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Iteration of Three Entire Functions With Finite Iterated Order Sharing Three Values IM

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Abstract
Considering the relative iterations of three entire functions we investigate about the growth of iterated entire functions with finite iterated order and positive iterated lower order sharing three values IM on the base of the paper by Banerjee, Mandal [1].

Keywords: Iterated order, Iterated lower order, Relative iteration, Value sharing.

Introduction

If $f(z)$ be an entire function, in [6] Kinnunen introduced the notion of iterated order and iterated lower order as follows.

Definition 1.1: The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f) = \limsup_{r\to\infty} \frac{\log^{[i+1]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[i]} T(r,f)}{\log r}, \ (i \in \mathbb{N}).$$

Similarly, the iterated i lower order $\mu_i(f)$ of f is defined by

$$\mu_i(f) = \liminf_{r\to\infty} \frac{\log^{[i+1]} M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[i]} T(r,f)}{\log r}, \ (i \in \mathbb{N}).$$

Definition 1.2 {[2], [6]}: The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0, & \text{if } f(z) \text{ is a polynomial;} \\ \min\{k \in \{1,2,3,\cdots\}, \rho_k(f) < \infty\}, & \text{if } f(z) \text{ is transcendental;} \\ \infty, & \text{when } \rho_k(f) = \infty, \forall k. \end{cases}$$

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We use the standard notations and definitions of Nevanlinna's theory available in [5]. Also we mean that $f(z)$ is entire function having finite iterated order if $\rho_p(f) < \infty$ and positive iterated lower order if $\mu_p(f) > 0$ for some $p \in \mathbb{N}$.

For three entire functions $f(z)$, $g(z)$ and $h(z)$, we are introducing the idea of relative iterations of $f(z)$ with respect to $g(z)$ and $h(z)$ respectively as follows.

$$F_0(z) = z$$

$$F_1(z) = f(z)$$

$$F_2(z) = f\left(g(z)\right)$$

$$F_3(z) = f\left(g\left(h(z)\right)\right)$$

$$F_4(z) = f\left(g\left(h\left(f(z)\right)\right)\right)$$

$$F_5(z) = f\left(g\left(h\left(f\left(g(z)\right)\right)\right)\right)$$

$$F_6(z) = f\left(g\left(h\left(f\left(g\left(h(z)\right)\right)\right)\right)\right)$$

$$F_7(z) = f\left(g\left(h\left(f\left(g\left(h\left(f(z)\right)\right)\right)\right)\right)\right)$$

$$\vdots$$

$$\text{and so } F_n(z) = \begin{cases} F_k\left(F_{n-k}(z)\right), & \text{if } k = 3k_1 \\ F_k\left(G_{n-k}(z)\right), & \text{if } k = 3k_1 + 1 \\ F_k\left(H_{n-k}(z)\right), & \text{if } k = 3k_1 + 2, \end{cases}$$

where $k \leqslant n,\ n \geqslant 3$ and $k_1 \in \mathbb{N}$.

Similarly relative iteration of $g(z)$ with respect to $h(z)$ and $f(z)$ respectively is as follows.

$$G_0(z) = z$$

$$G_1(z) = g(z)$$

$$G_2(z) = g\left(h(z)\right)$$

$$G_3(z) = g\left(h\left(f(z)\right)\right)$$

$$\vdots$$

$$\text{and so } G_n(z) = \begin{cases} G_k\left(G_{n-k}(z)\right), & \text{if } k = 3k_1 \\ G_k\left(H_{n-k}(z)\right), & \text{if } k = 3k_1 + 1 \\ G_k\left(F_{n-k}(z)\right), & \text{if } k = 3k_1 + 2, \end{cases}$$

where $k \leqslant n,\ n \geqslant 3$ and $k_1 \in \mathbb{N}$.

And relative iteration of $h(z)$ with respect to $f(z)$ and $g(z)$ respectively is as follows.

$$H_0(z) = z$$

$$H_1(z) = h(z)$$

$$H_2(z) = h\left(f(z)\right)$$

$$H_3(z) = h\left(f\left(g(z)\right)\right)$$

$$\vdots$$

$$\text{and so } H_n(z) = \begin{cases} H_k\left(H_{n-k}\left(z\right)\right), & \text{if } k = 3k_1 \\ H_k\left(F_{n-k}\left(z\right)\right), & \text{if } k = 3k_1 + 1 \\ H_k\left(G_{n-k}\left(z\right)\right), & \text{if } k = 3k_1 + 2, \end{cases}$$

where $k \leqslant n,\ n \geqslant 3$ and $k_1 \in \mathbb{N}$.

Definition 1.3: We say that three non-constant meromorphic functions $f(z)$, $g(z)$ and $h(z)$ mutually share a finite complex value 'a' CM (or IM) if $f(z)$ and $g(z)$ share 'a' CM (or IM) and if $g(z)$ and $h(z)$ share 'a' CM (or IM) and if $h(z)$ and $f(z)$ share 'a' CM (or IM).

Example 1.1: Take $f(z) = z$, $g(z) = z^2$, $h(z) = z^3$. Then f, g, h mutually share 0 IM.

Example 1.2: Take $f(z) = z$, $g(z) = 2z$, $h(z) = 3z$. Then f, g, h mutually share 0 CM.

Known Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1 [4]: If f and g share three values IM, then

$$\frac{1}{3}T(r,g)\left(1+o(1)\right) \leqslant T(r,f) \leqslant 3T(r,g)\left(1+o(1)\right) \text{ as } r \to \infty \text{ and } r \notin E,$$

where E is a set of finite linear measure.

Lemma 2.2 [7]: Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\epsilon}{\epsilon} \mid g(0) \mid$ for any $\epsilon > 0$, then

$$T\left(r, f\left(g\right)\right) < \left(1+\epsilon\right) T\left(M\left(r,g\right), f\right).$$

Lemma 2.3 [5]: Let $f(z)$ and $g(z)$ be entire functions. Then $\forall$ sufficiently large values of r

$$M\left(r, f\left(g\right)\right) \leqslant M\left(M\left(r,g\right), f\right).$$

Lemma 2.4 [5]: Let $f(z)$ be an entire function. Then $\forall$ sufficiently large values of r

$$T(r,f) \leqslant \log M(r,f) \leqslant 3T(2r,f).$$

Lemma 2.5 [3]: Let $f(z)$ and $g(z)$ be entire functions. Then $\forall$ sufficiently large values of r

$$M(r,f(g)) \geqslant M\left(\frac{1}{9}M\left(\frac{r}{2},g\right),f\right).$$

Main Results

Our main results are the following theorems.

