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A Note on a Simple Polynomial-Sine Copula

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Abstract
Copula modeling is beneficial and popular in a wide range of statistical fields, including banking, hydrology, geostatistics, engineering, economics, climatology, and so on. In this study, we introduce a new bivariate copula involving tuning parameters, polynomial and sine functions. It generalizes a previously known but understudied sine copula in the literature. We present it mathematically and graphically, and show how it can be used to generate new bivariate distributions with diverse support. In the remaining part, we investigate some ordering, stability, and dependence properties. Interesting relationships between it and the Farlie-Gumbel-Morgensten copula are discussed.

Keywords: Copulas; Farlie-Gumbel-Morgenstern Copula; Copula Measures of Dependence; Sinc Function; Stability; Tail Dependence.

Introduction

The notion of copula is, first of all, mathematical. A copula is a multivariate cumulative distribution function (CDF) whose each marginal CDF is the CDF of the unit uniform distribution. It can be used to (i) create multivariate distribution functions with a variety of dependency structures, or (ii) study the dependence structure of an existing multivariate distribution independently of its marginal distributions. Reference [18] published the first version of copula in 1959, and the concept has received a great deal of attention since then, becoming the subject of numerous scientific studies. Applications of copulas to multivariate survival analysis can be found in Reference [9]. We may also refer specifically to Reference [8] for applications in finance, Reference [11] for applications in geostatistics, Reference [22] for applications in hydrology, Reference [20] for applications in engineering, and Reference [17] for applications in floods.

As introduced in Reference [18], a bivariate copula is a bivariate function $C(u, v)$, $(u, v) \in [0, 1]^2$ that satisfies the following conditions:

- Boundary conditions: for any $(u, v) \in [0, 1]^2$,

$$C(u, 0) = C(0, v) = 0, \quad C(u, 1) = u, \quad C(1, v) = v, \quad C(1, 1) = 1.$$

- Two-increasing condition: for any $(u_1, u_2, v_1, v_2) \in [0, 1]^4$, such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

According to Reference [21], for any twice differentiable copula, this condition is equivalent to the more tractable condition: $\partial^2 C(u, v)/\partial u \partial v \geq 0$.

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As a major interest in the concept of copula, the theorem of Sklar states that, if $F(x, y)$ is a joint CDF of a continuous bivariate distribution with marginal CDFs given by $F_X(x)$ and $F_Y(y)$, respectively, then there exists a copula $C(u, v)$ such that $F(x, y) = C(F_X(x), F_Y(y))$. Over time, a variety of copulas, or methods for constructing copulas, have been proposed. Examples of copula are the Ali-Mikhail-Haq copula, Joe copula, Clayton copula, Frank copula, Gumbel-Hougaard copula, Farlie-Gumbel-Morgensten (FGM) copula, normal-type copulas, cubic-type copulas, and Cuadras-Augé copula. The complete list of classical copulas can be found in Reference [14]. Modern developments can be found in References [10], [4], [2] and [3]. However, only a few copulas, to our knowledge, use trigonometric functions. We may mention Reference [1, Example 1 (point 4)], Reference [7, Example 5], Reference [5], or the bivariate sine-type copulas developed in Reference [13]. In particular, the copula presented in Reference [1, Example 1 (point 4)] is defined as

$$C_*(u, v) = uv + \theta \frac{1}{\pi^2} \sin(\pi u) \sin(\pi v), \quad (u, v) \in [0, 1]^2, \tag{1}$$

where $\theta \in [-1, 1]$ is a dependence parameter. We call it the simple PS (SPS) copula. It can modelize large dependences, according to Reference [1]; several measures of dependence are provided in [1, Table 1 (point 4)] to support this claim. Despite its undeniable interest, the SPS copula appears to have received little attention in the literature. The aim of this research is to highlight its characteristics and to go beyond them by proposing a simple extension of this copula. The proposed extension intends to add more functionalities by introducing two tuning parameters. It remains simple, but more flexible than the SPS copula in terms of modulation of the trigonometric functions, and better suited for the development of new statistical models in a variety of fields. We illustrate this claim by studying some of its properties, such as three equivalent expressions involving different functions, the generation of new interesting bivariate distributions, a possible multivariate extension, some ordering properties, including one involving the FGM copula, a stability property, the determination of various measures of dependence, and diverse tail dependence characteristics.

All these aspects are examined in detail, with the following organisation. The proposed copula is defined in Section . Section focuses on its major properties. Finally, in Section , the research findings are summarized.

New polynomial-sine copula

Presentation

The following proposition presents a new simple copula derived from Equation (1). It is based on tuning parameters, polynomial and sine functions.

Proposition 1 *Let $(m, n) \in [1, +\infty)^2$, $\theta \in [-1, 1]$, and $C_*(u, v)$, $(u, v) \in [0, 1]^2$ be the following bivariate function:*

$$C_*(u, v) = uv + \theta \frac{1}{\pi^2 mn} [\sin(\pi u)]^m [\sin(\pi v)]^n. \tag{2}$$

Then $C_*(u, v)$ is a copula.

Proof. Let us check that the boundary and two-increasing conditions defining a copula are satisfied.

- Boundary conditions: since $\sin(0) = 0$, it is immediate that $C_*(u, 0) = C_*(0, v) = 0$. Now, since $\sin(\pi) = 0$, we get the other conditions: $C_*(u, 1) = u$, $C_*(1, v) = v$, and $C_*(1, 1) = 1$.

- Two-increasing condition: since $C_*(u, v)$ is twice differentiable and $\cos(x)$, $\sin(x)$, $\theta \in [-1, 1]$, it is clear that

$$\frac{\partial^2 C_*(u, v)}{\partial u \partial v} = 1 + \theta \cos(\pi u) \cos(\pi v) [\sin(\pi u)]^{m-1} [\sin(\pi v)]^{n-1} \geq 1 - |\theta| \geq 0.$$

  The two-increasing condition is established.

