k_2k_3k_4k_5;\\
\Psi_{12} &= \left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)k_4k_5+r\rho\,\tau_{dc}-k_3k_4k_5;\\
\Psi_{11} &= \left(\beta_c\mu\,\eta_2+\left(\left(\mu_c-\mu_d\right)\alpha+\mu_d\right)\left(k_1+k_2\right)\right)k_4k_5\\
&\quad +r\left(\rho\,\tau_{dc}k_1+\rho\,\tau_{dc}k_2\right)-\left(\mu+k_2+k_1\right)k_3k_4k_5;\\
\Psi_{10} &= \left(k_3\beta_c\mu+k_3\mu_ck_1\right)k_4k_5+rk_3\left(\rho\,\tau_c+Q_ck_4\right)k_1\\
&\quad -\left(\mu\,k_2+k_2k_1\right)k_3k_4k_5;\\
\Psi_{02} &= \left(\beta_c\mu\,\eta_2+\mu_dk_3-k_1k_3\right)k_4k_5\\
&\quad +rk_3\left(\rho\,\tau_d+Q_dk_4\right);\\
\Psi_{01} &= \left(\beta_c\mu\,\eta_2\left(k_1+k_2\right)+\mu_dk_2k_3\right)k_4k_5;\\
&\quad +rk_3\left(\rho\,\tau_d+Q_dk_4\right)k_2-\left(\mu+k_2\right)k_1k_3k_4k_5;\\
\Psi_{00} &= \mu\,k_1k_3k_4k_5\left(\beta_c-k_2\right)
\end{aligned}$$

The model (1) has the following equilibrium points of interests:

1. The Vice-free equilibrium $\mathscr{E}_0 = \left(\frac{\Lambda}{\mu},\ 0,\ 0,\ 0,\ 0,\ 0\right)$. At this equilibrium, there is neither drug abuse nor crime and only susceptible population exists.

2. The Crime-free equilibrium $\mathscr{E}_{CFE} = \left(S^{CFE},\ D^{CFE},\ 0,\ 0,\ T^{CFE},\ R^{CFE}\right)$. At this equilibrium, individuals do not commit crime and hence $C = DC = 0$ and hence $\lambda_2 = 0$. The drug-abuse initiation force at this equilibrium then becomes

$$\lambda_1^{CFE} = \frac{\mu k_1 k_4 k_5 \left(\mathscr{R}_{D0} - 1\right)}{k_4 k_5 \left(Q_d + \tau_d + \mu\right) - r\left(\rho\, \tau_d + Q_d k_4\right)}$$

and the components of $\mathscr{E}_{CFE}$ are given by

$$\left.\begin{aligned} S^{CFE} &= \tfrac{k_5 k_4 k_1 \Lambda}{\mu k_1 k_4 k_5 - [r(\rho\, \tau_d + Q_d k_4) - k_5 k_4 k_1]\lambda_1^{CFE}}, \\ D^{CFE} &= \tfrac{\lambda_1^{CFE} S^{CFE}}{k_1}, \\ T^{CFE} &= \tfrac{\tau_d \lambda_1^{CFE} S^{CFE}}{k_1 k_4}, \\ R^{CFE} &= \tfrac{S^{CFE} \lambda_1^{CFE}}{k_5}\left(\tfrac{Q_d}{k_1} + \tfrac{\rho\, \tau_d}{k_1 k_4}\right). \end{aligned}\right\}$$

