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The Equivalence Theorem for $QTAG$-Modules

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Abstract
We establish an equivalence theorem for countably generated submodules of a class of modules called folding modules. More precisely, we find necessary and sufficient conditions for two countably generated submodules of a folding module to be equivalent. A $QTAG$-module M is a folding module if every countably generated submodule has countably generated closure in the h-topology. The equivalence theorem is then used to obtain some structural results and homological properties of folding modules.

Keywords: QTAG-Modules; Folding Modules; Nice Submodules.

Introduction and Preliminary Terminology

Let R be any ring. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

In 1976 Singh [11] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [1, 13]. In 1987 Singh made an improvement and studied the modules satisfying only the condition (I) and called them $QTAG$-modules. Other authors also have considered the problem of detecting finite direct sums of uniserial modules [2, Theorem 4]. The study of $QTAG$-modules and their structure began with work of Singh in [12]. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed, introducing several notions, interesting properties and different characterizations of submodules. Many interesting results have

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been obtained, but still there remains a lot to explore.

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital $QTAG$-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. The module M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is called separable if every finitely generated submodule of M can be embedded in a summand of M. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to $g\left(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\right)$.

Imitating [9], the submodules $H_k(M)$, $k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty}(N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

For an ordinal σ, a submodule N of M is said to be σ-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ and a submodule N of M is said to be isotype in M, if it is σ-pure for every ordinal σ [10]. A submodule $N \subset M$ is nice [7] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height. If N is both isotype and nice, then N is called a balanced submodule of M. Observe that if N is closed, then N is necessarily nice.

An h-reduced module M is totally projective if it has a collection $\mathcal{N}$ of nice submodules such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated. Call a collection $\mathcal{N}$ of nice submodules of M which satisfies conditions (i), (ii) and (iii) a nice system (see [8]) for M.

Mehran et al. [10] proved that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules. Many results of this paper are motivated from the paper [5]. Our notations and terminology are standard and may be found in the texts [3, 4].

Main Theorem

Two submodules N_1 and N_2 of a $QTAG$-module M are equivalent [6, Definition 3.3], if $\exists$ an automorphism ψ of M such that $\psi(N_1) = N_2$. Equivalence theorems (concluding that submodules are equivalent under certain hypotheses) have played an important role in determining the structure of various classes of modules. Virtually all the equivalence theorems that have been established up to present time for $QTAG$-modules apply only to submodules of totally projective modules. However, we will prove an equivalence theorem for the class of modules called folding modules. This result defeats the notion that equivalence theorems are associated only with totally projective modules.

For convenience of notation and terminology, hereafter we call a $QTAG$-module M a folding module if every countably generated submodule has a countably generated closure in the h-topology. This is equivalent to saying that $H_\omega(M/N)$ is countably generated whenever N is a countably generated submodule of M. Perhaps it should be emphasized here that $H_\omega(M)$ need not be zero; in other words if we use the terminology of [3], we do not assume that M is separable. Since $H_\omega(M)$ is the closure of zero, the following proposition is a mere observation.

Proposition 2.1. *If M is a folding module, then $H_\omega(M)$ is countably generated and $M/H_\omega(M)$ is a separable folding module. Conversely, if $H_\omega(M)$ is countably generated and $M/H_\omega(M)$ is a folding module, then M, itself, must be a folding module.*

The next proposition is a rather immediate consequence of Proposition 2.1 that any separable folding module has the property that every countably generated submodule is contained in a countably generated balanced submodule.

