ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2022) 2022:3

Best Proximity Point Theorems for $\alpha-$Proximal $\theta - \phi$-Non-Self Mappings

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Abstract
In this paper, we search some best proximity point for a novel class of non-self-mappings called $\alpha-$proximal $\theta - \phi-$mapping. Our results generalize and extend many recent results appearing in the literature. Several consequences are derived. As an application, we explore the existence of best proximity points for a metric space endowed with asymmetric binary relation.

Keywords: Proximity Point; $\alpha-$Proximal Admissible; $\alpha-$Proximal $\theta - \phi-$Non-self Mappings.

Introduction

One of the fundamental results in fixed point theory is the Banach contraction principle [1]. Due to its importance, various mathematics steadied many interesting extensions and generalizations of this principle[2, 5–7].

The Banach contraction theorem states that if (X, d) is a complete metric space and $T : X \to X$ is self-mapping with contraction, then T has a unique fixed point.

On the other hand for given non-empty closed subsets A and B of a complete metric space (X, d), a contraction for non-self mapping $T : A \to B$ does not necessarily guarantee that it will have a fixed point. In this case, it is quite natural to investigate an element $x \in A$ such that $d(x, Tx) > 0$ is in some sense minimum, more precisely a point $x \in A$ for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T.

In this paper, we prove the existence and uniqueness of best proximity point for $\alpha-$proximal $\theta-\phi-$non-self mapping defined on a closed subset of a complete metric space. Also, we prove the existence and uniqueness of best proximity point on metric space endowed with symmetric binary relations.

preliminaries

Let (A, B) be a pair of non empty subsets of a metric space (X, d). We adopt the following notations:

$d(A, B) = \{\inf d(a, b) : a \in A, b \in B\}$;

$A_0 = \{ a \in A$ there exists $b \in A$ such that $d(a, b) = d(A, B)\}$;

$B_0 = \{ b \in B$ there exists $a \in A$ such that $d(a, b) = d(A, B)\}$.

Received: 2021-4-29

Definition 1. *[4]. Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if*

$$d(x^*, Tx^*) = d(A, B).$$

Definition 2. *[9]. Let (A, B) be a pair of non empty subsets of a metric space (X, d) such that A_0 is non empty. Then the pair (A, B) is to have P-property if and only*

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B)$$

$$\text{then } d(x_1, x_2) = d(y_1, y_2), \text{ where } x_1, x_2 \in A \text{ and } y_1, y_2 \in B.$$

Definition 3. *[4]. Let $\alpha : A \times A \to [0, +\infty[$. We say that T is said to be α proximal admissible if $\alpha(x_1, x_2) \geq 1$ and*

$$d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B) \Rightarrow \alpha(u_1, u_2) \geq 1 \; for \; all \; x_1, x_2, u_1, u_2 \in A.$$

Definition 4. *[3] Let Θ be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ such that*

(θ_1) *θ is increasing,*

(θ_2) *For each sequence $x_n \in]0, +\infty[$;*

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) *θ is continuous.*

Definition 5. *[10] Let Φ be the family of all functions ϕ: $[1, +\infty[\to [1, +\infty[$, such that*

(ϕ_1) *ϕ is increasing;*

(ϕ_2) *For each $t \in]1, +\infty[$, $lim_{n\to\infty} \phi^n(t) = 1$;*

(ϕ_3) *ϕ is continuous.*

Lemma 6. *[10] If $\phi \in \Phi$ Then $\phi(1)$=1, and $\phi(t) < t$.*

Definition 7. *[10] Let (X, d) be a metric space and $T : X \to X$ be a mapping.*
T is said to be a $\theta - \phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(d(x, y))],$$

Main result

We introduce the following concept which is a generalization of the definition of $\theta - \phi-$mapping.

Definition 8. *Let (X, d) be a metric space and (A, B) be pair of nonempty subset of X. A non-self mapping $T : A \to B$ is called $\alpha-$proximal $\theta - \phi-$mapping, where $\alpha : A \times A \to [0, +\infty[$, if there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,*

$$\alpha(x, y)\theta(d(Tx, Ty)) \leq \phi[\theta(d(x, y))]$$

Theorem 9. *Let (A, B) be pair of nonempty closed subset of a complete metric space (X, d) such that A_0 is nonemty. Let $\alpha : A \times A \to [0, +\infty[$, $\theta \in \Theta$ and $\phi \in \Phi$. Consider an $\alpha-$proximal $\theta - \phi-$non-self mapping $T : A \to B$ satisfying the following assertion:*

(1) $T(A_0) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is α -proximal admissible;

(3) there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(4) T is continuous.