Theorem 3.1 : Let $f(z)$, $g(z)$ and $h(z)$ be three non-constant entire functions. If F_k, G_k and H_k $(k \geqslant 3)$ are of finite iterated order, positive iterated lower order and mutually share three values IM, then for any $n\ (\geqslant k)$

$$\rho_{mp}(F_n) = \rho_{mp}(G_n) = \rho_{mp}(H_n), \text{ if } n = mk;\ m,\ n,\ p = i(F_k) \in \mathbb{N}.$$

Proof : Since F_k and G_k share three values IM, we have from Lemma 2.1

$$\frac{1}{3}T(r,G_k)(1+o(1)) \leqslant T(r,F_k) \leqslant 3T(r,G_k)(1+o(1)) \text{ as } r \to \infty \text{ and } r \notin E.$$

So there exists $p_1 \in \mathbb{N}$ such that

$$0 < \rho_{p_1}(G_k) \leqslant \rho_{p_1}(F_k) \leqslant \rho_{p_1}(G_k) < \infty.$$

$$\text{Thus } \rho_{p_1}(F_k) = \rho_{p_1}(G_k) > 0. \tag{3.1}$$

Again since G_k and H_k share three values IM, from Lemma 2.1 we similarly obtain a $p_2 \in \mathbb{N}$ such that

$$\rho_{p_2}(G_k) = \rho_{p_2}(H_k) > 0. \tag{3.2}$$

From (1) and (2) it is clear that $p_1 = p_2 = p$ (say).

Hence $\rho_p(F_k) = \rho_p(G_k) = \rho_p(H_k)$. $\quad$ (3.3)

Also we have for given $\epsilon(>0)$ and for sufficiently large r

$$\left.\begin{aligned} T(r,F_k) &\leqslant exp^{[p-1]}\left\{r^{\rho_p(F_k)+\epsilon}\right\}; \\ M(r,F_k) &\leqslant exp^{[p]}\left\{r^{\rho_p(F_k)+\epsilon}\right\}, \\ T(r,G_k) &\leqslant exp^{[p-1]}\left\{r^{\rho_p(G_k)+\epsilon}\right\}; \\ M(r,G_k) &\leqslant exp^{[p]}\left\{r^{\rho_p(G_k)+\epsilon}\right\}, \\ \text{and } T(r,H_k) &\leqslant exp^{[p-1]}\left\{r^{\rho_p(H_k)+\epsilon}\right\}; \\ M(r,H_k) &\leqslant exp^{[p]}\left\{r^{\rho_p(H_k)+\epsilon}\right\}. \end{aligned}\right\} \tag{3.4}$$

Again since $\mu_p(F_k) > 0$, $\exists$ a sequence $\{r_l\}$ tending to infinity such that for given ϵ $\left(o < \epsilon < \mu_p(F_k) \leqslant \rho_p(F_k)\right)$ and for sufficiently large r_l, we have

$$\log^{[p+1]} M(r_l, F_k) \geqslant \left(\rho_p(F_k) - \epsilon\right) \log r_l$$

$$\left.\begin{array}{r} \text{i.e., } M(r_l, F_k) \geqslant \exp^{[p]}\{r_l^{\rho_p(F_k)-\epsilon}\}. \\ \text{Similarly there exist sequences } \{s_l\} \text{ and } \{t_l\} \text{ s.t.} \\ M(s_l, G_k) \geqslant \exp^{[p]}\{s_l^{\rho_p(G_k)-\epsilon}\} \\ \text{and } M(t_l, H_k) \geqslant \exp^{[p]}\{t_l^{\rho_p(H_k)-\epsilon}\}. \end{array}\right\} \tag{3.5}$$

Also for sufficiently large r

$$\left.\begin{array}{r} M(r, F_k) \geqslant \exp^{[p]}\{r^{\mu_p(F_k)-\epsilon}\}, \\ M(r, G_k) \geqslant \exp^{[p]}\{r^{\mu_p(G_k)-\epsilon}\} \\ \text{and } M(r, H_k) \geqslant \exp^{[p]}\{r^{\mu_p(H_k)-\epsilon}\}. \end{array}\right\} \tag{3.6}$$

Now we consider the following cases.

Case 1 : Let $k = 3k_1, k_1 \in \mathbb{N}$. Then for sufficiently large r, using Lemma 2.2, Lemma 2.3 and (3.4), we have

$$\begin{aligned} T(r, F_n) &< 2\,T\left(M\left(r, F_{(m-1)k}\right), F_k\right) \\ &\leqslant 2\exp^{[p-1]}\left(M\left(r, F_{(m-1)k}\right)\right)^{\rho_p(F_k)+\epsilon} \\ &= 2\exp^{[p]}\left(\rho_p(F_k)+\epsilon\right)\log\left(M\left(r, F_{(m-1)k}\right)\right) \\ &\leqslant \exp^{[p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M\left(M\left(r, F_{(m-2)k}\right), F_k\right)\right) \\ &\leqslant \exp^{[p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(\exp^{[p]}\left(M\left(r, F_{(m-2)k}\right)\right)^{\rho_p(F_k)+\epsilon}\right) \\ &\leqslant \exp^{[2p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M\left(r, F_{(m-2)k}\right)\right) \\ &\qquad\vdots \\ &\leqslant \exp^{[(m-1)p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M(r, F_k)\right) \\ &\leqslant \exp^{[(m-1)p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(\exp^{[p]} r^{\rho_p(F_k)+\epsilon}\right) \\ &\leqslant \exp^{[mp]}\log\left(r^{\rho_p(F_k)+2\epsilon}\right). \end{aligned}$$

$$\text{So, } \frac{\log^{[mp]} T(r, F_n)}{\log r} < \rho_p(F_k) + 2\epsilon. \tag{3.7}$$