3. The Drug-free equilibrium $\mathscr{E}_{DFE} = \left(S^{DFE},\ 0,\ C^{DFE},\ 0,\ T^{DFE},\ R^{DFE}\right)$. At this equilibrium, individuals do not abuse drugs and hence $D = DC = 0$ and hence $\lambda_1 = 0$. The criminal-initiation force at this equilibrium then becomes

$$\lambda_2^{DFE} = \frac{\mu k_2 k_4 k_5 \left(\mathscr{R}_{C0} - 1\right)}{k_4 k_5 \left(Q_c + \tau_c + \mu\right) - r\left(\rho\, \tau_c + Q_c k_4\right)}$$

and the components of $\mathscr{E}_{DFE}$ are given by

$$\left.\begin{aligned} S^{DFE} &= \tfrac{k_5 k_4 k_2 \Lambda}{\mu k_2 k_4 k_5 + [k_5 k_4 k_2 - r(\rho\, \tau_c + Q_c k_4)]\lambda_2^{DFE}}; \\ C^{DFE} &= \tfrac{1}{k_2}\lambda_2^{DFE} S^{DFE}, \\ T^{DFE} &= \tfrac{\tau_c}{k_2 k_4}\lambda_2^{DFE} S^{DFE}, \\ R^{DFE} &= \left(\tfrac{Q_c}{k_2 k_5} + \tfrac{\rho\, \tau_c}{k_5 k_4 k_2}\right) S^{DFE} \lambda_2^{DFE} \end{aligned}\right\}$$

4. The Vice-persistent equilibrium, $\mathscr{E}^*_{VPE} = \left(S^{VPE},\ D^{VPE},\ C^{VPE},\ DC^{VPE},\ T^{VPE},\ R^{VPE}\right)$, which is not explicitly determined because of its mathematical intractability.

Basic Initiation Numbers

In infectious disease models, the basic reproduction number, $\mathscr{R}_0$ is a threshold quantity that measures the average number of secondary infections that are caused by a single infectious individual throughout its period of infectiousness, when it is introduced into an initially infection-free population. The quantity is very important as it helps give an indication of possibility of eradication or otherwise of an infection. Normally, when $\mathscr{R}_0 < 1$, the disease is expected to die out after some time as less and less people get infected over time. In this paper, a threshold quantity called **basic initiation number** that is akin to the basic reproduction number is introduced. The technique of [1] is employed to determine $\mathscr{R}_0$ for the sub-models and the full model. The infected state of the Drug-only model is given by

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \lambda_1 S - k_1 D$$

so that the transmission and transition matrices are respectively given by

$$\mathscr{F}_D = \beta_d, \quad \text{and} \quad \mathscr{V}_D = k_1,$$

so that the basic imitation number of the Drug-only model is given by

$$\mathcal{R}_{D0} = \mathcal{F}_D \mathcal{V}_D^{-1} = \frac{\beta_d}{k_1}$$

Similarly, the basic initiation number for the Crime-only model is given by

$$\mathcal{R}_{C0} = \mathcal{F}_C \mathcal{V}_C^{-1} = \frac{\beta_c}{k_2}$$

To find the basic initiation number of the Drug-Crime model we consider the activity related compartmental equations as follows

$$\begin{aligned}
\tfrac{\mathrm{d}D}{\mathrm{d}t} &= \lambda_1 S - (\lambda_2 + Q_d + \tau_d + \mu_d + \mu)\, D; \\
\tfrac{\mathrm{d}C}{\mathrm{d}t} &= \lambda_2 S - (\lambda_1 + Q_c + \tau_c + \mu_c + \mu)\, C; \\
\tfrac{\mathrm{d}DC}{\mathrm{d}t} &= \lambda_1 C + \lambda_2 D - [\tau_{dc} + \mu + \alpha\mu_c + (1-\alpha)\mu_d]\, DC.
\end{aligned}$$

so that the transmission and transision matrices are respectively given by

$$\mathcal{F} = \begin{bmatrix} \beta_d - k_1 & 0 & \beta_d \eta_1 \\ 0 & \beta_c - k_2 & \beta_c \eta_2 \\ 0 & 0 & -k_3 \end{bmatrix}, \text{ and } \mathcal{V} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

The basic initiation number of the full model can be shown to be given by

$$\mathcal{R}_0 = \max\{1,\ \mathcal{R}_{D0} - 1,\ \mathcal{R}_{C0} - 1\}.$$

Remark 1. The Crime-free $\mathcal{E}_{CFE}$ and Drug-free equilibria $\mathcal{E}_{DFE}$ exists only when the alternate basic initiation numbers are more than unity.