Then T has a unique best proximity point $x^* \in A$ such that $d\left(x^*, Tx^*\right)$=$d\left(A, B\right)$.

Proof. From condition (3), there exist elements $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \;\; and \;\; \alpha(x_0, x_1) \geq 1.$$

Since $T(A_0) \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$.

Now, we have

$$d(x_1, Tx_0) = d(A, B), \;\; \alpha(x_0, x_1) \geq 1 \; and \; d(x_2, Tx_1) = d(A, B).$$

Since T is $\alpha-$ proximal admissible, this implies that $\alpha(x_1, x_2) \geq 1$. Thus, we have

$$d(x_2, Tx_1) = d(A, B) \;\; and \;\; \alpha(x_1, x_2) \geq 1.$$

Again, Since $T(A_0) \in B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B).$$

Continuing this process, by induction, we construct a sequence $x_n \in A_0$ such that

$$d\left(x_{n+1}, Tx_n\right) = d(A, B) \quad and \quad \alpha(x_n, x_{n+1}) \geq 1, \forall n \in \mathbb{N}. \;\; (1)$$

Since (A, B) satisfies the P property, we conclude from (1) that

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1}), \forall n \in \mathbb{N}. \tag{1}$$

We shall prove that the sequence x_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

As T is α-proximal $(\theta - \phi)-$ mapping and $\alpha(x_n, x_{n+1}) \geq 1$. Then

$$\theta\left[d\left(x_n, x_{n+1}\right)\right] = \theta\left[d(Tx_{n-1}, Tx_n)\right] \leq \alpha(x_{n-1}, x_n)\, \theta\left[d\left(Tx_{n-1}, Tx_n\right)\right]$$

As θ is increasing and using the Lemma 6, we conclude

$$d\left(x_n, x_{n+1}\right) < d\left(x_{n-1}, x_n\right).$$

Therefore, $\{d\left(x_{n+1,}x_n\right)\}_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d\left(x_{n+1,}x_n\right) = \lambda.$$

Now, we claim that $\lambda = 0$. Arguing by contraction, we assume that $\lambda > 0$. Since $d\left(x_{n+1,}x_n\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$d\left(x_{n+1,}x_n\right) \geq \lambda \quad \forall n \in \mathbb{N}.$$

From assumption of the theorem we get,

$$\begin{aligned}\theta\left[d\left(x_n, x_{n+1}\right)\right] &= \theta\left[d\left(Tx_{n-1}, Tx_n\right)\right] \\ &\leq \phi\left[\theta\left(d\left(Tx_{n-1}, Tx_n\right)\right)\right] \\ &\leq \dots \\ &\leq \phi^n\left[\theta\left(d\left(x_0, x_1\right)\right)\right]\end{aligned}$$

By (θ_1) and (ϕ_2) and letting $n \to \infty$, we obtain

$$1 < \theta(\lambda) \leq 1.$$

Which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n \in \mathbb{N}$. Suppose to the contrary that exists $\varepsilon > 0$ and sequences $n_{(k)}$ and $m_{(k)}$ of natural numbers such that

$$m_{(k)} > n_{(k)} > k, \quad D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon, \quad D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon. \tag{3}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n_{(k)}}\right) \tag{4}$$

$$< \varepsilon + d\left(x_{n(k)-1}, x_{n_{(k)}}\right). \tag{5}$$

Then, by 2 and 4, it follows that

$$\lim_{k\to\infty} d\left(m_{(k)}, n_{(k)}\right) = \varepsilon. \tag{6}$$

Using again the triangular inequality,

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(x_{m(k)}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right). \tag{7}$$

On the other hand, using triangular inequality, we have

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \tag{8}$$

Letting $k \to \infty$ in inequality 7 and 8, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{9}$$

Substituting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in assumption of the theorem, we get,

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]. \tag{10}$$

Letting Letting $k \to \infty$ the above inequality 10, from lemma 6 and using (θ_1) and (θ_3) , we obtain

$$\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta \lim_{k\to\infty}\left(d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

Hence

$$\varepsilon < \varepsilon.$$

Which is a contradiction. Thus, the sequence $\{x_n\}$ is a Cauchy sequence in the closed subset A of the metric space (X, d). Since (X, d) is compete and A is closed assures that the sequence $\{x_n\}$ converges to element $x^* \in A$.