This ends the proof of Proposition 1. □

To our knowledge, the copula defined as Equation (2) is new in the literature. It is reduced to the SPS copula for $m = n = 1$. We call it the polynomial-sine (PS) copula. Note that, for $\theta = 0$, we have $C_*(u, v) = uv$; $C_*(u, v)$ is thus reduced to the "independent" copula.

The PS copula density is given as

$$c_*(u, v) = \frac{\partial^2 C_*(u, v)}{\partial u \partial v} = 1 + \theta \cos(\pi u) \cos(\pi v) [\sin(\pi u)]^{m-1} [\sin(\pi v)]^{n-1}.$$

It is clear that m and n affect the possible shapes of this function in a sophisticated manner. Basically, for $m \in (1, +\infty)$ or $n \in (1, +\infty)$, we have $c_*(0, 0) = c_*(1, 1) = 1$, which is not the case when $m = n = 1$, where $c_*(0, 0) = c_*(1, 1) = 1 + \theta$, which strongly depends on θ.

For $m = n$, $C_*(u, v) = C_*(v, u)$ for any $(u, v) \in [0, 1]^2$; the PS copula is symmetric. It is not symmetric for $m \neq n$. For $\theta \in [0, 1]$, we have $C_*(u, v) \geq uv$, implying that any random variables X and Y having $C_*(u, v)$ as copula are positively quadrant dependent (PQD). For $\theta \in [-1, 0]$, the reversed inequality holds; they are negatively quadrant dependent (NQD). The reflected copula associated to $C_*(u, v)$ is given by $\widehat{C}_*(u, v) = u + v - 1 + C_*(1 - u, 1 - v)$, so

$$\begin{aligned}
\widehat{C}_*(u, v) &= u + v - 1 + (1 - u)(1 - v) + \theta \frac{1}{\pi^2 mn} [\sin(\pi(1 - u))]^m [\sin(\pi(1 - v))]^n \\
&= uv + \theta \frac{1}{\pi^2 mn} [\sin(\pi u)]^m [\sin(\pi v)]^n = C(u, v).
\end{aligned}$$

This shows that the PS copula is radially symmetric. As an additional property, like any copula, the PS copula satisfies the Fréchet-Hoeffding inequalities: for any $(u, v) \in [0, 1]^2$, we have $\max(u + v - 1, 0) \leq C_*(u, v) \leq \min(u, v)$.

More secondary, from the analytical standpoint, we can express $C_*(u, v)$ in different manners. Two of them are presented below.

- First, by introducing the normalized sinc function defined by $\text{sinc}(x) = \sin(\pi x)/(\pi x)$, we have

  $$C_*(u, v) = uv \left\{ 1 + \theta \frac{\pi^{m+n-2}}{mn} u^{m-1} v^{n-1} [\text{sinc}(u)]^m [\text{sinc}(v)]^n \right\}.$$

- Second, the following elementary trigonometric formula: $\sin(\pi u) = 2 \sin((\pi/2)u) \sin((\pi/2)(1 - u))$, gives

  $$C_*(u, v) = uv + \theta \frac{2^{m+n}}{\pi^2 mn} \left[\sin\left(\frac{\pi}{2} u\right) \right]^m \left[\sin\left(\frac{\pi}{2}(1 - u)\right) \right]^m \left[\sin\left(\frac{\pi}{2} v\right) \right]^n \left[\sin\left(\frac{\pi}{2}(1 - v)\right) \right]^n. \tag{3}$$

  Some similarities between this representation in its simplest form and the FGM copula will be discussed later.

We complete this description with the three-dimensional and contour plots of the PS copula for selected values of the parameters θ, m and n.

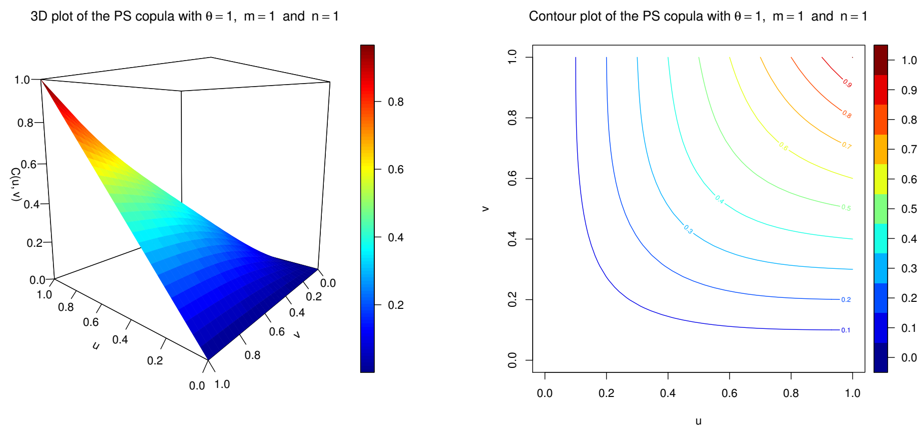

Figure 1: Three-dimensional and contour plots of the PS copula defined with $\theta = 1$ and $m = n = 1$.

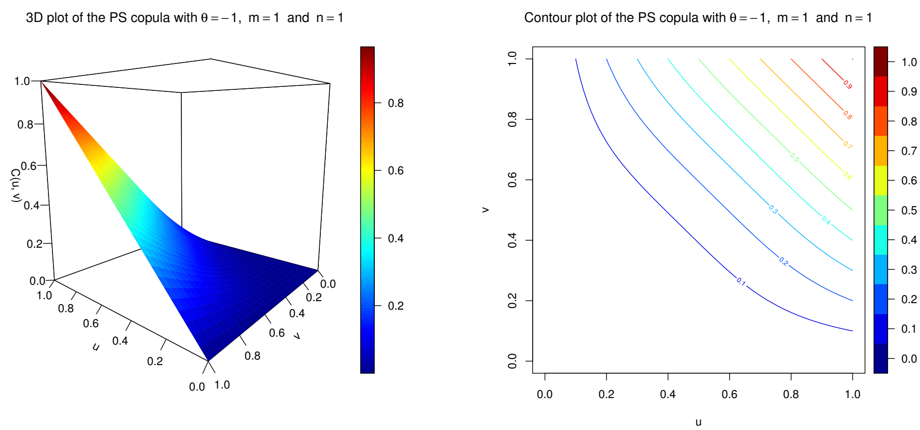

Figure 2: Three-dimensional and contour plots of the PS copula defined with $\theta = -1$ and $m = n = 1$.