Now for opposite inequality we have for sufficiently large r_l, using Lemma 2.4, Lemma 2.5, (3.5) and (3.6)

$$\begin{aligned} T(r_l, F_n) &\geqslant \frac{1}{3}\log\left(M\left(\frac{r_l}{2}, F_{mk}\right)\right) \\ &\geqslant \frac{1}{3}\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^2}, F_{(m-1)k}\right), F_k\right)\right) \\ &\geqslant \frac{1}{3}\log\exp^{[p]}\left(\frac{1}{9}M\left(\frac{r_l}{2^2}, F_{(m-1)k}\right)\right)^{\mu_p(F_k)-\epsilon} \\ &\geqslant \exp^{[p]}\left(\mu_p(F_k)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^2}, F_{(m-1)k}\right)\right) \\ &\geqslant \exp^{[p]}\left(\mu_p(F_k)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^3}, F_{(m-2)k}\right), F_k\right)\right) \\ &\geqslant \exp^{[2p]}\left(\mu_p(F_k)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^3}, F_{(m-2)k}\right)\right) \end{aligned}$$

$$\vdots$$
$$\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^m},F_k\right)\right)$$
$$\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(\exp^{[p]}\left(\frac{r_l}{2^m}\right)^{\rho_p(F_k)-\epsilon}\right)$$
$$\geqslant \exp^{[mp]}\log\left(r_l^{\rho_p(F_k)-2\epsilon}\right).$$

So, $\dfrac{\log^{[mp]}T(r_l,F_n)}{\log r_l}\geqslant\rho_p\left(F_k\right)-2\epsilon.$ (3.8)

From (3.7) and (3.8) we have $\rho_{mp}\left(F_n\right)=\rho_p\left(F_k\right).$ (3.9)

Similarly we have $\rho_{mp}\left(G_n\right)=\rho_p\left(G_k\right)$ and $\rho_{mp}\left(H_n\right)=\rho_p\left(H_k\right).$ (3.10)

Hence from (3.3), (3.9) and (3.10)

$$\rho_{mp}\left(F_n\right)=\rho_{mp}\left(G_n\right)=\rho_{mp}\left(H_n\right).$$

Case 2 : Let $k=3k_1+1$, $k_1\in\mathbb{N}$. Then for sufficiently large r, using Lemma 2.2, Lemma 2.3 and (3.4), we have

$$\begin{aligned}T\left(r,F_n\right)&<2T\left(M\left(r,G_{(m-1)k}\right),F_k\right)\\&\leqslant 2\exp^{[p]}\left(\rho_p\left(F_k\right)+\epsilon\right)\log\left(M\left(r,G_{(m-1)k}\right)\right)\\&\leqslant \exp^{[p]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(M\left(M\left(r,H_{(m-2)k}\right),G_k\right)\right)\\&\leqslant \exp^{[p]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(\exp^{[p]}\left(M\left(r,H_{(m-2)k}\right)\right)^{\rho_p(G_k)+\epsilon}\right)\\&\leqslant \exp^{[2p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(M\left(M\left(r,F_{(m-3)k}\right),H_k\right)\right)\\&\leqslant \exp^{[3p]}\left(\rho_p\left(H_k\right)+2\epsilon\right)\log\left(M\left(r,F_{(m-3)k}\right)\right).\end{aligned}$$

Now we consider the following subcases.

Subcase 2.1 : Let $m=3k_2,\ k_2\in\mathbb{N}$. Then

$$\begin{aligned}T\left(r,F_n\right)&<\exp^{[(m-1)p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(M\left(r,H_k\right)\right)\\&\leqslant \exp^{[(m-1)p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(\exp^{[p]}r^{\rho_p(H_k)+\epsilon}\right)\\&\leqslant \exp^{[mp]}\log\left(r^{\rho_p(H_k)+2\epsilon}\right).\end{aligned}$$

Subcase 2.2 : Let $m=3k_2+1,\ k_2\in\mathbb{N}$. Then similarly as Subcase 2.1

$$T\left(r,F_n\right)<\exp^{[mp]}\log\left(r^{\rho_p(F_k)+2\epsilon}\right).$$

Subcase 2.3 : Let $m=3k_2+2,\ k_2\in\mathbb{N}$. Then similarly as Subcase 2.1

$$T\left(r,F_n\right)<\exp^{[mp]}\log\left(r^{\rho_p(G_k)+2\epsilon}\right).$$

Combining the above three subcases we have

$$\frac{\log^{[mp]}T(r,F_n)}{\log r}<\begin{cases}\rho_p\left(H_k\right)+2\epsilon; & \text{if } m=3k_2,k_2\in\mathbb{N}\\ \rho_p\left(F_k\right)+2\epsilon; & \text{if } m=3k_2+1,k_2\in\mathbb{N}\\ \rho_p\left(G_k\right)+2\epsilon; & \text{if } m=3k_2+2,k_2\in\mathbb{N}.\end{cases}\tag{3.11}$$

For the opposite inequality, proceeding as 2nd part of Case 1, using Lemma 2.4, Lemma 2.5, (3.5) and (3.6), for large r

$$\begin{aligned}
T\left(r, F_n\right) &\geqslant \frac{1}{3}\log\left(M\left(\frac{r}{2}, F_{mk}\right)\right) \\
&\geqslant \frac{1}{3}\log\left(M\left(\frac{1}{9}M\left(\frac{r}{2^2}, G_{(m-1)k}\right), F_k\right)\right) \\
&\geqslant \frac{1}{3}\log\exp^{[p]}\left(\frac{1}{9}M\left(\frac{r}{2^2}, G_{(m-1)k}\right)\right)^{\mu_p(F_k)-\epsilon} \\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r}{2^2}, G_{(m-1)k}\right)\right) \\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r}{2^3}, H_{(m-2)k}\right), G_k\right)\right) \\
&\geqslant \exp^{[2p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{r}{2^3}, H_{(m-2)k}\right)\right) \\
&\geqslant \exp^{[2p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r}{2^4}, F_{(m-3)k}\right), H_k\right)\right) \\
&\geqslant \exp^{[3p]}\left(\mu_p\left(H_k\right)-2\epsilon\right)\log\left(M\left(\frac{r}{2^4}, F_{(m-3)k}\right)\right).
\end{aligned}$$

Next we consider the following subcases.