Proposition 2.2. *Let M be a folding module (separable or not). Then every countably generated submodule of M is contained in a countably generated balanced submodule of M.*

Proof. Let N_1 be a countably generated submodule of the folding module M. Since $H_\omega(M)$ is countably generated, there exists a countably generated submodule N_2 of M that contains N_1 with the property that $H_\omega(N_2) = H_\omega(M)$. According to the statement of this proposition, there exists a countably generated submodule B of M containing the countably generated submodule N_2 such that $B/H_\omega(M)$ is balanced in $M/H_\omega(M)$. Obviously, B must be nice in M since M/B is separable. Moreover, since $H_\omega(M) \supseteq H_\omega(B) \supseteq H_\omega(N_2) \supseteq H_\omega(M)$, we conclude that $H_\omega(B) = H_\omega(M)$ and B is isotype in M. Hence B is balanced. □

We let c denote the cardinality of the continuum. If N is a submodule of the $QTAG$-module M, the coset valuation on the quotient module M/N is defined by

$$V_{x+N} = \sup\{H_M(x+y)+1 : x \in M, y \in N\}.$$

We shall apply the following criterion, which is the one normally used for $QTAG$-modules.

Criterion 2.1. *Let $\mathcal{P}$ be a partially ordered set that satisfies countable antichain condition. If $\mathcal{D} = \{\mathcal{D}_\alpha\}$ is a collection of fewer than c dense subsets of $\mathcal{P}$, there exists a directed subset $\mathcal{P}_1$ of $\mathcal{P}$ which is $\mathcal{D}$-generic ($\mathcal{P}_1 \cap \mathcal{D}_\alpha \neq \phi$ for each α)*

We are now ready to prove the equivalence theorem for folding modules.

Theorem 2.1. *Suppose that the $QTAG$-modules M and M' are folding modules of cardinality less than c. Let N and N' be isotype submodules of M and M', respectively, which are countably generated and have the same Ulm invariants. If $\pi : M/N \twoheadrightarrow M'/N'$ is an isomorphism that preserves the coset valuation, then there exists an isomorphism $\pi' : M \twoheadrightarrow M'$ that maps N onto N' and induces π. In particular, if $M = M'$ then N and N' are equivalent submodules of M.*

Proof. If K and K' are submodules of M and M', respectively, call a map $f : K \to K'$ admissible if it satisfies the following conditions.

- (i) f is an isomorphism from K to K'.
- (ii) f preserve heights computed in M and M'.
- (iii) f induces π, that is, $f(a) + N' = \pi(a+N)$ for all $a \in K$.

Define $\mathcal{P}$ to be the set of all admissible maps $f : K \twoheadrightarrow K'$ where K is any finitely generated submodule of M. We consider $\mathcal{P}$ partially ordered in the usual way: $f \geq g$, where $f : K \twoheadrightarrow K'$ and $g : L \twoheadrightarrow L'$, if $K \supseteq L$ and $f = g$ when restricted to L.

For finitely generated submodules L and L' of M and M', respectively, define

$$\mathcal{D}_{(L,L')} = \{f : K \twoheadrightarrow K', \text{ where } f \in \mathcal{P} \text{ and } K \supseteq L, K' \supseteq L'\}$$

It follows that $\mathcal{D}_{(L,L')}$ is h-dense in $\mathcal{P}$ for any pair of finitely generated submodules L and L' of M and M'. This simply means that given any $g : Q \twoheadrightarrow Q'$ in $\mathcal{P}$ that there exists $f \in \mathcal{D}_{(L,L')}$ for which $f \geq g$. Let

$\mathscr{D} = \{\mathscr{D}_{(L,L')}\}$ where L and L' range over all the finitely generated submodules of M and M', respectively. Since, by hypothesis, M and M' have cardinality less than c and since L and L' are finitely generated, we conclude that $\mathscr{D}$ is a collection of fewer than c dense subsets of $\mathscr{P}$.