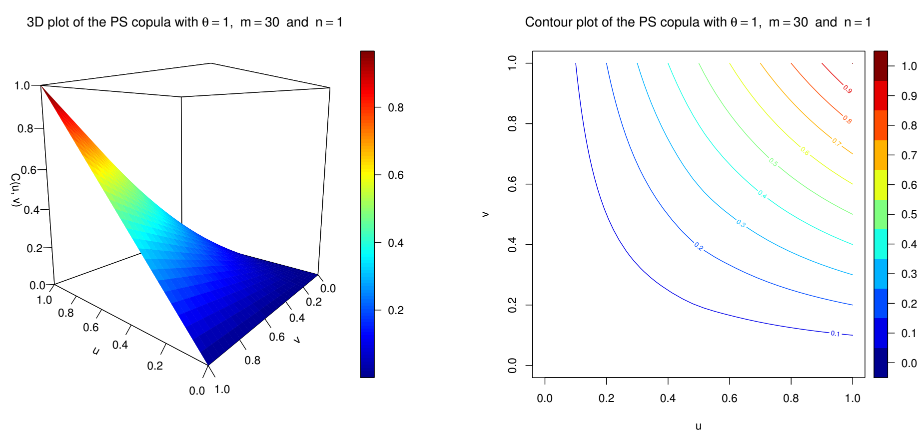

Figure 3: Three-dimensional and contour plots of the PS copula defined with $\theta = 1$, $m = 30$ and $n = 1$.

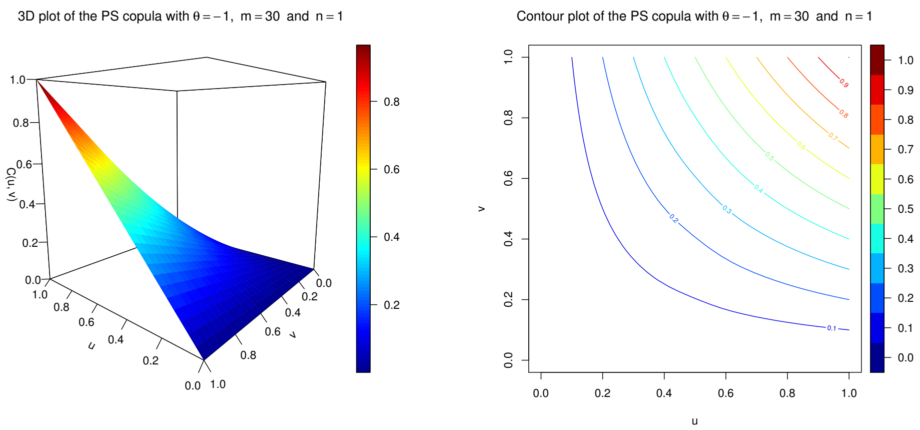

Figure 4: Three-dimensional and contour plots of the PS copula defined with $\theta = -1$, $m = 30$ and $n = 1$.

From Figures 1, 2, 3 and 4, we see that the PS copula has the aspect of a triangle that deforms itself depending on the values of the parameters. Also, a more or less pronounced bump is sometimes observed, certainly due to the effect of the sine terms combined with the power parameters m and n.

Remark 2 *As new facts on the existing SPS copula, it can be expressed in terms of cosine functions as*

$$C_*(u, v) = uv + \theta \frac{1}{\pi^2} \left[\cos(\pi u)\cos(\pi v) - \cos(\pi(u + v))\right],$$

or in terms of the normalized sinc function as

$$C_*(u, v) = uv \left[1 + \theta \operatorname{sinc}(u) \operatorname{sinc}(v)\right],$$

or in terms of more sine functions as

$$C_*(u, v) = uv + \theta \frac{4}{\pi^2} \sin\left(\frac{\pi}{2}u\right) \sin\left(\frac{\pi}{2}v\right) \sin\left(\frac{\pi}{2}(1-u)\right) \sin\left(\frac{\pi}{2}(1-v)\right).$$

We mention that, as the parental PS copula, it covers the independent case, satisfies the PQD property for $\theta \in [0, 1]$ and the NQD property for $\theta \in [-1, 0]$, and is radially symmetric. Furthermore, it is symmetric. It was previously plotted in Figures 1 and 2 with $\theta = 1$ and $\theta = -1$, respectively. Last but not least, the SPS copula density is quite simple; it is given as

$$c_*(u, v) = 1 + \theta \cos(\pi u) \cos(\pi v).$$

Some generalized bivariate distributions

On the other hand, the PS copula can be utilized to create generalized bivariate distributions. A few examples are given below.