Proof.
It is easy to see that the denominators of λ_1^{CFE} and λ_2^{DFE} are positive and hence λ_1^{CFE} and λ_2^{DFE} are positive only when $\mathcal{R}_{D0} > 1$ and $\mathcal{R}_{C0} > 1$, concluding the proof. □

It is noted that $\mathcal{R}_0 \geq 1$ for the model (1), and the following remark is thus established.

Remark 2. Drug-abuse and drug-related crime are persistent.

Local Stability of Equilibria

In this section, the centre manifold theory as discussed in [2] is used to study the local dynamics of the sub-models and the full Drug-Crime model.
Considering the Drug-only model and choosing $\beta_d^* = k_1$ as the bifurcation parameter, we observe that the Jacobian of the Drug-only model evaluated at $\left(\beta_d^*,\ \mathcal{E}_0\right)$ is given by

$$\mathcal{J}\left(\beta_d^*,\ \mathcal{E}_0\right) = \begin{bmatrix} -\mu & -\beta_d & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & \tau_d & -k_4 & 0 \\ 0 & Q_d & \rho & -k_5 \end{bmatrix}$$

It is easy to see that $\mathcal{J}\left(\beta_d^*,\ \mathcal{E}_0\right)$ has a simple zero eigenvalue whose right and left eigenvectors are respectively given by

$$\mathbf{w} = \begin{bmatrix} \frac{r(\rho\,\tau_d + Q_d k_4) - k_1 k_4 k_5}{k_4 k_5 \mu}, & 1, & \frac{\tau_d}{k_4}, & \frac{\rho\,\tau_d + Q_d k_4}{k_4 k_5} \end{bmatrix}^T, \text{ and } \mathbf{v} = \begin{bmatrix} 0, & 1, & 0, & 0 \end{bmatrix}.$$

From [2], the bifurcation parameters are calculated as follows:

$$\mathbf{a} = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}\left(\beta_d^*, \mathscr{E}_0\right), \quad \text{and} \quad \mathbf{b} = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right),$$

where $x_1 = S,\ x_2 = D,\ x_3 = T,\ x_4 = R,\ f_1 = \frac{\mathrm{d}S}{\mathrm{d}t},\ f_2 = \frac{\mathrm{d}D}{\mathrm{d}t},\ f_3 = \frac{\mathrm{d}T}{\mathrm{d}t}$ and $f_4 = \frac{\mathrm{d}R}{\mathrm{d}t}$.
The only non-zero derivatives in the expression of **a** are

$$\frac{\partial^2 f_2}{\partial x_2^2} = -\frac{2\beta_d \mu}{\Lambda},\ \frac{\partial^2 f_2}{\partial x_2 \partial x_3} = -\frac{\beta_d \mu}{\Lambda},\ \frac{\partial^2 f_2}{\partial x_2 \partial x_4} = -\frac{\beta_d \mu}{\Lambda}$$

The only non-zero derivatives in the expression of **b** are $\frac{\partial^2 f_1}{\partial x_2 \partial \beta_d^*} = -1,\ \frac{\partial^2 f_2}{\partial x_2 \partial \beta_d^*} = 1$.
Now, since all but $v_2 = 1$ are zero, the bifurcation coefficient **a** is given by

