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Results of Semigroup of Linear Operators on Perturbation of Hille-Yosida Operator

Jude Babatunde Omosowon[1], Akinola Yussuff Akinyele[1]

[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria

Correspondence should be addressed to Akinola Yussuff Akinyele: olaakinyele04@gmail.com

Abstract
In this paper, we show that ω-order preserving partial contraction mapping is the infinitesimal generator of a C_0-semigroup on perturbation of Hille-Yosida operator. We reveal that the addition of a bounded linear operator B to an infinitesimal generator A of semigroup of linear does not destroy the property of A. In particular, $X_0 = \overline{D(A)}$ and A_0 is the part of A in X_0 and we used variation of constants formula to prove some of our results.

Keywords: ω-OCP_n, Perturbation, C_0-semigroup, Closed linear operator.

Introduction

Hille–Yosida theorem characterizes the generators of strongly continuous one-parameter semigroups of linear operators on Banach spaces. Perturbation theory comprises methods for finding an approximate solution to a problem, in perturbation theory, the solution is expressed as a power series in a small parameter ε. The first term is the known solution to the solvable problem. Successive terms in the series at higher powers of ε usually become smaller. Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating some results of perturbation of Hille-Yosida operator.

Akinyele *et al.* [1], revealed some perturbation results of the infinitesimal generator in the semigroup of the linear operator, also in [2] Akinyele *et al.*, obtained some results of semigroup of linear operator in spectra theory. Batty *et al.* [3], proved some asymptotic behavior of semigroup of operators. Balakrishnan [4], introduced an operator calculus for infinitesimal generators of semigroup. Banach [5], established and introduced the concept of Banach spaces. Chill and Tomilov [6], established some resolvent approach to stability operator semigroup. Davies [7], introduced linear operators and their spectra. Engel and Nagel [8], deduced one-parameter semigroup for linear evolution equations. Nagel *et al.* [9], identified extrapolation spaces for unbounded operators. Neerven [10], presented some results on adjoint of semigroup of linear operators. Omosowon *et al.* [11], proved some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [12], Omosowon *et al.*, deduced dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Rauf and Akinyele [13], introduced ω-order preserving partial contraction mapping and established its properties, also in [14], Rauf *et al.* presented some results

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of stability and spectra properties on semigroup of linear operator. Vrabie [15], proved some results of C_0-semigroup and its applications. Yosida [16], obtained some results on differentiability and representation of one-parameter semigroup of linear operators.

Preliminaries

Definition 2.1 (C_0-Semigroup) [15]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-OCP_n) [13]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in$ Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (Perturbation) [1]
Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A is perturbed by operator B or that B is a perturbation of A.
Definition 2.4 (closed linear operator) [16]
Let X, Y be two Banach spaces. A linear operator $A : D(A) \subseteq X \to Y$ is closed if for every sequence x_n in $D(A)$ converging to x in X such that $Ax_n \to y \in Y$ as $n \to \infty$, one has $x \in D(A)$ and $Ax = y$. Equivalently, A is closed if its graph is closed in the direct sum $X \oplus Y$.
Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Theorem 2.1 Hille-Yoshida [15]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{1}$$

Main Results

This section present results of semigroup of linear operators on perturbation of Hille-Yosida operator generated by ω-OCP_n using variation of constants formula:

Theorem 3.1
Suppose $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a Hille-Yosida operator and let $B : X_0 \to X$ be a bounded operator such that $A, B \in \omega - OCP_n$. Then:

(i) $A + B$ is a Hille-Yosida operator;

(ii) the part $A + B$ in X_0 generates a C_0-semigroup $T_0^B(t)$ on X_0.