- The PS normal (PSN) distribution: Let $\mu \in \mathbb{R}$, $\sigma \in (0, +\infty)$ and $\Phi(x; \mu, \sigma)$ be the CDF of the normal distribution with mean μ and standard deviation σ, i.e., $\Phi(x; \mu, \sigma) = (1/(\sigma\sqrt{2\pi})) \int_{-\infty}^{x} \exp(-(t - \mu)^2/(2\sigma^2))dt, x \in \mathbb{R}$. Then, the following CDF defines the PSN distribution: $F(x, y) = C_*(\Phi(x; \mu_1, \sigma_1), \Phi(y; \mu_2, \sigma_2))$, $(x, y) \in \mathbb{R}^2$, with $(\mu_1, \mu_2) \in \mathbb{R}^2$ and $(\sigma_1, \sigma_2) \in (0, +\infty)^2$. In an expanded form, we have

$$F(x, y) = \Phi(x; \mu_1, \sigma_1)\Phi(y; \mu_2, \sigma_2) + \theta \frac{1}{\pi^2 mn} \left[\sin\left(\pi\Phi(x; \mu_1, \sigma_1)\right)\right]^m \left[\sin\left(\pi\Phi(y; \mu_2, \sigma_2)\right)\right]^n,$$

$(x, y) \in \mathbb{R}^2$.

As an example of representation, Figure 5 shows the three-dimensional and contour plots for the PSN distribution defined with the following arbitrary configuration: $\theta = 0.5$, $m = 3$, $n = 1$, $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.

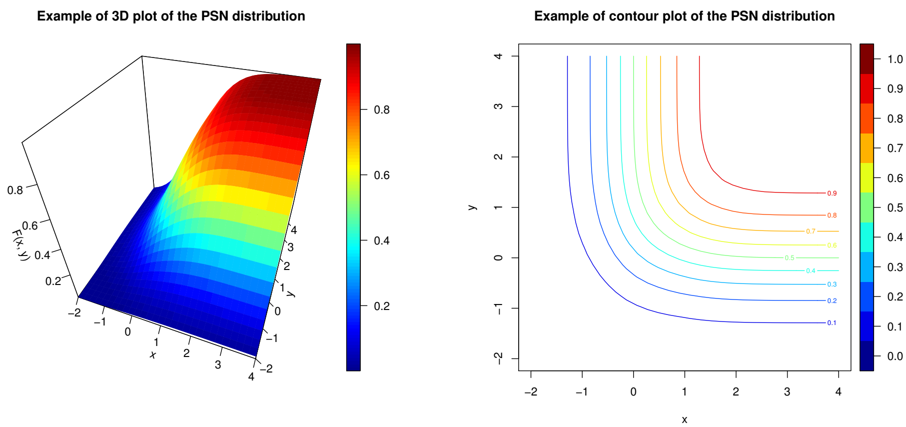

Figure 5: Three-dimensional and contour plots of the PSN distribution defined with $\theta = 0.5$, $m = 3$, $n = 1$, $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.

- The PS Weibull (PSW) distribution: Let $G(x; \beta, \lambda)$ be the CDF of the Weibull distribution, i.e., $G(x; \beta, \lambda) = 1 - \exp(-\lambda x^{\beta})$ for $x \in (0, +\infty)$, and $G(x; \beta, \lambda) = 0$ elsewhere. Then, the following CDF defines the PSW distribution: $F(x, y) = C_*(G(x; \beta_1, \lambda_1), G(y; \beta_2, \lambda_2))$, $(x, y) \in \mathbb{R}^2$, with $(\beta_1, \beta_2, \lambda_1, \lambda_2) \in (0, +\infty)^4$. In an expanded form, this gives

$$F(x, y) = \left(1 - \exp(-\lambda_1 x^{\beta_1})\right)\left(1 - \exp(-\lambda_2 y^{\beta_2})\right)$$
$$+ \theta \frac{1}{\pi^2 mn}\left[\sin\left(\pi \exp(-\lambda_1 x^{\beta_1})\right)\right]^m \left[\sin\left(\pi \exp(-\lambda_2 y^{\beta_2})\right)\right]^n, \quad (x, y) \in (0, +\infty)^2,$$

and $F(x, y) = 0$ elsewhere.

Figure 6 displays the three-dimensional and contour plots for the PSW distribution defined with the following arbitrary configuration: $\theta = 0.5$, $m = 3$, $n = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 1.5$, and $\beta_1 = \beta_2 = 2$.

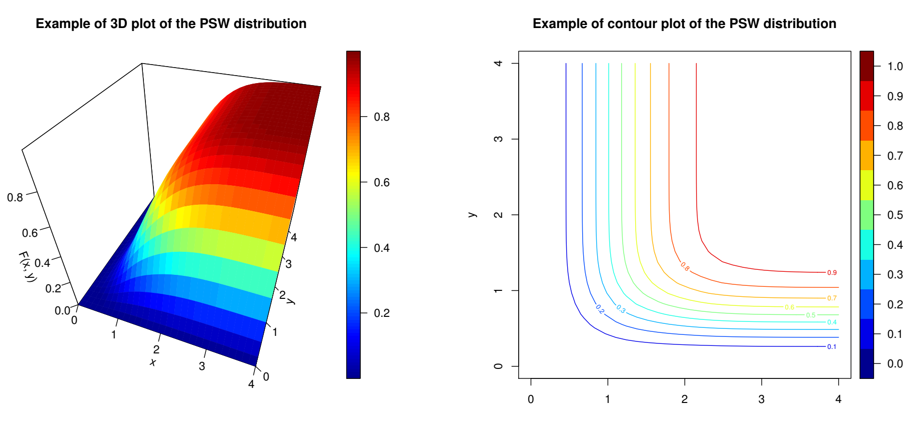

Figure 6: Three-dimensional and contour plots of the PSW distribution defined with $\theta = 0.5$, $m = 3$, $n = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 1.5$, and $\beta_1 = \beta_2 = 2$.