$$\begin{aligned}
\mathbf{a} &= \sum_{i,j=1}^{4} v_2 w_i w_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}\left(\beta_d^*, \mathscr{E}_0\right) \\
&= w_1\left[w_1 \frac{\partial^2 f_2}{\partial {x_1}^2}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial x_1 \partial x_3}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial x_1 \partial x_4}\left(\beta_d^*, \mathscr{E}_0\right)\right] \\
&+w_2\left[w_1 \frac{\partial^2 f_2}{\partial x_2 \partial x_1}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 \frac{\partial^2 f_2}{\partial {x_2}^2}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial x_2 \partial x_3}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial x_2 \partial x_4}\left(\beta_d^*, \mathscr{E}_0\right)\right] \\
&+w_3\left[w_1 \frac{\partial^2 f_2}{\partial x_3 \partial x_1}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 \frac{\partial^2 f_2}{\partial x_3 \partial x_2}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial {x_3}^2}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial x_3 \partial x_4}\left(\beta_d^*, \mathscr{E}_0\right)\right] \\
&+w_4\left[w_1 \frac{\partial^2 f_2}{\partial x_4 \partial x_1}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 \frac{\partial^2 f_2}{\partial x_4 \partial x_2}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial x_4 \partial x_3}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial {x_4}^2}\left(\beta_d^*, \mathscr{E}_0\right)\right] \\
&= w_2\left[w_2 \frac{\partial^2 f_2}{\partial {x_2}^2}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial x_2 \partial x_3}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial x_2 \partial x_4}\left(\beta_d^*, \mathscr{E}_0\right)\right] \\
&= w_2^2 \frac{\partial^2 f_2}{\partial {x_2}^2}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 w_3 \frac{\partial^2 f_2}{\partial x_2 \partial x_3}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 w_4 \frac{\partial^2 f_2}{\partial x_2 \partial x_4}\left(\beta_d^*, \mathscr{E}_0\right) \\
&= -2w_2^2 \frac{\beta_d \mu}{\Lambda} - w_2 w_3 \frac{\beta_d \mu}{\Lambda} - w_2 w_4 \frac{\beta_d \mu}{\Lambda} \\
&= -\frac{\beta_d \mu}{\Lambda}\left(2 + w_3 + w_4\right) = -\frac{\beta_d \mu}{\Lambda}\left(2 + \frac{\tau_d}{k_4} + \frac{\rho\,\tau_d + Q_d k_4}{k_4 k_5}\right) < 0
\end{aligned}$$

Similarly, the bifurcation parameter, **b** becomes

$$\begin{aligned}
\mathbf{b} &= \sum_{i=1}^{4} v_2 w_i \frac{\partial^2 f_2}{\partial x_i \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) \\
&= w_1 \frac{\partial^2 f_2}{\partial x_1 \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) + w_2 \frac{\partial^2 f_2}{\partial x_2 \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) + w_3 \frac{\partial^2 f_2}{\partial x_3 \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) + w_4 \frac{\partial^2 f_2}{\partial x_4 \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) \\
&= w_2 \frac{\partial^2 f_2}{\partial x_2 \partial \beta_d^*}\left(\beta_d^*, \mathscr{E}_0\right) = 1 > 0.
\end{aligned}$$

Using item (iv) of **Theorem 4.1** of Castillo-Chavez and Song [2], the following result is established.

Lemma 2. [Local Dynamics of Drug-Crime Model and Sub-models]

1. When $\mathscr{R}_{D0} < 1$, the Drug-only model has a locally asymptotically stable, vice-free equilibrium, and when $\mathscr{R}_{D0} > 1$, then $\mathscr{E}_0$ loses its stability to the drug-persistent equilibrium $\mathscr{E}^{CFE}$.

2. Same dynamics are observed for the drug-free equilibrium point $\mathscr{E}^{DFE}$.

3. Both Drug-only and Crime-only models exhibit forward bifurcation at their respective basic initiation numbers being unity.