Proof:
Assume A is of type (M, ω). By replacing A by $A - \omega I$ and passing to an equivalent norm, we may assume that A is of type $(1, 0)$. Because

$$\lambda - (A + B) = (I - BR(\lambda, A))(\lambda, A) \tag{2}$$

and

$$\|BR(\lambda, A)\| \leq \|B\|\lambda^{-1},$$

for λ large enough and $A, B \in \omega - OCP_n$, the operator $I - BR(\lambda, A)$ is invertible and

$$\|(I - BR(\lambda, A))^{-1}\|\|R(\lambda, A)\| \leq \lambda^{-1}(I - \|B\|\lambda^{-1}) = (\lambda - \|B\|)^{-1}. \tag{3}$$

So that for this λ, the operator $\lambda - (A + B)$ is invertible and its inverse satisfies

$$\|R(\lambda, A + B)\| \leq (\lambda - \|B\|)^{-1},$$

which proves (i).

To prove (ii). Let us denote the semigroup generated by A_0 as $T_0(t)$. We can prove by a variation of constants formula for $T_0^B(t)$. To avoid encountering a difficulty, we have the usual formulas as

$$\begin{aligned} T_0^B(t)x_0 &= T_0(t)x_0 + \int_0^t T_0^B(t-s)BT_0(s)x_0 ds \\ &= T_0(t)x_0 + \int_0^t T_0(t-s)BT_0^B(s)x_0 ds \end{aligned} \tag{4}$$

for all $x_0 \in X_0$ and $B \in \omega - OCP_n$.

Since B maps X_0 out of X_0. Then we need to pass to the extrapolation space X_{-1} and to prove that both $T_0(t)$ and $T_0^B(t)$ extend to C_0-semigroup on X_{-1}. Having done that, then the variation of constants formulas above make sense if we replace $T_0^B(t-s)$ and $T_0^B(t-s)$ by their extension to X_{-1}, then we have that $A+B$ in X_0 generates a C_0-semigroup $T_0^B(t)$ on X_0. Hence the prove is completed.

Theorem 3.2

Assume $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a Hille-Yosida operator and $B : X_0 \to X$ is bounded for all $A, B \in \omega - OCP_n$. Then:

(i) The identity map on X extends to an isomorphism $(X, A)_{-1} \simeq (X, A+B)_{-1}$;

(ii) $(A+B)_{-1} = A_{-1} + B$;

(iii) The semigroup $T_0^B(t)$ extends to a C_0-semigroup on X_{-1} with generator $A_{-1} + B$.

Proof:

For any $x \in X$ and $A, B \in \omega - OCP_n$, we have

$$\begin{aligned} \|R(\lambda_0, A)x\| &= \|R(\lambda_0, A)x\|_{X_0} = \|R(\lambda_0, A)(\lambda_0 - A - B)R(\lambda_0, A+B)x\|_{X_0} \\ &= \|R(\lambda_0, A)(\lambda_0 - A)R(\lambda_0, A+B)x - R(\lambda_0, A)BR(\lambda_0, A+B)x\|_{X_0} \\ &\leq (1 + \|R(\lambda_0, A)B\|_{X_0})\|R(\lambda_0, A+B)x\|_{X_0} \\ &= (1 + \|R(\lambda_0, A)B\|_{X_0})\|R(\lambda_0, A+B)x\|. \end{aligned} \tag{5}$$

The converse estimate is obtained similarly. This shows that the $\|\cdot\|_{-1}$-norm with respect to A and $A+B$ are equivalent on X, hence also X_{-1}, and this proves (i).

To prove (ii), we have that since both operators are closed as operators on X_{-1}, have domain X_0 and agree on $D(A)$. But $A, B \in \omega - OCP_n$ and $D(A) \subset X_0$ is dense with respect to the graph norms by both $(A+B)_{-1}$ and $A_{-1} + B$. Therefore, the operators must be equal, and this achieves (ii).