- The PS power Lomax (PSPL) distribution: Let $H(x;\alpha,\beta,\lambda)$ be the CDF of the power Lomax distribution, i.e., $H(x;\alpha,\beta,\lambda) = 1-\lambda^{\alpha}(\lambda+x^{\beta})^{-\alpha}$ for $x \in (0,+\infty)$, and $H(x;\alpha,\beta,\lambda) = 0$ elsewhere. See Reference [15]. Then, the following CDF defines the PSPL distribution: $F(x,y) = C_*(H(x;\alpha_1,\beta_1,\lambda_1), H(y;\alpha_2,\beta_2,\lambda_2))$, $(x,y) \in \mathbb{R}^2$, with $(\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2) \in (0,+\infty)^6$. In an expanded form, we have

$$F(x,y) = \left(1-\lambda_1^{\alpha_1}(\lambda_1+x^{\beta_1})^{-\alpha_1}\right)\left(1-\lambda_2^{\alpha_2}(\lambda_2+y^{\beta_2})^{-\alpha_2}\right)$$
$$+\theta\frac{1}{\pi^2 mn}\left[\sin\left(\pi\lambda_1^{\alpha_1}(\lambda_1+x^{\beta_1})^{-\alpha_1}\right)\right]^m\left[\sin\left(\pi\lambda_2^{\alpha_2}(\lambda_2+y^{\beta_2})^{-\alpha_2}\right)\right]^n, \quad (x,y) \in (0,+\infty)^2,$$

and $F(x,y) = 0$ elsewhere.

Figure 7 displays the three-dimensional and contour plots for the PSPL distribution defined with the following arbitrary configuration: $\theta = -1$, $m = n = 1$, $\alpha_1 = \beta_1 = 2$, $\alpha_2 = \beta_2 = 3$, and $\lambda_1 = \lambda_2 = 1$.

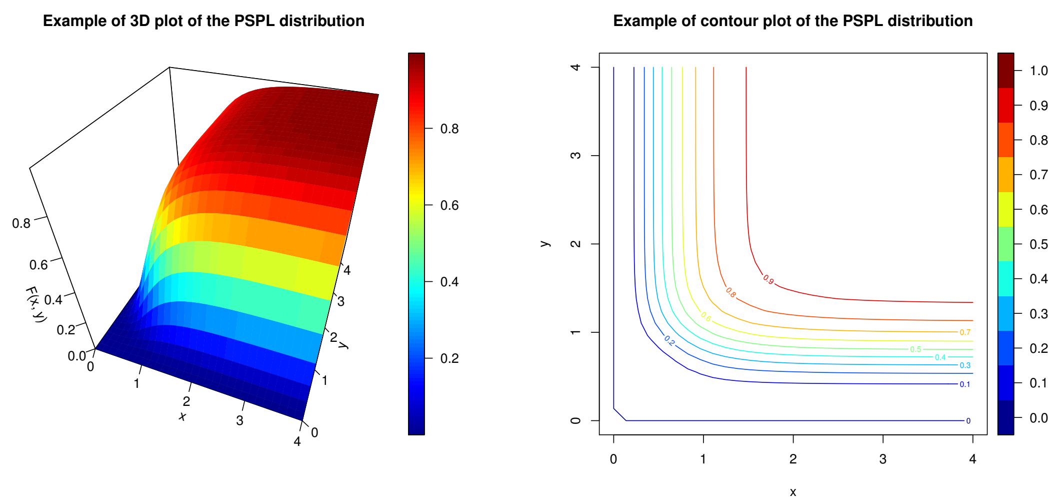

Figure 7: Three-dimensional and contour plots of the PSPL distribution defined with $\theta = -1$, $m = n = 1$, $\alpha_1 = \beta_1 = 2$, $\alpha_2 = \beta_2 = 3$, and $\lambda_1 = \lambda_2 = 1$.

Other examples can be presented in a similar manner. To our knowledge, these three bivariate distributions have not been examined in a statistical modelling scenario, and have a certain interest in this regard. This, however, demands further investigations that we leave for future work.

Multivariate case

We end this part by a corollary about a natural multivariate extension of the PS copula.

Corollary 1 *Let d be a strictly positive integer, $m_1, m_2, \ldots, m_d \in [1,+\infty)^d$, $\theta \in [-1,1]$, and $C_*(u_1,u_2,\ldots,u_d)$, $(u_1,u_2,\ldots,u_d) \in [0,1]^d$ be the following multivariate function:*

$$C_*(u_1,u_2,\ldots,u_d) = \left[\prod_{k=1}^{d} u_k\right] + \theta\frac{1}{\pi^d\prod_{k=1}^{d} m_k}\prod_{k=1}^{d}[\sin(\pi u_k)]^{m_k}.$$

Then $C_*(u_1,u_2,\ldots,u_d)$ is a copula.

The proof can be conducted as the proof of Proposition 1.

The multivariate version of the PS copula can be called the multivariate PS (MPS) copula. One can use it to create generalized multivariate distributions, and multivariate statistical models as well.

Specific properties

The PS and SPS copulas satisfy a number of special properties, which are detailed in this section.

Ordering properties

Some ordering properties satisfied by the PS copula are now being examined. The following result is about a direct hierarchical order involving the PS and SPS copulas.

Proposition 3 *Let $C_*(u,v) = C_*(u,v;m,n)$ be the PS copula with the specification of m and n, and $C_*(u,v;1,1)$ be the SPS copula. For any $\theta \in [0,1]$ and any $(u,v) \in [0,1]^2$, we have*

$$C_*(u,v;m,n) \leq C_*(u,v;1,1).$$

For any $\theta \in [-1,0]$, the reversed inequality holds.

Proof. For any $m \in [1,+\infty)$, we have $[\sin(\pi u)]^{m-1} \leq 1 \leq m$, implying that $[\sin(\pi u)]^m/m \leq \sin(\pi u)$. Therefore, for $\theta \in [0,1]$, since all the involved quantities are positive, we have

$$C_*(u,v;m,n) = uv + \theta\frac{1}{\pi^2 mn}[\sin(\pi u)]^m[\sin(\pi v)]^n \leq uv + \theta\frac{1}{\pi^2}\sin(\pi u)\sin(\pi v) = C_*(u,v;1,1).$$

The desired result is proved. The inequality is reversed by considering a negative value of θ. This ends the proof of Proposition 3. □

The following result shows a double inequality involving the PS copula and a function connected to the FGM copula, and some of its generalizations.