To complete the proof of the theorem, it remains only to show that $\mathscr{P}$ satisfies countable antichain condition. For if $\mathscr{P}$ satisfies countable antichain condition then criterion 2.1 implies the existence of a directed $\mathscr{D}$-generic subset $\mathscr{P}_1$ of $\mathscr{P}$. Since $\mathscr{P}_1$ is directed, $\pi' = \sup\{f \in \mathscr{P}_1\}$ is a well-defined map from a submodule of M to a submodule of M' because $f(x) = g(x)$ if $f : K \twoheadrightarrow K'$ and $g : L \twoheadrightarrow L'$ both belong to $\mathscr{P}_1$ and $x \in K \cap L$. Moreover, since $\mathscr{P}_1$ is $\mathscr{D}$-generic, for any pair (L, L') of finitely generated submodules of M and M', the set $\mathscr{D}_{(L,L')}$ has an element in common with $\mathscr{P}_1$. Therefore, $\pi' : M \twoheadrightarrow M'$ is an isomorphism that induces π in view of the fact that the elements of $\mathscr{P}$ are admissible maps. Since π' induces π, it must be the case that $\pi'(N) = N'$.

In order to verify that $\mathscr{P}$ satisfies countable antichain condition suppose that $\mathscr{C}$ is an uncountable subset of $\mathscr{P}$. We need to show that there are two distinct elements f_1 and f_2 of $\mathscr{C}$ which are compatible (in the sense that there exists $g \in \mathscr{P}$ that satisfies $g \geq f_1$ and $g \geq f_2$). Clearly, we may assume that $\mathscr{C} = \{\alpha : K_\alpha \twoheadrightarrow K'_\alpha\}$ is a subset of $\mathscr{P}$ indexed by ω_1, so α ranges over the countable ordinals. Moreover, we may assume without loss of generality that there is a fixed finitely generated submodule K such that $K_\alpha \cap K_\beta = K$ for all distinct α and β. Therefore, it follows readily that we may assume that $K_\beta \cap \sum_{\alpha<\beta} K_\alpha = K$ for all $\beta > 0$ because the elements in the countable set $(\sum_{\alpha<\beta} K_\alpha)\backslash K$ can not be contained in an uncountable number of the K_γ's without violating $K_\gamma \cap K_\delta = K$, and we can reach the desired result by successively eliminating the appropriate (countable number of) K_γ's as we inductively select the K_β's. Indeed, in view of the fact that M is a folding module, we can go significantly further. The initial step in the next move is to observe that since the closure of $\sum_{\alpha<\beta} K_\alpha$ is countably generated for any $\beta \in \omega_1$ we may, using the same strategy stated above, choose K_β so that $K_\beta \cap \overline{(\sum_{\alpha<\beta} K_\alpha)} = K$ when $\beta > 0$. This means that if $x \in K_\beta\backslash K$ then the heights of all elements in the set $S_{\beta,x} = \{y+x : y \in \sum_{\alpha<\beta} K_\alpha\}$ are under a finite bound, for otherwise x would be in the closure of $\sum_{\alpha<\beta} K_\beta$. Since K_β is finitely generated, there exists a positive integer I_β such that $H_M(y + x) \leq I_\beta$ for all $y \in \sum_{\alpha<\beta} K_\alpha$ and all $x \in K_\beta\backslash K$. Next, choose a positive integer I so that $I_\beta = I$ for uncountably many β. We may assume that $I_\beta = I$ for all β; in fact, the subsequence

$$K_{\beta(0)}, K_{\beta(1)} \ldots, K_{\beta(\lambda)} \ldots, \lambda < \omega_1,$$

satisfies this condition when $0 < \beta(0) < \beta(1) < \ldots \beta(\lambda) < \ldots$ is an increasing sequence of countable ordinals $\beta(\lambda)$ that satisfy $I_{\beta(\lambda)} = I$. We, of course, reindex and $K_{\beta(\lambda)}$ becomes simply K_λ. Henceforth, we shall assume that, for a fixed positive integer I, it is true for all $\beta \in \omega_1$ that $H_M(y+x) \leq I$ whenever $y \in \sum_{\alpha<\beta} K_\alpha$ and $x \in K_\beta\backslash K$.