On the other hand, T is a continuous mapping. Then we have $Tx_n \to Tx^*$ as $n \to \infty$. The continuity of the metric d implies that

$$d(A, B) = d(x_{n+1}, Tx_n) \to d(x^*, Tx^*).$$

Therefore,

$$d(x^*, Tx^*) = d(A, B).$$

Then T has a best proximity point.

Uniqueness. Now, suppose that $x^*, y^* \in A$ are two distinct best proximity points for T such that $x^* = y^*$. Since $d(x^*, Tx^*) = d(y^*, Ty^*) = d(A, B)$, using the P property, we conclude that

$$d(x^*, y^*) = d(Tx^*, Ty^*).$$

Since T is an $\alpha-$ proximal $\theta - \phi-$mapping, we obtain

$$\theta\left(d(Tx^*, Ty^*)\right) \leq \phi\left[\theta\left(d(x^*, y^*)\right)\right].$$

Therefore

$$\theta\left(d(A, B)\right) \leq \phi\left[\theta\left(d(A, B)\right)\right].$$

Then $d(A, B) < d(A, B)$, which is a contradiction. □

Theorem 10. *Let (A, B) be pair of nonempty closed subset of a complete metric space closed subsets of a complete metric space (X, d) such that A_0 is nonemty. Let $\alpha : A \times A \to [0, +\infty[$, and $\theta \in \Theta$ and $\phi \in \Phi$. Consider an α-proximal $\theta - \phi-$mapping $T : A \to B$ satisfying the following assertion:*

(1) $T(A_0) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is α -proximal admissible;

(3) there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(4) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x^*$ as $n \to \infty$, with $x^* \in A$, then there exists a subsequence $x_{n_{(k)}}$ of $\{x_n\}$ such that $\alpha(x_{n_{(k)}}, x^*) \geq 1$ for all k.

Then T has a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*)$=$d(A, B)$.

Proof. Following the proof of Theorem 9, there exists a Cauchy sequence $\{x_n\} \in A$ such that

$$(x_{n+1}, Tx_n) = d(A, B) \quad and \quad \alpha(x_n, x_{n+1}) \geq 1, \forall n \in \mathbb{N}. \tag{11}$$

and $(x_n) \to x^*$ as $n \to \infty$, with $x^* \in A$. From the condition (4) of the theorem, there exists a subsequence $x_{n_{(k)}}$ of $\{x_n\}$ such that $\alpha(x_{n_{(k)}}, x^*) \geq 1$ for all k.

Since T is a $\alpha-$ proximal $\theta - \phi-$ mapping, then we have

$$\theta\left(d(Tx_{n_{(k)}}, Tx^*)\right) \leq \phi\left[\theta\left(d(x_{n_{(k)}}, x^*)\right)\right] \ \forall k.$$

By of the lemma (6), we get

$$\theta\left(d(Tx_{n_{(k)}}, Tx^*)\right) < \theta\left(d(x_{n_{(k)}}, x^*)\right) \forall k.$$

As θ is increasing we conclude that

$$d(Tx_{n_{(k)}}, Tx^*) < (d(x_{n_{(k)}}, x^*) \ \ for \ \ all \ \ k. \tag{12}$$

By the triangular inequality, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n_{(k)+1}}) + d(x_{n_{(k)+1}}, Tx_{n_{(k)}}) + d(Tx_{n_{(k)}}, Tx^*) \\ &= d(x^*, x_{n_{(k)+1}}) + d(A, B) + d(Tx_{n_{(k)}}, Tx^*). \end{aligned}$$

We obtain that

$$d(x^*, Tx^*) - d(A, B) - d(x^*, x_{n_{(k)+1}}) \leq d(Tx_{n_{(k)}}, Tx^*)). \tag{13}$$

Using (12) and (13), we get

$$d(x^*, Tx^*) - d(A, B) - d(x^*, x_{n_{(k)+1}}) < (d(x_{n_{(k)}}, x^*). \tag{14}$$

By letting $k \to \infty$ in inequality (14), we obtain

$$d(x^*, Tx^*) = d(A, B).$$

Therefore x^* is a best proximity point for the non-self mapping T.