Subcase 2.4 : Let $m = 3k_2,\ k_2 \in \mathbb{N}$. Then for sufficiently large t_l

$$\begin{aligned}
T\left(t_l, F_n\right) &\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{t_l}{2^m}, H_k\right)\right) \\
&\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(\exp^{[p]}\left(\frac{t_l}{2^m}\right)^{\rho_p(H_k)-\epsilon}\right) \\
&\geqslant \exp^{[mp]}\log\left({t_l}^{\rho_p(H_k)-2\epsilon}\right).
\end{aligned}$$

Subcase 2.5 : Let $m = 3k_2+1,\ k_2 \in \mathbb{N}$. Then similarly as Subcase 2.4, for sufficiently large r_l

$$T\left(r_l, F_n\right) \geqslant \exp^{[mp]}\log\left({r_l}^{\rho_p(F_k)-2\epsilon}\right).$$

Subcase 2.6 : Let $m = 3k_2+2,\ k_2 \in \mathbb{N}$. Then similarly as Subcase 2.4, for sufficiently large s_l

$$T\left(s_l, F_n\right) \geqslant \exp^{[mp]}\log\left({s_l}^{\rho_p(G_k)-2\epsilon}\right).$$

Combining the above three subcases we have

$$\left.\begin{aligned}
\frac{\log^{[mp]}T(t_l, F_n)}{\log t_l} &\geqslant \rho_p\left(H_k\right)-2\epsilon;\ \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\
\frac{\log^{[mp]}T(r_l, F_n)}{\log r_l} &\geqslant \rho_p\left(F_k\right)-2\epsilon;\ \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\
\frac{\log^{[mp]}T(s_l, F_n)}{\log s_l} &\geqslant \rho_p\left(G_k\right)-2\epsilon;\ \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N}.
\end{aligned}\right\} \tag{3.12}$$

From (3.11) and (3.12) we have

$$\rho_{mp}\left(F_n\right) = \begin{cases} \rho_p\left(H_k\right); & \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\ \rho_p\left(F_k\right); & \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\ \rho_p\left(G_k\right); & \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N}. \end{cases} \tag{3.13}$$

Similarly we have

$$\left.\begin{array}{l} \rho_{mp}\left(G_n\right) = \begin{cases} \rho_p\left(F_k\right); & \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\ \rho_p\left(G_k\right); & \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\ \rho_p\left(H_k\right); & \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N} \end{cases} \\ \text{and } \rho_{mp}\left(H_n\right) = \begin{cases} \rho_p\left(G_k\right); & \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\ \rho_p\left(H_k\right); & \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\ \rho_p\left(F_k\right); & \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N}. \end{cases} \end{array}\right\} \tag{3.14}$$

Hence from (3.3), (3.13) and (3.14)

$$\rho_{mp}\left(F_n\right) = \rho_{mp}\left(G_n\right) = \rho_{mp}\left(H_n\right).$$

Case 3 : Let $k = 3k_1 + 2,\ k_1 \in \mathbb{N}$. Then for sufficiently large r, proceeding as Case 2, using Lemma 2.2, Lemma 2.3 and (3.4), we have

$$\frac{\log^{[mp]} T(r, F_n)}{\log r} < \begin{cases} \rho_p\left(G_k\right) + 2\epsilon; & \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\ \rho_p\left(H_k\right) + 2\epsilon; & \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\ \rho_p\left(F_k\right) + 2\epsilon; & \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N}. \end{cases} \tag{3.15}$$

For the opposite inequality, proceeding as 2nd part of Case 2, we have for sufficiently large r_l, s_l and t_l

$$\left.\begin{array}{l} \dfrac{\log^{[mp]} T(s_l, F_n)}{\log s_l} \geqslant \rho_p\left(G_k\right) - 2\epsilon;\ \text{if } m = 3k_2, k_2 \in \mathbb{N} \\ \dfrac{\log^{[mp]} T(t_l, F_n)}{\log t_l} \geqslant \rho_p\left(H_k\right) - 2\epsilon;\ \text{if } m = 3k_2+1, k_2 \in \mathbb{N} \\ \dfrac{\log^{[mp]} T(r_l, F_n)}{\log r_l} \geqslant \rho_p\left(F_k\right) - 2\epsilon;\ \text{if } m = 3k_2+2, k_2 \in \mathbb{N}. \end{array}\right\} \tag{3.16}$$

From (3.15) and (3.16) we have

$$\rho_{mp}\left(F_n\right) = \begin{cases} \rho_p\left(G_k\right); & \text{if } m = 3k_2,\ k_2 \in \mathbb{N} \\ \rho_p\left(H_k\right); & \text{if } m = 3k_2+1,\ k_2 \in \mathbb{N} \\ \rho_p\left(F_k\right); & \text{if } m = 3k_2+2,\ k_2 \in \mathbb{N}. \end{cases} \tag{3.17}$$

Similarly we have

$$\left.\begin{array}{l} \rho_{mp}\left(G_n\right) = \begin{cases} \rho_p\left(H_k\right); & \text{if } m = 3k_2, k_2 \in \mathbb{N} \\ \rho_p\left(F_k\right); & \text{if } m = 3k_2+1, k_2 \in \mathbb{N} \\ \rho_p\left(G_k\right); & \text{if } m = 3k_2+2, k_2 \in \mathbb{N} \end{cases} \\ \text{and } \rho_{mp}\left(H_n\right) = \begin{cases} \rho_p\left(F_k\right); & \text{if } m = 3k_2, k_2 \in \mathbb{N} \\ \rho_p\left(G_k\right); & \text{if } m = 3k_2+1, k_2 \in \mathbb{N} \\ \rho_p\left(H_k\right); & \text{if } m = 3k_2+2, k_2 \in \mathbb{N}. \end{cases} \end{array}\right\} \tag{3.18}$$

Hence from (3.3), (3.17) and (3.18)

$$\rho_{mp}\left(F_n\right) = \rho_{mp}\left(G_n\right) = \rho_{mp}\left(H_n\right).$$

Hence the theorem.