The implication from Lemma 2 is that reducing the associated initiation numbers below unity will eventually eradicate Drug and Crime in the population.
Next, the local stability of the vice-free equilibrium of the full model is discussed, using the Centre manifold

theory [2]. Chosing $\mathcal{R}_{D0}$ as the bifurcation parameter (or alternatively $\beta_d^* = k_1$ such that $\mathcal{R}_{D0} = 1$), the Jacobian of the full model (1) evaluated at $\mathcal{E}_0$ is given by

$$\mathcal{J}\left(\mathcal{E}_0, \beta_d^*\right) = \begin{bmatrix} -\mu & -k_1 & -\beta_c & -\beta_c\eta_2 - k_1\eta_1 & 0 & r \\ 0 & 0 & 0 & k_1\eta_1 & 0 & 0 \\ 0 & 0 & \beta_c - k_2 & \beta_c\eta_2 & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 & 0 \\ 0 & \tau_d & \tau_c & \tau_{dc} & -k_4 & 0 \\ 0 & Q_d & Q_c & 0 & \rho & -k_5 \end{bmatrix}$$

whose left and right eigenvectors are given by $v = \begin{pmatrix} 0, & v_2, & 0, & \frac{k_1\eta_1 v_2}{k_3}, & 0, & 0 \end{pmatrix}$ and $w = \begin{pmatrix} \frac{w_2(r(\rho\,\tau_d + Q_d k_4) - k_5 k_4 k_1)}{k_4 k_5 \mu}, & w_2, & 0, & 0, & \frac{\tau_d w_2}{k_4}, & \frac{w_2(\rho\,\tau_d + Q_d k_4)}{k_4 k_5} \end{pmatrix}^T$, where v_2 and w_2 are chosen to satisfy $w \bullet v = 1$.

The bifurcation coefficients of the full model are thus obtained as

$$\mathbf{a} = -2\,\frac{\mu\, k_1 [(Q_d + k_5)k_4 + \tau_d(\rho + k_5)] v_2 {w_2}^2}{k_4 k_5 \Lambda}, \text{ and } \mathbf{b} = w_2 v_2 = 1$$

From (iv) of Theorem 4.1 of [2], since $\mathbf{a} < 0$, $\mathbf{b} > 0$, we can conclude that the vice-free equilibrium of the drug-crime model when $\mathcal{R}_{D0} < 1$ is asysmptotically stable but losses its when $\mathcal{R}_{D0} > 1$. The model exhibits forward bifurcation.

Sensitivity Analysis

We present the sensitivities of the basic initiation numbers of the sub-models to marginal changes in the model parameters using the normalised forward sensitivity index defined as follows:

$$\Upsilon_y^{p_i} = \frac{\partial y}{\partial p_i} \times \frac{p_i}{y}.$$

where y depends functionally on parameter p_i. Thus, the sensitivity indices of the basic initiation numbers relative to the parameters are given as follows:

$$\begin{aligned} &\Upsilon_{\mathcal{R}_{D0}}^{\beta_d} = 1, \quad \Upsilon_{\mathcal{R}_{C0}}^{\beta_c} = 1, \\ &\Upsilon_{\mathcal{R}_{D0}}^{p_i} = \frac{-p_i}{k_1},\ p_i \in \{Q_d,\ \tau_d,\ \mu_d,\ \mu\}\ , \\ &\Upsilon_{\mathcal{R}_{C0}}^{q_i} = \frac{-q_i}{k_2},\ q_i \in \{Q_c,\ \tau_c,\ \mu_c,\ \mu\}\ . \end{aligned}$$

Using the values in Table 1, the numerical sensitivity indexes are obtained as follows:

$$\begin{aligned} &\Upsilon_{\mathcal{R}_{D0}}^{\beta_d} = 1, && \Upsilon_{\mathcal{R}_{C0}}^{\beta_c} = 1, \\ &\Upsilon_{\mathcal{R}_{D0}}^{Q_d} = -0.045209, && \Upsilon_{\mathcal{R}_{D0}}^{\tau_d} = -0.951643, && \Upsilon_{\mathcal{R}_{D0}}^{\mu_d} = -0.0000085467, && \Upsilon_{\mathcal{R}_{D0}}^{\mu} = -0.00313949, \\ &\Upsilon_{\mathcal{R}_{C0}}^{Q_c} = -0.71675904, && \Upsilon_{\mathcal{R}_{C0}}^{\tau_c} = -0.14335181, && \Upsilon_{\mathcal{R}_{C0}}^{\mu_c} = -0.0403392, && \Upsilon_{\mathcal{R}_{C0}}^{\mu} = -0.09554995. \end{aligned}$$