Proposition 4 *With the notations used for the PS copula, let us set*

$$\Lambda(u,v;\theta) = uv + \theta u^m(1-u)^m v^n(1-v)^n, \quad (u,v) \in [0,1]^2.$$

For any $\theta \in [0,1]$ and any $(u,v) \in [0,1]^2$, the PS copula satisfies the following double inequality:

$$\Lambda\left(u,v;\theta\frac{2^{m+n}}{\pi^2 mn}\right) \leq C_*(u,v) \leq \Lambda\left(u,v;\theta\frac{\pi^{2(m+n-1)}}{2^{m+n}mn}\right).$$

For $\theta \in [-1,0]$, the reversed double inequality holds.

Proof. Let us have in mind the representation of the PS copula in Equation (3). The following double inequality holds: for $y \in [0,\pi/2]$, $(2/\pi)y \leq \sin(y) \leq y$. Therefore

$$\begin{aligned} & u^m(1-u)^m v^n(1-v)^n \leq \left[\sin\left(\frac{\pi}{2}u\right)\right]^m \left[\sin\left(\frac{\pi}{2}(1-u)\right)\right]^m \left[\sin\left(\frac{\pi}{2}v\right)\right]^n \left[\sin\left(\frac{\pi}{2}(1-v)\right)\right]^n \\ & \leq \frac{\pi^{2(m+n)}}{2^{2(m+n)}} u^m(1-u)^m v^n(1-v)^n. \end{aligned}$$

We thus obtain the desired result by applying this double inequality to Equation (3). The double inequality is reversed by considering a negative value of θ. This ends the proof of Proposition 4. □

Let us now discuss some ordering relationships between the PS and FGM copulas. First, the FGM copula is defined by

$$C_\circ(u,v) = uv + \theta uv(1-u)(1-v), \quad (u,v) \in [0,1]^2,$$

with $\theta \in [-1,1]$. Hence, for $m = n = 1$, we have $\Lambda(u,v;\theta) = C_\circ(u,v)$. Therefore, for some values of θ, m and n, Proposition 4 provides direct ordering properties between the SPS and FGM copulas. In addition, for any $(m,n)\in[1,+\infty)^2$, we have $\Lambda(u,v;\theta)\le C_\circ(u,v)$. Therefore, for some values of θ, m and n, Proposition 4 shows that the PS copula can be compared to the FGM copula. In addition, ordering properties between the parental PS copula and some general forms of the FGM copula involving powers of u, $1-u$, v and $1-v$ can be established, such as the one proposed in Reference [12] by putting the power parameters equal.

Stability property

In this subsection, we investigate a special stability property satisfied by the PS copula. It involves a special product presented below. Let $C^{(1)}(u,v)$ and $C^{(2)}(u,v)$ be two copulas. Then we define the copula product of $C^{(1)}(u,v)$ and $C^{(2)}(u,v)$ by

$$(C^{(1)}\star C^{(2)})(u,v)=\int_0^1\frac{\partial C^{(1)}(u,t)}{\partial v}\frac{\partial C^{(2)}(t,v)}{\partial u}dt. \tag{4}$$

It is proved in Reference [14] that $(C^{(1)}\star C^{(2)})(u,v)$ is also a copula. In the following result, we establish a stability property of the PS copula with respect to the product above.

Proposition 5 *For any $q\in(0,+\infty)$, let us set*

$$\mathfrak{I}(q)=\int_0^1[\sin(\pi x)]^q dx. \tag{5}$$

Let $C_*(u,v)=C_*(u,v;\theta)$ be the PS copula with the specification of θ, $C_*^{(1)}(u,v)=C_*(u,v;\theta_1)$ and $C_*^{(2)}(u,v)=C_*(u,v;\theta_2)$ with $\theta_1,\theta_2\in[-1,1]^2$. Then $(C_*^{(1)}\star C_*^{(2)})(u,v)=C_*(u,v;\theta_\star)$, where

$$\theta_\star=\theta_1\theta_2\left[\mathfrak{I}(m+n-2)-\mathfrak{I}(m+n)\right].$$

Thus, the copula product of two PS copulas is also a PS copula.

Proof. Based on Equation (1), we have

$$\frac{\partial C_*(u,v)}{\partial u}=v+\theta\frac{1}{\pi n}\cos(\pi u)[\sin(\pi u)]^{m-1}[\sin(\pi v)]^n$$

and

$$\frac{\partial C_*(u,v)}{\partial v}=u+\theta\frac{1}{\pi m}\cos(\pi v)[\sin(\pi u)]^{m}[\sin(\pi v)]^{n-1}.$$

Therefore, based on the copula product in Equation (4), we have

$$\begin{aligned}
&(C_*^{(1)}\star C_*^{(2)})(u,v)\\
&=\int_0^1\left[u+\theta_1\frac{1}{\pi m}\cos(\pi t)[\sin(\pi u)]^m[\sin(\pi t)]^{n-1}\right]\left[v+\theta_2\frac{1}{\pi n}\cos(\pi t)[\sin(\pi t)]^{m-1}[\sin(\pi v)]^n\right]dt\\
&=uv+\theta_2\frac{1}{\pi n}u[\sin(\pi v)]^n\int_0^1\cos(\pi t)[\sin(\pi t)]^{m-1}dt\\
&+\theta_1\frac{1}{\pi m}v[\sin(\pi u)]^m\int_0^1\cos(\pi t)[\sin(\pi t)]^{n-1}dt\\
&+\theta_1\theta_2\frac{1}{\pi^2 mn}[\sin(\pi u)]^m[\sin(\pi v)]^n\int_0^1[\cos(\pi t)]^2[\sin(\pi t)]^{m+n-2}dt\\
&=uv+0+0+\theta_1\theta_2\frac{1}{\pi^2 mn}[\sin(\pi u)]^m[\sin(\pi v)]^n\left[\mathfrak{I}(m+n-2)-\mathfrak{I}(m+n)\right].
\end{aligned}$$