Theorem 3.2 : Let $f(z)$, $g(z)$ and $h(z)$ be three non-constant entire functions. If F_k, G_k and H_k $(k \geqslant 3)$ are of finite iterated order, positive iterated lower order with $i(F_q) \neq i(G_q),\ i(F_q) \neq i(H_q),\ i(G_q) \neq i(H_q)$ and mutually share three values IM, then for any $n = mk + q$

(i) $\rho_{mp+i(F_q)}(F_n) + \rho_{mp+i(G_q)}(G_n) + \rho_{mp+i(H_q)}(H_n) = \rho_{i(F_q)}(F_q) + \rho_{i(G_q)}(G_q) + \rho_{i(H_q)}(H_q)$,
if mk is multiple of 3 and

(ii) $\rho_{mp+i(F_q)}(G_n) - \rho_{mp+i(F_q)}(H_n) = \rho_{mp+i(G_q)}(H_n) - \rho_{mp+i(G_q)}(F_n) = \rho_{mp+i(H_q)}(F_n) - \rho_{mp+i(H_q)}(G_n)$,
if mk is not a multiple of 3, where $p = i(F_q)$, $m,\ n,\ q\ \in \mathbb{N}$ with $1 \leqslant q < n$.

Proof : From Definition 1.1 and Definition 1.2 we have

$$\rho_{i(F_q)}(F_q) = \limsup_{r\to\infty} \frac{\log^{[i(F_q)+1]} M(r, F_q)}{\log r}.$$

So, for chosen $\epsilon\ (> 0)$ and for sufficiently large r as (3.4) we similarly have

$$\left.\begin{array}{r} M(r, F_q) \leqslant \exp^{[i(F_q)]} r^{\rho_{i(F_q)}(F_q)+\epsilon}. \\ \text{Similarly for } G_q \text{ and } H_q \text{ we have} \qquad\qquad \\ M(r, G_q) \leqslant \exp^{[i(G_q)]} r^{\rho_{i(G_q)}(G_q)+\epsilon} \\ \text{and } M(r, H_q) \leqslant \exp^{[i(H_q)]} r^{\rho_{i(H_q)}(H_q)+\epsilon}. \end{array}\right\} \qquad (3.19)$$

Again since $\rho_{i(F_q)}(F_q) > 0$, there exists a sequence $\{r_l\}$ tending to infinity such that for sufficiently large r_l, as (3.5) we similarly have

$$\left.\begin{array}{r} \log^{[p+1]} M(r_l, F_q) \geqslant \left(\rho_{i(F_q)}(F_q) - \epsilon\right) \log r_l \\ \text{i.e., } M(r_l, F_q) \geqslant \exp^{[p]}\{r_l^{\rho_{i(F_q)}(F_q)-\epsilon}\}. \\ \text{Similarly we get sequences } \{s_l\} \text{ and } \{t_l\} \text{ s.t.} \qquad\qquad \\ M(s_l, G_q) \geqslant \exp^{[p]}\{s_l^{\rho_{i(G_q)}(G_q)-\epsilon}\} \\ \text{and } M(t_l, H_q) \geqslant \exp^{[p]}\{t_l^{\rho_{i(H_q)}(H_q)-\epsilon}\}. \end{array}\right\} \qquad (3.20)$$

(i) When $mk = 3k_1,\ k_1 \in \mathbb{N}$. Then we have the following subcases.

Subcase 1 : Let $m = 3k_2,\ k_2 \in \mathbb{N}$. Then m can be any natural number. Now proceeding similarly as Case 1 of Theorem 3.1 for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$\begin{aligned} T(r, F_n) &< 2T\left(M\left(r, F_{(m-1)k+q}\right), F_k\right) \\ &\leqslant 2\exp^{[p-1]}\left(M\left(r, F_{(m-1)k+q}\right)\right)^{\rho_p(F_k)+\epsilon} \\ &= 2\exp^{[p]}\left(\rho_p(F_k)+\epsilon\right)\log\left(M\left(r, F_{(m-1)k+q}\right)\right) \\ &\leqslant \exp^{[p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M\left(M\left(r, F_{(m-2)k+q}\right), F_k\right)\right) \\ &\leqslant \exp^{[p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(\exp^{[p]}\left(M\left(r, F_{(m-2)k+q}\right)\right)^{\rho_p(F_k)+\epsilon}\right) \\ &\leqslant \exp^{[2p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M\left(r, F_{(m-2)k+q}\right)\right) \\ &\qquad\vdots \\ &\leqslant \exp^{[(m-1)p]}\left(\rho_p(F_k)+2\epsilon\right)\log\left(M\left(r, F_{k+q}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \exp^{[(m-1)p]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(\exp^{[p]}M\left(r,F_q\right)^{\rho_p\left(F_k\right)+\epsilon}\right)\\
&\leqslant \exp^{[mp]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(\exp^{\left[i\left(F_q\right)\right]}r^{\rho_{i\left(F_q\right)}\left(F_q\right)+\epsilon}\right)\\
&\leqslant \exp^{\left[mp+i\left(F_q\right)\right]}\left(\rho_{i\left(F_q\right)}\left(F_q\right)+2\epsilon\right)\log r.
\end{aligned}
$$

So, $$\frac{\log^{\left[mp+i\left(F_q\right)\right]}T\left(r,F_n\right)}{\log r}<\rho_{i\left(F_q\right)}\left(F_q\right)+2\epsilon. \tag{3.21}$$