The results here mean that a 10% increase (decrease) in β_d or β_c will lead to a 10% increase(decrease) in $\mathcal{R}_{D0}$ or $\mathcal{R}_{C0}$ respectively. Also, a 10% increase (decrease) in Q_d or Q_c will lead to a $\frac{Q_d}{k_1}\%$ or $\frac{Q_c}{k_2}\%$ decrease(increase) in $\mathcal{R}_0$ respectively. This indicates that, in order to control the spread of Drug and Drug-related Crimes, efforts should be made to decrease β_d,β_c,Q_d and Q_c. We can reduce β_d and β_c by increasing η_1 and η_2 respectively. Moreso, inncrease in Q_d, Q_c, τ_d, τ_c, μ_d, μ_c and μ respectively will reduce $\mathcal{R}_0$ thereby controlling the menance. However, increasing μ_d, μ_c and μ is undesirable and should not be employed.

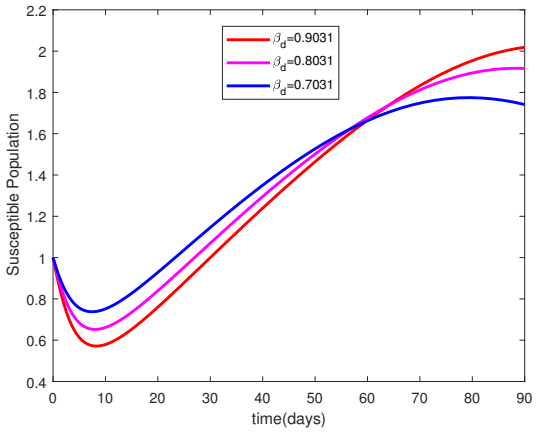
Numerical Simulation

The model (1) was numerically solved using the fourth-order Runge-Kutta scheme with the parameter values in Table 1. The numerical simulation is performed to illustrate some of the analytical results established and to study the impact of some model parameters on the Drug-Crime dynamics. THe simulation results are presented graphically and the trends observed.

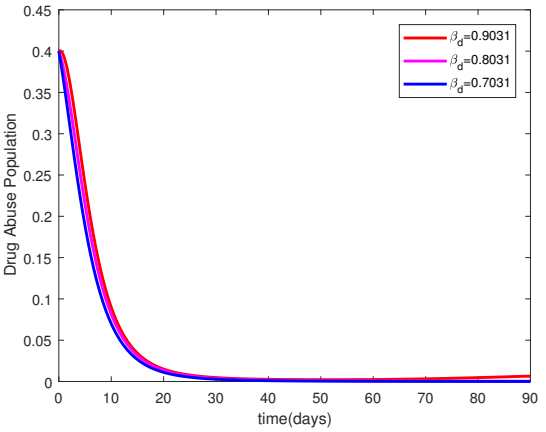
Parameter	Value	Source
Λ	0.025	GSS2010
α	0.001	Assumed
β_c	0.09	[12]
β_d	0.9031	[11]
η_1		Variable
η_2		Variable
μ	$\frac{1}{720}$/month	Estimated from 60years average lifespan
μ_c	5.628×10^{-4}/month	Estimated from crime death rate in 2015 [4]
μ_d	3.781×10^{-6}/month	Estimated from drug-related deaths in 2019[3]
Q_c	0.01	Assumed
Q_d	0.02	Assumed
ρ	0.003	Assumed
τ_c	0.002	Assumed
τ_d	0.4210	[11]
τ_{dc}	0.6420	Assumed