Uniqueness: follow similarly as Theorem 9. □

Example 11. *Let $X = \mathbb{R}$ endowed with the standard metric for all $x, y \in A$.*

Let $A = [2, 4]$ and $B = \left[\frac{1}{4}, \frac{1}{2}\right]$.

Consider the non-self mapping $T : A \to B$ such that $T(a) = \frac{1}{a}$ for all $a \in A$.

Therefore, $T(A) = B$.

On the other hand A and B are closed subsets on the complete space $(\mathbb{R}, d)$.

It is easy to see that the couple (A, B) satisfies the P property. Let the function $\alpha(x, y) = 1$ for all $x, y \in A$. We have

$$d(T(2), 2) = d\left(\frac{1}{2}, 2\right) = \frac{3}{2} = d(A, B).$$

So hypotheses (1), (2) (3), and (4) of the theorem are satisfied.

Now, let the functions ϕ: $[1, +\infty[\to [1, +\infty[$ defined by

$$\phi(t) = \frac{t+1}{2}.$$

And define $\theta :]0, +\infty[\to]1, +\infty[$ by

$$\theta(t) = \sqrt{t} + 1.$$

Obviously, $\phi \in \Phi$ and $\theta \in \Theta$.

In what follows, we prove that T is a $(\theta - \phi)-$ proximal mapping. We consider four two cases.

***Case. 1.** $x = y$. In this case, we have*

$$\theta\left(d(Tx, Ty)\right) = 0 \leq \phi\left[\theta\left(d\left(x, y\right)\right)\right].$$

***Case. 2.** $x \neq y$. In this case, we have*

$$\theta\left(d(Tx, Ty)\right) = \sqrt{d(Tx, Ty)} + 1 = \sqrt{\left|\frac{1}{x} - \frac{1}{y}\right|} + 1 = \sqrt{\left|\frac{x-y}{xy}\right|} + 1.$$

On the other hand

$$\phi\left[\theta\left(d\left(x, y\right)\right)\right] = \sqrt{\left|\frac{x-y}{4}\right|} + 1.$$

Since,

$$\left[\sqrt{\left|\frac{x-y}{xy}\right|} + 1\right] - \left[\sqrt{\left|\frac{x-y}{4}\right|} + 1\right] = \frac{(|x-y|)(4-xy)}{\left[\sqrt{\left|\frac{x-y}{xy}\right|} + 1\right] + \left[\sqrt{\left|\frac{x-y}{4}\right|} + 1\right]} \leq 0 \;\; for\ all\ x, y \in A.$$

Thus, T is a α-proximal $(\theta - \phi)-$mapping. So the conclusion is the existence and uniqueness of best proximity point of the mapping T which is 2.

Consequences

For the case $\alpha = 1$, the definition of $\theta - \phi-$mapping is the following. We have following best proximity point result.

Definition 12. *Let (X, d) be a metric space and (A, B) be pair of nonempty subset of X. A non-self mapping $T : A \longrightarrow B$ is called proximal $\theta - \phi-$ mapping if there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,*

$$\theta \left(d \left(Tx, Ty\right)\right) \leq \phi \left(\theta \left[d \left(x, y\right)\right]\right)$$

Theorem 13. *Let (A, B) be pair of nonempty closed subset of a complete metric space (X, d) such that A_0 is nonempty. Let $\theta \in \Theta$ and $\phi \in \Phi$. Suppose that $T : A \longrightarrow B$ is a non-self mapping satisfying the following assertion:*

(1) $T(A_0) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is a proximal $\theta - \phi-$mapping,

then T has a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*)$=$d(A, B)$.

Proof. Consider the mapping $\alpha : A \times A \longrightarrow [0, +\infty[$ defined by: $\alpha(x, y) = 1, \forall x, y \in A$.

From the definition of α, clearly T is $\alpha-$proximal admissible and also it is $\alpha-$ proximal $\theta - \phi-$mapping. On the other hand, for any $x \in A_0$, since $T(A_0) \in B_0$, there exists $y \in B$ such that $d(Tx, y) = d(A, B)$.