Now for opposite inequality, again proceeding similarly as 2nd part of Case 1 of Theorem 3.1, for sufficiently large r_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$
\begin{aligned}
T\left(r_l,F_n\right) &\geqslant \frac{1}{3}\log\left(M\left(\frac{r_l}{2},F_{mk+q}\right)\right)\\
&\geqslant \frac{1}{3}\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^2},F_{(m-1)k+q}\right),F_k\right)\right)\\
&\geqslant \frac{1}{3}\log\exp^{[p]}\left(\frac{1}{9}M\left(\frac{r_l}{2^2},F_{(m-1)k+q}\right)\right)^{\mu_p\left(F_k\right)-\epsilon}\\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^2},F_{(m-1)k+q}\right)\right)\\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^3},F_{(m-2)k+q}\right),F_k\right)\right)\\
&\geqslant \exp^{[2p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^3},F_{(m-2)k+q}\right)\right)\\
&\qquad\vdots\\
&\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^m},F_{k+q}\right)\right)\\
&\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^{m+1}},F_q\right),F_k\right)\right)\\
&\geqslant \exp^{[mp]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^{m+1}},F_q\right)\right)\\
&\geqslant \exp^{\left[mp+i\left(F_q\right)\right]}\left(\rho_{i\left(F_q\right)}\left(F_q\right)-2\epsilon\right)\log r_l.
\end{aligned}
$$

So, $$\frac{\log^{\left[mp+i\left(F_q\right)\right]}T\left(r_l,F_n\right)}{\log r_l}\geqslant\rho_{i\left(F_q\right)}\left(F_q\right)-2\epsilon. \tag{3.22}$$

From (3.21) and (3.22) we have $\rho_{mp+i\left(F_q\right)}\left(F_n\right)=\rho_{i\left(F_q\right)}\left(F_q\right)$. (3.23)

Similarly we have $\rho_{mp+i\left(G_q\right)}\left(G_n\right)=\rho_{i\left(G_q\right)}\left(G_q\right)$, $\rho_{mp+i\left(H_q\right)}\left(H_n\right)=\rho_{i\left(H_q\right)}\left(H_q\right)$. (3.24)

Subcase 2 : Let $k=3k_2+1,\ k_2\in\mathbb{N}$. Then $m=3k_3,\ k_3\in\mathbb{N}$. Now proceeding similarly as Case 2 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$
\begin{aligned}
T\left(r,F_n\right) &< 2T\left(M\left(r,G_{(m-1)k+q}\right),F_k\right)\\
&\leqslant 2\exp^{[p]}\left(\rho_p\left(F_k\right)+\epsilon\right)\log\left(M\left(r,G_{(m-1)k+q}\right)\right)\\
&\leqslant \exp^{[p]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(M\left(M\left(r,H_{(m-2)k+q}\right),G_k\right)\right)\\
&\leqslant \exp^{[p]}\left(\rho_p\left(F_k\right)+2\epsilon\right)\log\left(\exp^{[p]}\left(M\left(r,H_{(m-2)k}+q\right)\right)^{\rho_p\left(G_k\right)+\epsilon}\right)\\
&\leqslant \exp^{[2p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(M\left(r,F_{(m-3)k+q}\right),H_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \exp^{[3p]}\left(\rho_p\left(H_k\right)+2\epsilon\right)\log\left(M\left(r, F_{(m-3)k+q}\right)\right)\\
&\qquad\vdots\\
&\leqslant \exp^{[(m-1)p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(M\left(r, H_{k+q}\right)\right)\\
&\leqslant \exp^{[(m-1)p]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(\exp^{[p]}\left(M\left(r, F_q\right)\right)^{\rho_p\left(H_k\right)+\epsilon}\right)\\
&\leqslant \exp^{[mp]}\left(\rho_p\left(H_k\right)+2\epsilon\right)\log\left(\exp^{[i(F_q)]} r^{\rho_{i(F_q)}\left(F_q\right)+\epsilon}\right)\\
&\leqslant \exp^{[mp+i(F_q)]}\left(\rho_{i(F_q)}\left(F_q\right)+2\epsilon\right)\log r.
\end{aligned}
$$

So, $$\frac{\log^{[mp+i(F_q)]}T(r,F_n)}{\log r} < \rho_{i(F_q)}\left(F_q\right)+2\epsilon. \tag{3.25}$$

Now for opposite inequality, again proceeding similarly as 2nd part of Case 2 of Theorem 3.1, for sufficiently large r_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$
\begin{aligned}
T\left(r_l, F_n\right) &\geqslant \frac{1}{3}\log\left(M\left(\frac{r_l}{2}, F_{mk+q}\right)\right)\\
&\geqslant \frac{1}{3}\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^2}, G_{(m-1)k+q}\right), F_k\right)\right)\\
&\geqslant \frac{1}{3}\log\exp^{[p]}\left(\frac{1}{9}M\left(\frac{r_l}{2^2}, G_{(m-1)k+q}\right)\right)^{\mu_p\left(F_k\right)-\epsilon}\\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^2}, G_{(m-1)k+q}\right)\right)\\
&\geqslant \exp^{[p]}\left(\mu_p\left(F_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^3}, H_{(m-2)k+q}\right), G_k\right)\right)\\
&\geqslant \exp^{[2p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^3}, H_{(m-2)k+q}\right)\right)\\
&\qquad\vdots\\
&\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^m}, H_{k+q}\right)\right)\\
&\geqslant \exp^{[(m-1)p]}\left(\mu_p\left(G_k\right)-2\epsilon\right)\log\left(M\left(\frac{1}{9}M\left(\frac{r_l}{2^{m+1}}, F_q\right), H_k\right)\right)\\
&\geqslant \exp^{[mp]}\left(\mu_p\left(H_k\right)-2\epsilon\right)\log\left(M\left(\frac{r_l}{2^{m+1}}, F_q\right)\right)\\
&\geqslant \exp^{[mp+i(F_q)]}\left(\rho_{i(F_q)}\left(F_q\right)-2\epsilon\right)\log r_l.
\end{aligned}
$$

So, $$\frac{\log^{[mp+i(F_q)]}T(r_l,F_n)}{\log r_l} \geqslant \rho_{i(F_q)}\left(F_q\right)-2\epsilon. \tag{3.26}$$

From (3.25) and (3.26) we have $\rho_{mp+i(F_q)}\left(F_n\right)=\rho_{i(F_q)}\left(F_q\right)$. (3.27)

Similarly we have $\rho_{mp+i(G_q)}\left(G_n\right)=\rho_{i(G_q)}\left(G_q\right)$, $\rho_{mp+i(H_q)}\left(H_n\right)=\rho_{i(H_q)}\left(H_q\right)$. (3.28)