We now make another crucial observation. Since N and N' are countably generated and since $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$ induces π, the set $\{f_\alpha(b) : \alpha \in \omega_1\}$ is at most countable for a fixed element $b \in K$. In other words, π allows only a countable number of choices for $f(b)$ whenever f induces π since $f(b) + N' = \pi(b + N)$. Therefore, we may assume that all the maps $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$, $\alpha \in \omega_1$, are identical on K; $f_\alpha(x) = f_\beta(x)$ if $x \in K = \bigcap_{\alpha<\omega_1} K_\alpha$.

Naturally, all the normalizing assumptions that we have made for the K_α's can also be made for the K_α's (with respect to the maps $f_\alpha^{-1} : K'_\alpha \twoheadrightarrow K_\alpha$). In particular, if $K' = \bigcap_{\alpha<\omega_1} K'_\alpha$, then we may assume that $H_M(y' + x') \leq I$ whenever $y' \in \sum_{\alpha<\beta} K'_\alpha$ and $x' \in K'_\beta\backslash K'$. Moreover, we may assume that $f_\alpha^{-1}(x) = f_\beta^{-1}(x)$ whenever $x \in K' = \bigcap_{\alpha<\omega_1} K'_\alpha$. At this point, it is clear under the above assumptions that it must be the case that $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$ maps K to K' for each α (and when restricted to K all the f_α's are identical).

Before we are in a position to conclude that $\mathscr{P}$ satisfies countable antichain condition we need one more refinement of the maps $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$ belonging to our uncountable collection $\mathscr{C}$. Nothing we have done thus far actually requires that the f_α's are distinct (but of course we will need to use the fact that they are distinct before we can prove that there are two different ones that are compatible). For each positive integer $n \leq I$ and each $\alpha \in \omega_1$, define the map

$$f_{\alpha,n} : (K_\alpha + H_n(M))/H_n(M) \twoheadrightarrow (K'_\alpha + H_n(M'))/H_n(M')$$

by $f_{\alpha,n}(x+H_n(M)) = f_\alpha(x)+H_n(M')$ where $x \in K_\alpha$ and $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$. Note that $f_{\alpha,n}$ induces the natural map $\pi_n : M/(N + H_n(M)) \twoheadrightarrow M'/(N' + H_n(M'))$ induced by π. Also, notice that $f_{\alpha,n}$ is to π_n as f_α is to π. Letting $\overline{K_n} = \bigcap_{\alpha<\omega_1} (K_\alpha + H_n(M))/H_n(M)$ and $\overline{K'_n} = \bigcap_{\alpha<\omega_1} (K'_\alpha + H_n(M'))/H_n(M')$, we may assume, by the arguments previously presented, that

$$\sum_{\alpha<\beta} (K_\alpha + H_n(M))/H_n(M) \cap (K_\beta + H_n(M))/H_n(M) = \overline{K_n}$$

and

$$\sum_{\alpha<\beta} (K'_\alpha + H_n(M'))/H_n(M') \cap (K'_\beta + H_n(M'))/H_n(M') = \overline{K'_n}$$

hold for each $0 < \beta < \omega_1$ and all $n \leq I$. Finally, we may assume, for all $\alpha < \beta \in \omega_1$, that

$$f_{\alpha,n}(x + H_n(M)) = f_{\beta,n}(x + H_n(M)), \quad \text{if } (x + H_n(M)) \in \overline{K_n}$$

and

$$f^{-1}_{\alpha,n}(x' + H_n(M')) = f^{-1}_{\beta,n}(x' + H_n(M')), \quad \text{if } (x' + H_n(M')) \in \overline{K'_n}.$$

Now, we have all we need. If $\alpha < \beta \in \omega_1$, then not only do $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$ and $f_\beta : K_\beta \twoheadrightarrow K'_\beta$ have a common extension $f : (K_\alpha + K_\beta) \twoheadrightarrow (K'_\alpha + K'_\beta)$ by virtue of the fact that f_α and f_β agree on $K = K_\alpha \cap K_\beta$, but f is admissible and therefore belongs to $\mathcal{P}$. Recall that f is admissible if:

(i) f is an isomorphism from $(K_\alpha + K_\beta)$ onto $(K'_\alpha + K'_\beta)$.