To prove (iii), let us denote the extensions of $T_0(t)$ and $T_0^B(t)$ to the common space X_{-1} by $T_{-1}(t)$ and $T_{-1}^B(t)$ respectively. Then we have the following variation of constants formula, we have

$$\begin{aligned} T_0^B(t)x_0 &= T_0(t)x_0 + \int_0^t T_{-1}^B(t-s)BT_0(s)x_0 ds \\ &= T_0(t)x_0 + \int_0^t T_{-1}(t-s)BT_0^B(s)x_0 ds, \end{aligned} \tag{6}$$

for all $x_0 \in X_0$, $A, B \in \omega - OCP_n$ and the Bochner integrals being in X_{-1}.

Now, we have to prove the variation of constants formula.

For $x_0 \in X_0$, let fixed, define $h : [0, t] \to X_{-1}$ by

$$h(s) = T_{-1}(t-s)T_0^B(s)x_0. \tag{7}$$

Since $x_0 \in X_0 = D(A_{-1}) = D((A+B)_{-1})$, then this function is differentiable in X_{-1} with derivative

$$h'(s) = T_{-1}(t-s)BT_0^B(s)x_0.$$

Integrating this from 0 to t yields

$$T_0^B(t)x_0 = T_0(t)x_0 + \int_0^t T_{-1}(t-s)BT_0^B(s)x_0 ds.$$

With this we have proved the first formula. The second one follows by applying the same argument to

$$\widetilde{h}(s) = T_{-1}^B(t-s)T_0(s)x_0.$$

We will now prove a variant of these formulas avoiding reference to X_{-1}.

Suppose

$$\lim_{\lambda} \|\lambda R(\lambda, A) - \lambda R(\lambda, A+B)\| = 0. \tag{8}$$

Then we have the relation

$$R(\lambda, A) - R(\lambda, A+B) = R(\lambda, A)(I - (I - BR(\lambda, A))^{-1}).$$

Therefore the result follows from the estimate $\|R(\lambda, A)\| \leq M(\lambda - \omega)^{-1}$, and this achieved the proof.

Theorem 3.3

Let $A : D(A) \subseteq X \to X$ be the generator of a Hille-Yosida operator and $B : X_0 \to X$ is bounded. Then:

(i) The $\odot$-duals of X with respect to A and $A+B$ agree and also the $\odot$-duals of X_0 with respect to A_0 and $(A+B)_0$ agree for $A, B \in \omega - OCP_n$.

(ii) For all $x_0 \in X$ and $A, B \in \omega - OCP_n$, we have

$$\begin{aligned} T_0^B(t)x_0 &= T_0(t)x_0 + \lim_{\lambda} \int_0^t T_0^B(t-s)\lambda R(\lambda, A)BT_0(s)x_0 ds \\ &= T_0(t)x_0 + \lim_{\lambda} \int_0^t T_0(t-s)\lambda R(\lambda, A)BT_0^B(s)x_0 ds. \end{aligned}$$

Proof:

Suppose for all $x^{\odot} \in X$ and $A, B \in \omega - OCP_n$, we have

$$\lim_{\lambda} R(\lambda, A)x^{\odot} = \lim_{\lambda} R(\lambda, A+B)^{*}x^{\odot} = x^{\odot}$$

and for $x^{\odot} \in X^{\odot}$, we have

$$\lim_{\lambda} R(\lambda, A_0)^{*}x_0^{\odot} = \lim_{\lambda} R(\lambda, (A+B)_0)^{*}x_0^{\odot} = x_0^{\odot}$$

so that $X_A^{\odot}$ be the $\odot$-dual of X with respect to A. Then $\lambda R(\lambda, A^{*})x^{\odot} \to x^{\odot}$ holds for all $x^{\odot} \in R(\lambda, A)^{*}X^{*}$ by the resolvent identity, and for general $x^{\odot} \in X_A^{\odot}$, it follows by density and the uniform boundedness of $\lambda R(\lambda, A)$ for λ near ∞.