Subcase 3 : Let $k=3k_2+2,\ k_2\in\mathbb{N}$. Then $m=3k_3,\ k_3\in\mathbb{N}$. Now proceeding similarly as Case 3 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$
\begin{aligned}
T\left(r, F_n\right) &< \exp^{[mp]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(M\left(r, F_q\right)\right)\\
&\leqslant \exp^{[mp]}\left(\rho_p\left(G_k\right)+2\epsilon\right)\log\left(\exp^{[i(F_q)]} r^{\rho_{i(F_q)}\left(F_q\right)+\epsilon}\right)\\
&\leqslant \exp^{[mp+i(F_q)]}\left(\rho_{i(F_q)}\left(F_q\right)+2\epsilon\right)\log r.
\end{aligned}
$$

So, $\dfrac{\log^{[mp+i(F_q)]}T(r, F_n)}{\log r} < \rho_{i(F_q)}(F_q) + 2\epsilon.$ (3.29)

For the opposite inequality, again proceeding similarly as 2nd part of Case 3 of Theorem 3.1, for sufficiently large r_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$T(r_l, F_n) \geqslant \exp^{[mp+i(F_q)]}\left(\rho_{i(F_q)}(F_q) - 2\epsilon\right)\log r_l.$$

So, $\dfrac{\log^{[mp+i(F_q)]}T(r_l, F_n)}{\log r_l} \geqslant \rho_{i(F_q)}(F_q) - 2\epsilon.$ (3.30)

From (3.29) and (3.30) we have

$$\rho_{mp+i(F_q)}(F_n) = \rho_{i(F_q)}(F_q). \tag{3.31}$$

Similarly we have

$$\rho_{mp+i(G_q)}(G_n) = \rho_{i(G_q)}(G_q), \ \rho_{mp+i(H_q)}(H_n) = \rho_{i(H_q)}(H_q). \tag{3.32}$$

So from (3.23), (3.24), (3.27), (3.28), (3.31), (3.32) we get

$$\rho_{mp+i(F_q)}(F_n) + \rho_{mp+i(G_q)}(G_n) + \rho_{mp+i(H_q)}(H_n) = \rho_{i(F_q)}(F_q) + \rho_{i(G_q)}(G_q) + \rho_{i(H_q)}(H_q).$$

(ii) When mk is not a multiple of 3. Then we have the following two cases.

Case 1 : Let $mk = 3k_1 + 1, \ k_1 \in \mathbb{N}$. In this case we have the following subcases.

Subcase 1 : Let $k = 3k_2 + 1, \ k_2 \in \mathbb{N}$. Then $m = 3k_3 + 1, \ k_3 \in \mathbb{N} \cup \{0\}$. Now proceeding similarly as Case 2 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$\begin{aligned} T(r, F_n) &< \exp^{[mp]}\left(\rho_p(F_k) + 2\epsilon\right)\log\left(\exp^{[i(G_q)]} r^{\rho_{i(G_q)}(G_q)+\epsilon}\right) \\ &\leqslant \exp^{[mp+i(G_q)]}\left(\rho_{i(G_q)}(G_q) + 2\epsilon\right)\log r. \end{aligned}$$

So, $\dfrac{\log^{[mp+i(G_q)]}T(r, F_n)}{\log r} < \rho_{i(G_q)}(G_q) + 2\epsilon.$ (3.33)

For the opposite inequality, again proceeding similarly as 2nd part of Case 2 of Theorem 3.1, for sufficiently large s_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$\begin{aligned} T(s_l, F_n) &\geqslant \exp^{[mp]}\left(\mu_p(F_k) - 2\epsilon\right)\log\left(M\left(\frac{s_l}{2^{m+1}}, G_q\right)\right) \\ &\geqslant \exp^{[mp+i(G_q)]}\left(\rho_{i(G_q)}(G_q) - 2\epsilon\right)\log s_l. \end{aligned}$$

So, $\dfrac{\log^{[mp+i(G_q)]}T(s_l, F_n)}{\log s_l} \geqslant \rho_{i(G_q)}(G_q) - 2\epsilon.$ (3.34)

From (3.33) and (3.34) we have

$$\rho_{mp+i(G_q)}(F_n) = \rho_{i(G_q)}(G_q). \tag{3.35}$$

Similarly we have

$$\rho_{mp+i(H_q)}(G_n) = \rho_{i(H_q)}(H_q),\ \rho_{mp+i(F_q)}(H_n) = \rho_{i(F_q)}(F_q). \tag{3.36}$$

Subcase 2 : Let $k = 3k_2 + 2,\ k_2 \in \mathbb{N}$. Then $m = 3k_3 + 2,\ k_3 \in \mathbb{N} \cup \{0\}$. Now proceeding similarly as Case 2 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$\begin{aligned} T(r, F_n) &\leqslant \exp^{[mp]}\left(\rho_p(F_k) + 2\epsilon\right)\log\left(\exp^{[i(H_q)]} r^{\rho_{i(H_q)}(H_q)+\epsilon}\right) \\ &\leqslant \exp^{[mp+i(H_q)]}\left(\rho_{i(H_q)}(H_q) + 2\epsilon\right)\log r. \end{aligned}$$

So, $$\frac{\log^{[mp+i(H_q)]} T(r, F_n)}{\log r} < \rho_{i(H_q)}(H_q) + 2\epsilon. \tag{3.37}$$

For the opposite inequality, again proceeding similarly as 2nd part of Case 3 of Theorem 3.1, for sufficiently large t_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$\begin{aligned} T(t_l, F_n) &\geqslant \exp^{[mp]}\left(\mu_p(F_k) - 2\epsilon\right)\log\left(M\left(\frac{t_l}{2^{m+1}}, H_q\right)\right) \\ &\geqslant \exp^{[mp+i(H_q)]}\left(\rho_{i(H_q)}(H_q) - 2\epsilon\right)\log t_l. \end{aligned}$$

So, $$\frac{\log^{[mp+i(H_q)]} T(t_l, F_n)}{\log t_l} \geqslant \rho_{i(H_q)}(H_q) - 2\epsilon. \tag{3.38}$$

From (3.37) and (3.38) we have

$$\rho_{mp+i(H_q)}(F_n) = \rho_{i(H_q)}(H_q). \tag{3.39}$$

Similarly we have

$$\rho_{mp+i(F_q)}(G_n) = \rho_{i(F_q)}(F_q),\ \rho_{mp+i(G_q)}(H_n) = \rho_{i(G_q)}(G_q). \tag{3.40}$$

So from (3.23), (3.24), (3.27), (3.28), (3.31), (3.32) we get

$$\rho_{mp+i(F_q)}(G_n) - \rho_{mp+i(F_q)}(H_n) = \rho_{mp+i(G_q)}(H_n) - \rho_{mp+i(G_q)}(F_n) = \rho_{mp+i(H_q)}(F_n) - \rho_{mp+i(H_q)}(G_n).$$

Case 2 : Let $mk = 3k_1 + 2,\ k_1 \in \mathbb{N}$. In this case we have the following subcases.