(ii) f preserve heights computed in M and M'.

(iii) f induces π.

With conditions (i) and (iii) being immediate, only (ii) requires any discussion. To show that (ii) holds, let $y \in K_\alpha$ and $x \in K_\beta$. If $x \in K$, then $y + x \in K_\alpha$ and $f(y + x) = f_\alpha(y + x)$ has the same height as $y + x$ since f_α is admissible. Thus assume $x \notin K$. We know that $H_M(y + x) \leq I$ in this case. Let $H_M(y + x) = n \leq I$. This means that $x + H_n(M) \in \overline{K_n}$ since

$$\sum_{\gamma<\beta} (K_\gamma + H_n(M))/H_n(M) \cap (K_\beta + H_n(M))/H_n(M) = \overline{K_n}$$

and since

$$x + H_n(M) = -y + H_n(M) \in (K_\alpha + H_n(M))/H_n(M) \cap (K_\beta + H_n(M))/H_n(M)$$

The rest is routine:

$$\begin{aligned} f(y + x) + H_n(M') &= f_\alpha(y) + f_\beta(x) + H_n(M'), \\ &= f_{\beta,n}(x + H_n(M)) - f_{\alpha,n}(x + H_n(M)), \\ &= 0 + H_n(M) \end{aligned}$$

because $f_{\alpha,n}$ and $f_{\beta,n}$ agree on $\overline{K_n}$. Therefore, $H_M(f(y + x)) \geq n$, and equality follows by symmetry.

Since $f \geq f_\alpha$ and $f \geq f_\beta$, where $f \in \mathcal{P}$ and the maps $f_\alpha : K_\alpha \twoheadrightarrow K'_\alpha$ and $f_\beta : K_\beta \twoheadrightarrow K'_\beta$ are distinct elements of $\mathcal{C} \subseteq \mathcal{P}$, we have shown that $\mathcal{P}$ satisfies countable antichain condition.

This completes the proof. □

Applications

The purpose of the present section is to explore some structural consequences of Theorem 2.1. Several such applications are now presented. We begin with the following isomorphism theorem.

Theorem 3.2. *Let M be a folding module of cardinality not exceeding $\aleph_1$ and let σ be a countable limit ordinal. Suppose that M_1 and M_2 are arbitrary $QTAG$-modules such that $H_\sigma(M_1) \cong M' \cong H_\sigma(M_2)$, where M' is countably generated. if*

$$M_1/H_\sigma(M_1) \cong M \cong M_2/H_\sigma(M_2)$$

then M_1 and M_2 must be isomorphic.

Proof. First observe that M_1 and M_2 are folding modules since each is an extension of countably generated module by a folding module. Next, choose countably generated submodules K_1 and K_2 of M_1 and M_2, respectively, so that $H_\sigma(K_1) = H_\sigma(M_1)$ and $H_\sigma(K_2) = H_\sigma(M_2)$; this is certainly possible since M' and σ are countably generated and countable limit ordinal, respectively. According to Proposition 2.1, we can choose countably generated submodules N_1 and N_2 of M_1 and M_2, respectively, containing K_1 and K_2 with the property that, when identified as submodules of M, the quotient modules $N_1/H_\sigma(N_1)$ and $N_2/H_\sigma(N_2)$ represent the same balanced submodule B of M. It is easily verified that N_1 and N_2 have the same Ulm invariants (and are therefore isomorphic) since $H_\sigma(N_1) \cong M' \cong H_\sigma(N_2)$ and since $N_1/H_\sigma(N_1) \cong B \cong N_2/H_\sigma(N_2)$.