Table 1: Definition and baseline values of Drug-Crime Model Parameters

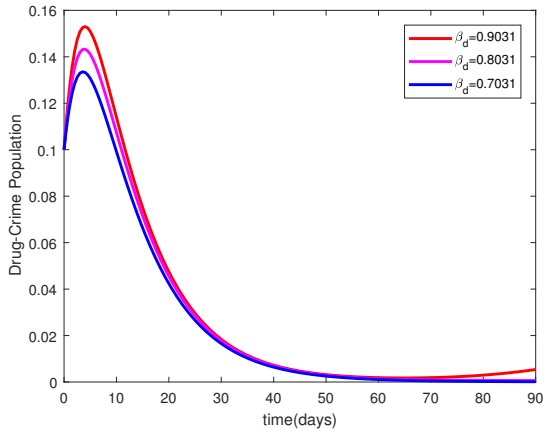
Figure 1 shows a time series plot of susceptible, drug abuse and drug-crime populations. It is observed in (1a) that the susceptible population increases exponentially towards the disease free equilibrium point with time as β_d decreases. The drug abuse and those involved in both drug abuse and crime however increase exponentially towards the endemic equilibrium point with time as β_d increases as seen in (1b) and (1c) respectively.



(a) Time series plots of the effect of varying β_d on Susceptible Population



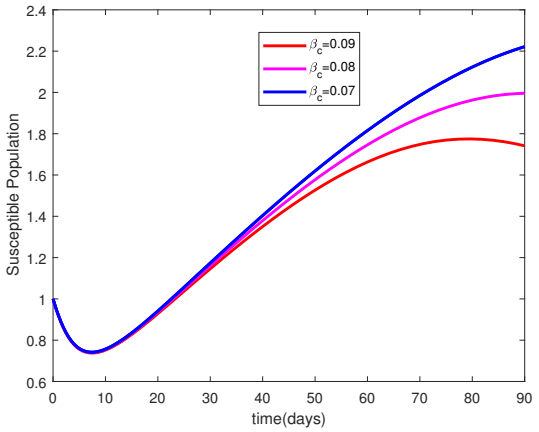
(b) Time series plots of the effect of varying β_d on Drug Abuse Population



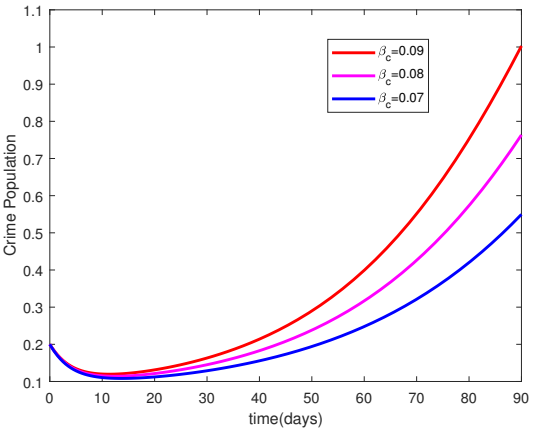
(c) Time series plots of the effect of varying β_d on Drug-Crime Population

Figure 1: Impact of varying β_d on Susceptible, Drug Abuse and Drug-Crime Populations

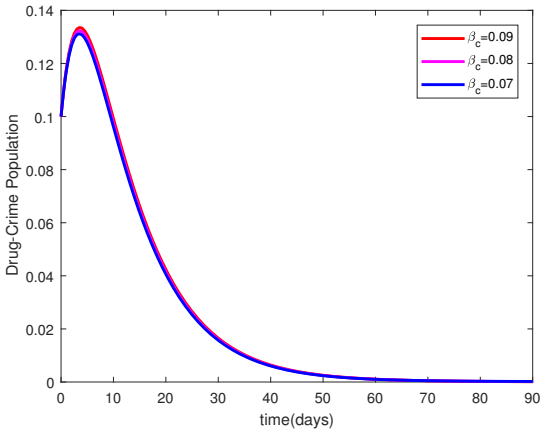
Figure 2 indicates time series plots of susceptible, Crime and Drug-Crime populations. It is observed in (2a) that the susceptible population increases towards crime free equilibrium point as β_c decreases. On the other hand, both Crime and Drug-Crime populations increases towards the endemic equilibrium points as β_c increases as observed in (2b) and (2c) respectively.