Moreover, by condition (2), T is a continuous mapping. Now all the hypotheses of Theorem 9 are satisfied of the existence and uniqueness best proximity point. □

Definition 14. *Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. A non-self mapping $T : A \longrightarrow B$ is called proximal-Browder contractive mapping, if there exists φ where $\varphi : [0, +\infty[\longrightarrow [0, +\infty[$ be an increasing and right continuous function such that $\varphi(t) < t$ for $t > 0$, We have*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \;\; for \; all \;\; x, y \in X.$$

Corollary 15. *Let (A, B) be pair of nonempty closed subset of a complete metric space (X, d) such that A_0 is nonempty. Consider non-self mapping $T : A \longrightarrow B$ satisfying the following assertion:*

(1) $T(A_0) \in B_0$ and the pair (A, B) satisfies the P property;

2) T is a proximal-Browder contractive mapping,

then T has a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*)$=$d(A, B)$.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = e^{\varphi(ln(t))}$ for all $t \in [1, +\infty[$. Obviously, $\phi \in \Phi$ and $\theta \in \Theta$.

By the definition of ϕ, we have

$$\phi\left(e^{(t)}\right) = e^{\varphi(t)}.$$

In what follows, we prove that T is a $(\theta - \phi)-$ proximal mapping.

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

So,

$$\begin{aligned} e^{d(Tx,Ty)} &= \theta\left(d(Tx, Ty)\right) \\ &\leq e^{\varphi(d(Tx,Ty))} \\ &= \phi\left(\theta\left[d\left(x, y\right)\right]\right). \end{aligned}$$

Therefore, from Theorem 9, T has a unique best proximity point $x^* \in A$. We can suppose that φ is a strictly increasing and continuous function. As in the proof of theorem 1 of [6] we conclude that T is a $\theta-\phi-$ proximal mapping. Therefore, from Theorem 9, T has a unique best proximity point $x^* \in A$. □

Definition 16. *Let (X, d) be a metric space and (A, B) be pair of nonempty subset of X. A non-self mapping $T : A \to B$ is called proximal $\theta-$ mapping if there exists $\theta \in \Theta$ and $k \in]0, 1[$ such that for any $x, y \in X$,*

$$\theta (d (Tx, Ty)) \leq [\theta (d (x, y))]^k ,$$

Corollary 17. *Let (A, B) be pair of nonempty closed subset of a complete metric space (X, d) such that A_0 is nonemty. Let $\theta \in \Theta$. Suppose that $T : A \to B$ is a non-self mapping satisfying the following assertion:*

(1) $T (A_0) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is a proximal $\theta-$mapping,

then T has a unique best proximity point $x^* \in A$ such that $d (x^*, Tx^*)$=$d (A, B)$.

Proof. Let $\phi(t) = t^k$. Obviously, $\phi \in \Phi$. So T is a $(\theta, \phi)-$ proximal mapping. Therefore, from Theorem 9, T has a unique best proximity point $x^* \in A$. □

Applications

Corollary 18. *Let (X, d) be a metric space and let T be a self mapping on X, suppose that there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$.*

If T satisfies the following inequality

$$\theta (d (Tx, Ty)) \leq \phi [\theta (d (x, y))] .$$

Then T has a unique fixed point.

Proof. By considering $A = B = X$ and the function $\alpha(x, y) = 1$ in Theorem 9, we guarantee the existence and uniqueness of a fixed point a self mapping T. □

We need some preliminaries to apply our results on the best proximity points in a metric space endowed with a symmetric binary relation.

Let $(X.d)$ be a metric space and $\mathcal{R}$ be a symmetric binary relation over X .

Definition 19. *[4]. A non-self mapping $T : A \to B$ is a proximal comparative mapping if $x\mathcal{R}y$ and $d(u_1, Tx) = d(u_2, Ty) = d(A, B)$ for all $x, y, u_1, u_2 \in A$, then $u_1\mathcal{R}u_2$.*

Definition 20. *[8]. We say that $(X, d, \mathcal{R})$ is regular if, for a sequence, we say that $(X, d, \mathcal{R})$ is regular if, for a sequence x_n in X, we have $x_n\ \mathfrak{R}\ x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d (x_n, x) = 0$ for some $x \in X$, then there exists a subsequence $n_{(k)}$ of x_n such that $n_{(k)}\mathcal{R}x$ for all $k \in \mathbb{N}$.*

Definition 21. *Let X be a nonempty set. A non-self mapping Let X be a nonempty set.Anon-self mapping $T : A \to B$ is called $(\theta - \phi)-$ contractive if there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that $x, y \in A : x\ \mathcal{R}\ y$, we have*

$$\theta (d (Tx, Ty)) \leq \phi [\theta (d (x, y))] .$$

We have the following best proximity point results.