Notice that M_1 and M_2 are folding modules of cardinality $\aleph_1 < c$ since M_1 and M_2 are extensions of the countably generated module M' by the folding module M of cardinality $\aleph_1$. Let $\pi : M_1/N_1 \longrightarrow M_2/N_2$ denote the composition of the isomorphisms $M_1/N_1 \cong M/B \cong M_2/N_2$. Since N_1 and N_2 are balanced submodules of M_1 and M_2 respectively, the mapping π necessarily preserve the coset valuation (required in Theorem 2.1). Therefore, all the hypotheses of Theorem 2.1 are satisfied, and consequently there exists an isomorphism $\pi' : M_1 \longrightarrow M_2$ that maps N_1 onto N_2. In particular, M_1 and M_2 are isomorphic, and the theorem is proved. □

As the next corollary demonstrates, there are further structural consequences of the preceding results.

Corollary 3.1. *Let M be a folding module of cardinality less than c. Then M is the direct sum of a countably generated module and a separable folding module. Indeed, if N is any countably generated submodule of M, there exists a decomposition $M = B \oplus T$, where B is a countably generated summand containing N and the summand T is a separable folding module.*

Proof. Suppose that M is an arbitrary folding module of cardinality less than c and that N is any countably generated submodule of M. Since $H_\omega(M)$ is countably generated, Proposition 2.2 implies the existence of a countably balanced submodule B of M that contains both N and $H_\omega(M)$. Theorem 2.1 implies that the balanced exact sequence

$$B \twoheadrightarrow M \twoheadrightarrow M/B$$

is equivalent to the split exact sequence

$$B \twoheadrightarrow (B \oplus M)/B \twoheadrightarrow M/B$$

that is, Theorem 2.1 asserts that there is an isomorphism π from M onto $(B \oplus M)/B$ that takes B onto itself. Therefore, the first sequence splits since the second one does. Hence the chosen balanced submodule B must be a direct summand of M, and we have that $M = B \oplus T$ where $T \cong M/B$ is a separable folding module. □

We conclude with the following uniqueness theorem.

Theorem 3.3. *Suppose that the $QTAG$-modules M_1 and M_2 are countably generated extension of the same folding module M having cardinality $\aleph_1$. Then M_1 and M_2 are isomorphic if and only if they have the same Ulm invariants.*

Proof. Choose countable submodules N_1 and N_2 of M_1 and M_2 respectively, so that $M_1 = N_1 + M$ and $M_2 = N_2 + M$. Without loss of generality, we may assume that N_1 and N_2 are isotype because because $H_\omega(M_1)$ and $H_\omega(M_2)$ are countably generated. Moreover, letting $f_{M_1}(\alpha)$ denote the α-th Ulm invariant of M_1, we may assume that

$$f_{N_1}(\alpha) = \begin{cases} f_{M_1}(\alpha) & \text{if } f_{M_1}(\alpha) \text{ is finite} \\ \aleph_0 & \text{if } f_{M_1}(\alpha) \text{ is infinite} \end{cases}$$

and the corresponding relation for $f_{N_2}(\alpha)$. Now, choose a countably generated, balanced submodule B of M that contains $M \cap N_1$ and $M \cap N_2$. Since B is countably generated, M/B is a folding module of cardinality $\aleph_1$ because M is. Therefore, B is a direct summand of M. Let $M = B \oplus K$. Observe that $M_1 = (N_1 + B) \oplus K$ and $M_2 = (N_2 + B) \oplus K$. But $N_1 + B$ and $N_2 + B$ are countably generated, and due to the choice of N_1 and N_2 we know that $N_1 + B$ and $N_2 + B$ necessarily have the Ulm-invariants (since as summands of M_1 and M_2 their invariants cannot exceed those of M_1 and M_2). Since $N_1 + B$ and $N_2 + B$ are isomorphic, the proof is finished. □

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