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Unique Solution of Operator Equations in Arbitrary Banach Spaces

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Abstract
We apply the Banach's contraction principle to obtain a unique solution of the operator equation $x - Tx = f$ in arbitrary Banach spaces.

Keywords: Banach's contraction principle; Contraction mappings; Picard iteration; Operator equation; Banach spaces.

Preliminaries

We require the followings definitions and the statement of the Banach's contraction principle.

Definition 1.1. [2] Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be Lipschitzian if there is a constant $k \geq 0$ such that for all $x, y \in M$,

$$d(T(x), T(y)) \leq kd(x, y).$$

The smallest number k for which the above inequality holds is called the Lipschitz constant of T.

Definition 1.2. [2] A Lipschitzian mapping $T : M \to M$ with Lipschitz constant $k < 1$ is said to be a contraction mapping.

Theorem 1.3. *[2] (Banach contraction principle) Let (M, d) be a complete metric space and let $T : M \to M$ be a contarction mapping, then T has a unique fixed point in M.*

Definition 1.4. [1] Let (X, d) be a metric space, and $T : X \to X$ a self map. For a given $x_0 \in X$, we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method,

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, ...$$

The sequence thus defined is known as the sequence of successive approximations or simply, Picard iteration.

Main Theorem

Theorem 1.5. *Let X be an arbitrary Banach space, f an element in X and $T : X \to X$ a contraction mapping, then the operator equation*

$$x - Tx = f$$

has a unique solution if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

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Proof. Let T_f be the mapping from X into X defined by

$$T_f(u) = Tu + f.$$

Then u is a unique solution of $x - Tx = f$ if and only if u is a unique fixed point of T_f. Since T is a contraction mapping, for $x, y \in X$, we have

$$\|T_f(x) - T_f(y)\| = \|T(x) - T(y)\| < \|x - y\|,$$

giving that T_f is also a contraction mapping.

Suppose T_f has a unique fixed point $u \in X$, then for all $n \in \mathbb{N}$

$$\|x_{n+1} - u\| = \|Tx_n + f - u\| = \|T_f(x_n) - T_f(u)\| < \|x_n - u\|,$$

hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and for each $x \in X$

$$B_d(x) = \{y \in X : \|x - y\| < d\}.$$

We define $C_n = \bigcap_{i \geq n} B_d(x_i)$. Now we have using that T is a contraction mapping and the given Picard iteration,

$$\begin{aligned} y \in B_d(x_n) &\Rightarrow \|y - x_n\| < d \\ &\Rightarrow \|Ty - Tx_n\| < d \\ &\Rightarrow \|Ty - [x_{n+1} - f]\| < d \\ &\Rightarrow \|(Ty + f) - x_{n+1}\| < d \\ &\Rightarrow (Ty + f) \in B_d(x_{n+1}). \end{aligned}$$

The above implications give the following:

$$\begin{aligned} T_f(C_n) &= T_f\Big(\bigcap_{i \geq n} B_d(x_i)\Big) \\ &\subseteq \bigcap_{i \geq n} T_f(B_d(x_i)) \\ &= \bigcap_{i \geq n} \{T_f(y) : \|y - x_i\| < d\} \\ &= \bigcap_{i \geq n} \{(T(y) + f) : \|y - x_i\| < d\} \\ &\subseteq \bigcap_{i \geq n+1} B_d(x_i) = C_{n+1}. \end{aligned}$$

Let $C = \overline{\bigcup_{n \in \mathbb{N}} C_n}$. Since C_n increases with n, C is a closed and bounded subset of X. We next consider

$$T_f(C) = T_f(\overline{\bigcup C_n}) \subseteq \overline{T_f(\bigcup C_n)} = \overline{\bigcup T_f(C_n)} \subseteq \overline{\bigcup C_{n+1}} = C,$$

giving that T_f maps C into itself. Now since C is a closed subset of the Banach space X, C is itself complete. Finally applying the Banach contraction principle to T_f and C, we get that T_f has a unique fixed point in C which proves the theorem. □

Conflict of interests: The author declares that there is no conflict of interests.

References

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