Towards the end part of the prove of (ii) of Theorem 3.2 shows that also $\lambda R(\lambda, A+B)x^{\odot} \to x^{\odot}$ for all $x^{\odot} \in X_A^{\odot}$ and $A, B \in \omega - OCP_n$. Particularly, every $x^{\odot}$ belongs to the closure $R(\lambda, A+B)^{*}X^{*}$, that is to $X_{A+B}^{\odot}$. Therefore, $X_A^{\odot} \subset X_{A+B}^{\odot}$. The converse inclusion follows by a similar argument. The assertions concerning $X_0^{\odot}$ follows from the corresponding ones for $X^{\odot}$, noting that

$$\lim_{\lambda} \|\lambda R(\lambda, A_0) - \lambda R(\lambda, (A+B)_0)\| = 0. \tag{9}$$

In fact, this follows from the restricting

$$\lim_{\lambda} \|\lambda R(\lambda, A) - \lambda R(\lambda, A+B)\| = 0 \quad \text{to } X_0,$$

and this complete the prove of (i).

To prove (ii), let us first make an observation concerning the variation constants formula below:

$$\begin{aligned} T_0^B(t)x_0 &= T_0(t)x_0 + \int_0^t T_{-1}^B(t-s)BT_0(s)x_0 ds \\ &= T_0(t)x_0 + \int_0^t T_{-1}(t-s)BT_0^B(s)x_0 ds \end{aligned} \tag{10}$$

for all $x_0 \in X_0$ and $B \in \omega - OCP_n$.

Since both $T_0(t)$ and $T_0^B(t)x_0$ belong to X_0, so does the integrals on the right hand sides. Moreover, we have

$$R(\lambda, A+B)\int_0^t = R(\lambda, (A+B)_{-1})\int_0^t$$
$$= \int_0^t R(\lambda, (A+B)_{-1}),$$

since the integrals are Bochner in X_{-1}. Now we have

$$\lim_{\lambda} \lambda R(\lambda, (A+B)_0)T_0^B(t)x_0 = T_0^B(t)x_0$$

and using

$$\lim_{\lambda} \|\lambda R(\lambda, A) - \lambda R(\lambda, A+B)\| = 0 \tag{11}$$

with relation

$$R(\lambda, A) - R(\lambda, A+B) = R(\lambda, A)(I - (I - BR(\lambda, A))^{-1}),$$

we have

$$\lim_{\lambda\to\infty} \lambda R(\lambda, (A+B)_0)T_0(t)x_0 = \lim_{\lambda\to\infty} \lambda R(\lambda, A_0)T_0(t)x_0 = T_0(t)x_0,$$

both strongly in X_0. Therefore, in view of the observation made by (iii) of Theorem 3.2 and (11), and the fact that

$$R(\lambda, A_{-1})\big|_X = R(\lambda, A),$$

we have

$$T_0^B(t)x_0 = T_0(t)x_0 + \lim_{\lambda} \int_0^t \lambda R(\lambda, (A+B)_{-1})T_{-1}^B(t-s)BT_0(s)x_0 ds$$
$$= T_0(t)x_0 + \lim_{\lambda} \int_0^t T_0^B(t-s)\lambda R(\lambda, A)BT_0(s)x_0 ds.$$

Hence the prove is completed.

Theorem 3.4
Let $A : D(A) \subseteq X \to X$ be a Hille-Yosida operator and $B : X_0 \to X$ be a bounded operator such that $A, B \in \omega - OCP_n$. Then

(i) $T_{-1}^B(t)$ satisfy variation of constants formula.

(ii) Suppose both A and $A+B$ are intertwining Hille-Yosida operator, then the operator satisfy variation of constants formula.

Proof:
Let

$$T_{-1}^B(t)x_{-1} = T_{-1}(t)x_{-1} + \lim_{\lambda} \int_0^t T_{-1}^B(t-s)B\lambda R(\lambda, A_{-1})T_{-1}(s)x_{-1} ds$$
$$= T_{-1}(t)x_{-1} + \lim_{\lambda} \int_0^t T_{-1}(t-s)B\lambda R(\lambda, A_{-1})T_{-1}^B(s)x_{-1} ds \tag{12}$$

for all $x_{-1} \in X_{-1}$ and the limits being in X_{-1}.