Subcase 1 : Let $k = 3k_2 + 1,\ k_2 \in \mathbb{N}$. Then $m = 3k_3 + 2,\ k_3 \in \mathbb{N} \cup \{0\}$. Now proceeding similarly as Case 2 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$T(r,F_n) < \exp^{[mp]}\left(\rho_p(G_k)+2\epsilon\right)\log\left(\exp^{[i(F_q)]}r^{\rho_{i(F_q)}(F_q)+\epsilon}\right)$$
$$\leqslant \exp^{[mp+i(H_q)]}\left(\rho_{i(H_q)}(H_q)+2\epsilon\right)\log r.$$

$$\text{So, } \frac{\log^{[mp+i(H_q)]}T(r,F_n)}{\log r} < \rho_{i(H_q)}(H_q)+2\epsilon. \tag{3.41}$$

For the opposite inequality, again proceeding similarly as 2nd part of Case 2 of Theorem 3.1, for sufficiently large t_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$T(t_l,F_n) \geqslant \exp^{[mp]}\left(\mu_p(G_k)-2\epsilon\right)\log\left(M\left(\frac{t_l}{2^{m+1}},H_q\right)\right)$$
$$\geqslant \exp^{[mp+i(H_q)]}\left(\rho_{i(H_q)}(H_q)-2\epsilon\right)\log t_l.$$

$$\text{So, } \frac{\log^{[mp+i(H_q)]}T(t_l,F_n)}{\log t_l} \geqslant \rho_{i(H_q)}(H_q)-2\epsilon. \tag{3.42}$$

From (3.41) and (3.42) we have

$$\rho_{mp+i(H_q)}(F_n) = \rho_{i(H_q)}(H_q). \tag{3.43}$$

Similarly we have

$$\rho_{mp+i(F_q)}(G_n) = \rho_{i(F_q)}(F_q),\ \ \rho_{mp+i(G_q)}(H_n) = \rho_{i(G_q)}(G_q). \tag{3.44}$$

Subcase 2 : Let $k = 3k_2+2,\ k_2 \in \mathbb{N}$. Then $m = 3k_3+1,\ k_3 \in \mathbb{N}\cup\{0\}$. Now proceeding similarly as Case 3 of Theorem 3.1, for sufficiently large r, using Lemma 2.2, Lemma 2.3, (3.4) and (3.19) we have

$$T(r,F_n) < \exp^{[mp]}\left(\rho_p(H_k)+2\epsilon\right)\log\left(\exp^{[i(G_q)]}r^{\rho_{i(G_q)}(G_q)+\epsilon}\right)$$
$$\leqslant \exp^{[mp+i(G_q)]}\left(\rho_{i(G_q)}(G_q)+2\epsilon\right)\log r.$$

$$\text{So, } \frac{\log^{[mp+i(G_q)]}T(r,F_n)}{\log r} < \rho_{i(G_q)}(G_q)+2\epsilon. \tag{3.45}$$

For the opposite inequality, again proceeding similarly as 2nd part of Case 3 of Theorem 3.1, for sufficiently large s_l, using Lemma 2.4, Lemma 2.5, (3.5), (3.6) and (3.20)

$$T(s_l,F_n) \geqslant \exp^{[mp]}\left(\mu_p(H_k)-2\epsilon\right)\log\left(M\left(\frac{s_l}{2^{m+1}},G_q\right)\right)$$
$$\geqslant \exp^{[mp+i(G_q)]}\left(\rho_{i(G_q)}(G_q)-2\epsilon\right)\log s_l.$$

$$\text{So, } \frac{\log^{[mp+i(G_q)]}T(s_l,F_n)}{\log s_l} \geqslant \rho_{i(G_q)}(G_q)-2\epsilon. \tag{3.46}$$

From (3.45) and (3.46) we have

$$\rho_{mp+i(G_q)}(F_n) = \rho_{i(G_q)}(G_q). \tag{3.47}$$

Similarly we have

$$\rho_{mp+i(H_q)}(G_n) = \rho_{i(H_q)}(F_q), \ \rho_{mp+i(F_q)}(H_n) = \rho_{i(F_q)}(F_q). \tag{3.48}$$

So from (3.43), (3.44), (3.47), (3.48) we get

$$\rho_{mp+i(F_q)}(G_n) - \rho_{mp+i(F_q)}(H_n) = \rho_{mp+i(G_q)}(H_n) - \rho_{mp+i(G_q)}(F_n) = \rho_{mp+i(H_q)}(F_n) - \rho_{mp+i(H_q)}(G_n).$$

Hence the theorem.

References

[1] D. Banerjee, N. Mondal, Iterated entire functions with finite iterated order sharing three values IM, Int. J. Math. Sci. Eng. Appl. 10 (2016), 133-147.

[2] L. G. Bernal, On growth k-order of solutions of a complex homogeneous linear differential equations, Proc. Amer. Math. Soc. 101 (1987), 317-322.

[3] J. Clunie, The composition of entire and meromorphic functions, Mathematical Essays Dedicated to A. J. Macintyre, Ohio University Press, (1970), 75-92.

[4] G. G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc. 20 (1979), 457-466.

[5] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, (1964).

[6] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. 22 (1998), 385-405.

[7] K. Niino, N. Smita, Growth of a composite function of entire functions, Kodai Math. J. 3 (1980), 374-379.