We recognize the PS copula with dependence parameter $\theta_\star=\theta_1\theta_2\left[\mathfrak{I}(m+n-2)-\mathfrak{I}(m+n)\right]$, ending the proof of Proposition 5. □

In particular, if $C_*^{(1)}(u,v)$ and $C_*^{(2)}(u,v)$ are two SPS copulas with parameters θ_1 and θ_2, then $(C_*^{(1)} \star C_*^{(2)})(u,v)$ is the SPS copula with parameter $\theta_1\theta_2/2$. A similar stability property also holds for the FGM copula (see Reference [19]). We end this part by discussing the integral $\mathfrak{I}(q)$ presented in Equation (5). Even if there is not a simple expression for this integral, it can be calculated for any reasonable value q via standard integral or numerical techniques. In order to illustrate this claim, Table 1 presents some of its values for selected integer values of q.

Table 1: Values of $\mathfrak{I}(q)$ for several integer values of q

q	0	1	2	3	4	5	6	7	8	9	10
$\mathfrak{I}(q)$	1	$\frac{2}{\pi}$	$\frac{1}{2}$	$\frac{4}{3\pi}$	$\frac{3}{8}$	$\frac{16}{15\pi}$	$\frac{5}{16}$	$\frac{32}{35\pi}$	$\frac{35}{128}$	$\frac{256}{315\pi}$	$\frac{63}{256}$

q	11	12	13	14	15	16	17	18	19	20
$\mathfrak{I}(q)$	$\frac{512}{693\pi}$	$\frac{231}{1024}$	$\frac{2048}{3003\pi}$	$\frac{429}{2048}$	$\frac{4096}{6435\pi}$	$\frac{6435}{32768}$	$\frac{65536}{109395\pi}$	$\frac{12155}{65536}$	$\frac{131072}{230945\pi}$	$\frac{46189}{262144}$

As basic properties, it is decreasing with respect to q. Note that $\mathfrak{I}(q)$ will also be involved in the dependence properties of the PS copula.

Dependency properties

In the bivariate scenario, measures of dependence are frequently used to describe a complex dependence structure. This part is devoted to some measures of dependence related to the PS copula.

Let X and Y be continuous random variables whose copula is denoted by $C(u,v)$. Then the dependence of X and Y can be evaluated by several measures, such as the Kendall or Spearman measures. The Kendall measure is defined by

$$\tau = 1 - 4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}dudv, \tag{6}$$

and the Spearman measures is specified by

$$\rho = 12\int_0^1\int_0^1 C(u,v)dudv - 3. \tag{7}$$

For alternative expressions and details on these measures, see Reference [16].

When the PS copula is considered, these two measures are determined in the next result.

Proposition 6 *The Kendall and Spearman measures of dependence related to the PS copula are given by*

$$\tau = 8\theta\frac{1}{\pi^2 mn}\mathfrak{I}(m)\mathfrak{I}(n), \qquad \rho = 12\theta\frac{1}{\pi^2 mn}\mathfrak{I}(m)\mathfrak{I}(n).$$

Proof. Based on Equation (2), we have

$$\frac{\partial C_*(u,v)}{\partial u} = v + \theta \frac{1}{\pi n} \cos(\pi u)[\sin(\pi u)]^{m-1}[\sin(\pi v)]^n$$

and

$$\frac{\partial C_*(u,v)}{\partial v} = u + \theta \frac{1}{\pi m} \cos(\pi v)[\sin(\pi u)]^{m}[\sin(\pi v)]^{n-1}.$$

Therefore, in view of using Equation (6), we calculate

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{\partial C_*(u,v)}{\partial u} \frac{\partial C_*(u,v)}{\partial v} dudv \\
&= \int_0^1 \int_0^1 \left(v + \theta \frac{1}{\pi n} \cos(\pi u)[\sin(\pi u)]^{m-1}[\sin(\pi v)]^n \right) \left(u + \theta \frac{1}{\pi m} \cos(\pi v)[\sin(\pi u)]^{m}[\sin(\pi v)]^{n-1} \right) dudv \\
&= \left[\int_0^1 udu \right] \left[\int_0^1 vdv \right] + \theta \frac{1}{\pi m} \left[\int_0^1 v \cos(\pi v)[\sin(\pi v)]^{n-1} dv \right] \left[\int_0^1 [\sin(\pi u)]^m du \right] \\
&+ \theta \frac{1}{\pi n} \left[\int_0^1 u \cos(\pi u)[\sin(\pi u)]^{m-1} du \right] \left[\int_0^1 [\sin(\pi v)]^n dv \right] \\
&+ \theta^2 \frac{1}{\pi^2 mn} \left[\int_0^1 \cos(\pi u)[\sin(\pi u)]^{2m-1} du \right] \left[\int_0^1 \cos(\pi v)[\sin(\pi v)]^{2n-1} dv \right].
\end{aligned}$$

By an integration by part, it comes

$$\int_0^1 v \cos(\pi v)[\sin(\pi v)]^{n-1} dv = -\frac{1}{n\pi} \Im(n).$$

The other integral terms can be calculated directly; we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{\partial C_*(u,v)}{\partial u} \frac{\partial C_*(u,v)}{\partial v} dudv = \frac{1}{4} - \theta \frac{1}{\pi^2 mn} \Im(n)\Im(m) - \theta \frac{1}{\pi^2 mn} \Im(m)\Im(n) + \theta^2 \frac{1}{\pi^2 mn} \times 0 \\
&= \frac{1}{4} - 2\theta \frac{1}{\pi^2 mn} \Im(m)\Im(n).
\end{aligned}$$