Theorem 22. *Let (A, B) be a pair of nonempty closed subsets a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a symmetric binary relation over X. Consider a non-self mapping $T : A \to B$ satisfies the following assertions:*

(1) $T (A_0) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is proximal comparative mapping;

(3) there exist elements $x_0, x_1 \in A$ such that $d(x_0, x_1) = d(A, B)$ and $x_0 \mathcal{R} x_1$;

(4) If $(A, B, \mathcal{R})$ is regular;

(5) There exists $\theta \in \phi$ and $\phi \in \Phi$ such that T is $\theta - \phi-$contractive.

then T has a unique best proximity point $x^* \in A$ such that $d\left(x^*, Tx^*\right)$=$d\left(A, B\right)$.

Proof. Let us introduce the function

$$\alpha : A \times A \longrightarrow [0, +\infty) \quad \text{by: } \alpha(x, y) = \begin{cases} 1 & \text{if } x\mathcal{R}y \\ 0 & \text{otherwise} \end{cases}$$

Suppose that

$$\begin{cases} \alpha\left(x_1, x_2\right) \geqslant 1; \\ d\left(u_1, Tx_1\right) = d(A, B); \\ d\left(u_2, Tx_2\right) = d(A, B). \end{cases}$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{cases} x\mathcal{R}y; \\ d\left(u_1, Tx_1\right) = d(A, B); \\ d\left(u_2, Tx_2\right) = d(A, B). \end{cases}$$

Condition (2) of Theorem implies $u_1\mathcal{R}u_2$, which gives us $\alpha(u_1, u_2) \geq 1$.

Thus we prove that T is $\alpha-$ proximal admissible.

Condition (3) implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

The condition $T : A \to B$ is $(\theta - \phi)-$ contractive means that T is an $\alpha-$proximal $\theta - \phi-$ mapping. Also the condition $(A, B, \mathcal{R})$ is regular implies that if x_n is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\lim_{n\to\infty} d\left(x_n, x\right) = x^* \in A$, then there exists a subsequence $n_{(k)}$ of x_n such that $\alpha(n_{(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.

Now all the hypotheses of Theorem 9 are satisfied, which implies the existence and uniqueness of a proximity point the non-self mapping. □

Theorem 23. *Let (A, B) be a pair of nonempty closed subsets a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a symmetric binary relation over X. Consider a non-self mapping $T : A \to B$ satisfies the following assertions:*

(1) $T\left(A_0\right) \in B_0$ and the pair (A, B) satisfies the P property;

(2) T is proximal comparative mapping;

(3) there exist elements $x_0, x_1 \in A$ such that $d(x_0 x_1) = d(A, B)$ and $x_0\mathcal{R}x_1$;

(4) there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$x\mathcal{R}y \Rightarrow \theta\left(d\left(Tx, Ty\right)\right) \leq \phi\left(\theta\left[d\left(x, y\right)\right]\right).$$

(5) T is continuous,

then T has a unique best proximity point $x^* \in A$ such that $d\left(x^*, Tx^*\right)$=$d\left(A, B\right)$.

Proof. Let us introduce the function

$$\alpha : A \times A \longrightarrow [0, +\infty) \quad \text{by: } \alpha(x, y) = \begin{cases} 1 & \text{if } x\mathcal{R}y \\ 0 & \text{otherwise} \end{cases}$$

Suppose that

$$\begin{cases} \alpha\left(x_1, x_2\right) \geqslant 1; \\ d\left(u_1, Tx_1\right) = d(A, B); \\ d\left(u_2, Tx_2\right) = d(A, B). \end{cases}$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{cases} x\mathcal{R}y; \\ d\left(u_1, Tx_1\right) = d(A, B); \\ d\left(u_2, Tx_2\right) = d(A, B). \end{cases}$$

Condition (2) of Theorem implies $u_1 \mathcal{R} u_2$, which gives us $\alpha(u_1, u_2) \geq 1$.

Thus we prove that T is $\alpha-$proximal admissible.

Condition (3) implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

by condition (5) T is continuous mapping.

Finally, condition (4) implies that

$$\alpha(x, y)\theta\left(d\left(Tx, Ty\right)\right) \leq \phi\left[\theta\left(d\left(x, y\right)\right)\right] for\ all\ x, y \in A,$$

Now all the hypotheses of Theorem 9 are satisfied, which implies the existence and uniqueness of a proximity point the non-self mapping. □

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