By (iii) of Theorem 3.2 and (11), we have

$$\begin{aligned}
T_{-1}^{B}(t)x_{-1} &= \lim_{\lambda} \lambda R(\lambda, (A+B)_{-1}) T_{-1}^{B}(t)x_{-1} \\
&= \lim_{\lambda} T_{0}^{B}(t) \lambda R(\lambda, (A+B)_{-1}) x_{-1} \\
&= \lim_{\lambda} T_{0}(t) \lambda R(\lambda, (A+B)_{-1}) x_{-1} \\
&\quad + \lim_{\lambda} \int_{0}^{t} T_{-1}^{B}(t-s) B T_{0}(s) \lambda R(\lambda, A+B_{-1}) x_{-1} ds \\
&= T_{-1}(t)x_{-1} + \lim_{\lambda} \int_{0}^{t} T_{-1}^{B}(t-s) B T_{0}(s) \lambda R(\lambda, (A+B)_{-1}) x_{-1} ds \\
&= T_{-1}(t)x_{-1} + \lim_{\lambda} \int_{0}^{t} T_{-1}^{B}(t-s) B T_{0}(s) \lambda R(\lambda, A_{-1}) x_{-1} ds
\end{aligned}$$

and this proves (i).

To prove (ii), we have to consider that the semigroups on X_0 and on X_{-1} have been taken care of and we need to look at the case where $T_{-1}(t)$ leaves X invariant, that is if A is an intertwining operator. Then for $x^{\odot} \in X^{\odot}$, $x \in X$ and $A, B \in \omega - OCP_n$, we have

$$\begin{aligned}
\langle x^{\odot}, T^{B}(t)x \rangle &= \langle x^{\odot}, T(t)x \rangle + \lim_{\lambda} \int_{0}^{t} \langle x^{\odot}, T^{B}(t-s) B \lambda R(\lambda, A) T(s)x \rangle ds \\
&= \langle x^{\odot}, T(t)x \rangle + \lim_{\lambda} \int_{0}^{t} \langle x^{\odot}, T(t-s) B \lambda R(\lambda, A) T^{B}(s)x \rangle ds.
\end{aligned}$$

Considering Theorem 3.3, (11) and the dominated convergence theorem, we have:

$$\begin{aligned}
\langle x^{\odot}, T^{B}(t)x \rangle &= \lim_{\lambda} \langle x^{\odot}, \lambda R(\lambda, A+B) T^{B}(t)x \rangle \\
&= \lim_{\lambda} \langle x^{\odot}, T_{0}^{B}(t) \lambda R(\lambda, A+B)x \rangle \\
&= \lim_{\lambda} \langle x^{\odot}, T_{0}(t) \lambda R(\lambda, A+B)x \rangle \\
&\quad + \lim_{\lambda} \lim_{\mu} \int_{0}^{t} \langle x^{\odot}, T_{0}^{B}(t-s) \mu R(\mu, A) B T_{0}(s) \lambda R(\lambda, A)x \rangle ds \\
&= \lim_{\lambda} \langle x^{\odot}, T_{0}(t) \lambda R(\lambda, A)x \rangle \\
&\quad + \lim_{\lambda} \lim_{\mu} \int_{0}^{t} \langle x^{\odot}, T_{0}^{B}(t-s) \mu R(\mu, A+B) B T_{0}(s) \lambda R(\lambda, A+B)x \rangle ds \\
&= \langle x^{\odot}, T(t)x \rangle + \lim_{\lambda} \int_{0}^{t} \langle x^{\odot}, T^{B}(t-s) B \lambda R(\lambda, A) T(s)x \rangle ds.
\end{aligned}$$

Hence the prove is completed.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generate some results on perturbation of Hille-Yosida operator by using variation of constants formula.

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