Thus, by Equation (6), the Kendall measure is obtained as

$$\tau = 1 - 4\left(\frac{1}{4} - 2\theta \frac{1}{\pi^2 mn} \Im(m)\Im(n) \right) = 8\theta \frac{1}{\pi^2 mn} \Im(m)\Im(n).$$

For the Spearman measure as described in Equation (7), the calculus is more direct; it follows from Equation (2) that

$$\begin{aligned}
\int_0^1 \int_0^1 C_*(u,v) dudv &= \left[\int_0^1 udu \right] \left[\int_0^1 vdv \right] + \theta \frac{1}{\pi^2 mn} \left[\int_0^1 [\sin(\pi u)]^m du \right] \left[\int_0^1 [\sin(\pi v)]^n dv \right] \\
&= \frac{1}{4} + \theta \frac{1}{\pi^2 mn} \Im(m)\Im(n).
\end{aligned}$$

Therefore

$$\rho = 12\left(\frac{1}{4} + \theta \frac{1}{\pi^2 mn} \Im(m)\Im(n) \right) - 3 = 12\theta \frac{1}{\pi^2 mn} \Im(m)\Im(n).$$

This ends the proof of Proposition 6. □

By taking $m = n = 1$ in Proposition 6, we rediscover the result for the SPS copula described in [1, Table 1 (point 4)]. That is

$$\tau = \frac{32}{\pi^4}\theta, \qquad \rho = \frac{48}{\pi^4}\theta.$$

It can be noted that $32/\pi^4 \approx 0.3285114$, and $48/\pi^4 \approx 0.4927671$, which give the following approximative dependence domains: $\tau \in [-0.33, 0.33]$ and $\rho \in [-0.5, 0.5]$. These domains extend the dependence domains associated with the FGM copula which are $\tau \in [-0.23, 0.23]$ and $\rho \in [-0.34, 0.34]$. Thus, the FGM copula does not allow the modeling of high dependences, whereas the SPS copula does. The SPS copula is preferable in this aspect. It is also preferable to some modern extensions of the FGM copula, as those presented in References [10] and [3]. The dependence domains of the more general PS copula are:

$$\tau \in \left[-8\frac{1}{\pi^2 mn}\Im(m)\Im(n),\, 8\frac{1}{\pi^2 mn}\Im(m)\Im(n)\right], \qquad \rho \in \left[-12\frac{1}{\pi^2 mn}\Im(m)\Im(n),\, 12\frac{1}{\pi^2 mn}\Im(m)\Im(n)\right].$$

Since $\Im(m)$ and $\Im(n)$ decrease as m and n increase, the maximal dependence domains are obtained for $m = n = 1$. The addition of m and n to the SPS copula provides more flexibility in the possible shapes of the main functions, but does not increase the range of the dependence domains.

Tail dependence

The level of dependency in the upper-right quadrant tail and lower-left quadrant tail of a bivariate distribution is referred to as tail dependence. It is a concept that is relevant to the research of extreme value reliance. Let X and Y be continuous random variables. Then the tail dependency between X and Y turns out to be a copula property; the amount of tail dependence is invariant under strictly increasing transformations of X and Y. See Reference [6]. By denoting $C(u, v)$ the copula of X and Y, the following notions are adopted:

- Let us consider
$$\lambda_U = \lim_{u\to 1} \frac{1 - 2u + C(u, u)}{1 - u}.$$
If λ_U exists, then $C(u, v)$ has an upper tail dependence if $\lambda_U \in (0, 1]$, and no upper tail dependence if $\lambda_U = 0$.

- Let us consider
$$\lambda_L = \lim_{u\to 0} \frac{C(u, u)}{u}.$$
If λ_L exists, then $C(u, v)$ has a lower tail dependence if $\lambda_L \in (0, 1]$, and no lower tail dependence if $\lambda_L = 0$.

Extreme value theory makes considerable use of these two measurements. Here, we investigate them for the PS copula in the next result.

Proposition 7 *The measures of tail dependence related to the PS copula are given by $\lambda_U = \lambda_L = 0$.*

Proof. Owing to Equation (2), when $u \to 1$, the following equivalence holds:

$$\frac{1 - 2u + C_*(u, u)}{1 - u} = 1 - u + \theta\frac{1}{\pi^2 mn}\frac{[\sin(\pi u)]^{m+n}}{1 - u} \sim (1 - u)\left[1 + \theta\frac{1}{mn}\pi^{m+n-2}(1 - u)^{m+n-2}\right].$$

Hence $\lambda_U = 0$.

Moreover, when $u \to 0$, we have

$$\frac{C_*(u, u)}{u} = u + \theta\frac{1}{\pi^2 mn}\frac{[\sin(\pi u)]^{m+n}}{u} \sim u\left[1 + \theta\frac{1}{mn}\pi^{m+n-2}u^{m+n-2}\right].$$

This entails that $\lambda_L = 0$. The proof of Proposition 7 is now complete. □

Since $\lambda_U = \lambda_L = 0$, the PS copula has no upper tail dependence or lower tail dependence.

Summary with discussion

In this article, we have provided a new bivariate copula extending the one in Reference [1, Example 1 (point 4)]. It is based on tuning parameters, polynomial and sine functions. We call it the polynomial-sine copula. We have described it in detail, and illustrated how it may be used to produce new bivariate distributions with various supports. In addition, we have discussed some connections with the Farlie-Gumbel-Morgensten copula, as well as various stability and dependence properties. Among its main features, the polynomial-sine copula has the following merits: (i) it has a simple mathematical structure, (ii) it is connected, in some senses, to the Farlie-Gumbel-Morgensten copula and some of its extensions, (iii) it may have a dependence domain wider than the one of the Farlie-Gumbel-Morgensten copula, and other well referenced copulas, and (iv) it has no upper tail dependence or lower tail dependence. Hence, the proposed copula is believed to be beneficial in modeling real data sets by practitioners. Meanwhile, the author is currently employing the polynomial-sine copula on some real data sets, with the findings to be published in a future study.

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