(a) Time series plots of the effect of varying β_c on Susceptible Population



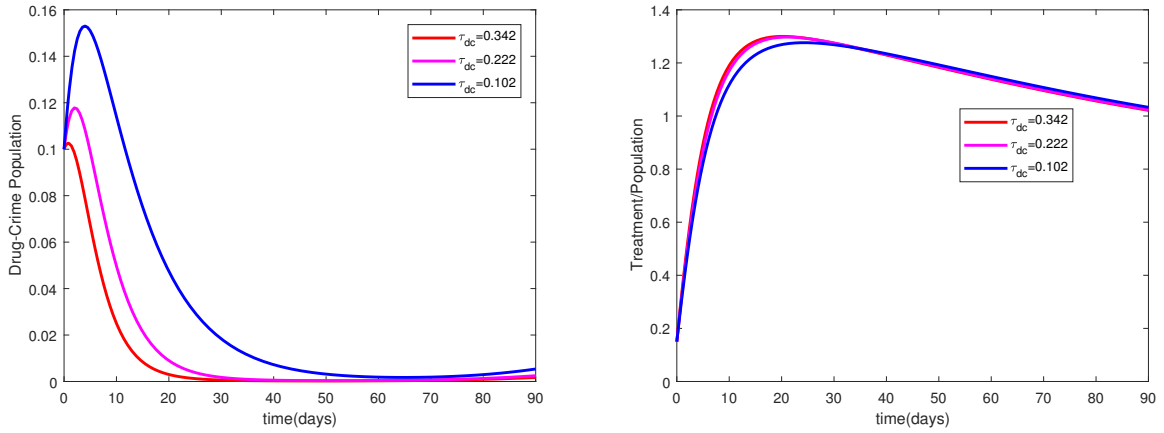
(b) Time series plots of the effect of varying β_c on Drug Abuser Population



(c) Time series plots of the effect of varying β_c on Drug-Crime Population

Figure 2: Impact of varying β_c on Susceptible, Drug Abuser and Drug-Crime Populations

Figure 3 indicates time series plots of Drug-Crime and Treatment/Rehabilitation populations. It is observed in (3a) that as τ_{dc} increases the Drug-Crime populations decreases towards the activity free equilibrium point. However,the same increase in τ_{dc} as observed in (3b) corresponds to increase in Treatment/Rehabilitation population.



(a) Time series plots of the effect of varying τ_{dc} on Susceptible Population

(b) Time series plots of the effect of varying τ_{dc} on Drug Abuser Population

Figure 3: Impact of varying τ_{dc} on Drug-Crime and Populations

Conclusion

We developed and analysed Drug Abuse-Crime model and its sub-models with six-state compartmental model.The drug and alcohol-abuse-free equilibria of the model and its sub-models are shown to be asymptotically stable when the respective basic initiation numbers are below unity(ie. $\mathcal{R}_0 < 1$).The models is shown to exhibit forward bifurcation at $\mathcal{R}_0 = 1$. It is observed that there are two parameters, namely; β_d and β_c with the most influence on the basic initiation number; $\mathcal{R}_0$. These parameters are associated with contact of susceptibles with drug-abusers and those that commit crime. Therefore, minimizing the values of these parameters can help to control the spread of Drug abuse and Crime in the community. The numerical simulation indicates that increasing τ_{dc} reduces the number of the Drug-Crime population. If Q_d, Q_c, are increased those involved in Drug abuse and Crime will be reduced.

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Conflict of Interest Statement

The authors declare that there is no conflict of interest regarding the manuscript.

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