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$*$-K-g-Frames in Hilbert Pro-C^*-Modules

Mohamed Rossafi[1], Roumaissae Eljazzar[2]

[1]LaSMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, B. P. 1796 Fes Atlas Morocco
[2]Laboratory of Partial Differential Equations, Spectral Algebra and Geometry Department of Mathematics, Faculty of Sciences, University Ibn Tofail, Kenitra, Morocco

Correspondence should be addressed to Mohamed Rossafi: rossafimohamed@gmail.com, mohamed.rossafi@usmba.ac.ma

Email(s): rossafimohamed@gmail.com (M. Rossafi), roumaissae.eljazzar@uit.ac.ma (R. Eljazzar)

Abstract
In this paper, we introduce the concept of $*$-K-g-frames in Hilbert modules over a pro-C^*-algebras. The analysis operator, the synthesis operator and the frame operator are presented. Also, we investigate the relationship between $*$-g-frames, and $*$-K-g-frames. We give some properties of them. Finally, we study the tensor product of $*$-K-g-frames for Hilbert pro-C^*-modules.

Keywords: Frame; $*$-K-g-frame; pro-C^*-algebra; Hilbert pro-C^*-modules; Tensor Product.

Introduction

Frame theory is recently an active research area in mathematics, computer science and engineering with many exciting applications in a variety of different fields. They are generalizations of bases in Hilbert spaces. Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaefer [4] for study some problems of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [3], and popularized from then on. Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

Pro-C^*-algebras also called locally C^*-algebra is a (projective) limit of C^*-algebras in the category of topological $*$-algebras. In this direction we mention, in particular, the works of Inoue [6], Zhuraev and Sharipov [11] and Phillips [9].

The aim of this paper is to introduce the notion of $*$-K-g-frame in Hilbert modules over pro-C^*-algebras and investigate some results for these frames. We extend some results about $*$-K-g-frames for Hilbert C^*-modules from [10].

This paper is divided into three sections. After recalling some fundamental definitions and notations of Hilbert pro-C^*-modules in section 2, we move on to definition of $*$-K-g-frame and we give some of its properties. Finally in section 4 we investigate the tensor product of Hilbert pro-C^*-modules,we show that tensor product of $*$-K-g-frame for Hilbert pro-C^*-modules $\mathcal{X}$ and $\mathcal{Y}$, present $*$-K-g-frame for $\mathcal{X} \otimes \mathcal{Y}$, and tensor product of their frame operators is the frame operator of the tensor product of $*$-K-g-frame.

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Preliminaries

In this section we briefly recall some definitions and properties of pro-C^*-algebras, which will be necessary to prove our results.

Recall that a pro-C^*-algebra is a complete Hausdorff complex topological $*$-algebra $\mathscr{A}$ whose topology is determined by its continuous C^*-seminorms in the sens that a net $\{a_\alpha\}$ converges to 0 if and only if $p(a_\alpha)$ converges to 0 for all continuous C^*-seminorm p on $\mathscr{A}$ and we have:

1) $p(ab) \leq p(a)p(b)$

2) $p(a^*a) = p(a)^2$

for all $a, b \in \mathscr{A}$.

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

We denote by $sp(a)$ the spectrum of a such that: $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathscr{A}} - a$ is not invertible $\}$ for all $a \in \mathscr{A}$. Where $\mathscr{A}$ is unital pro-C^*-algebra with unite $1_{\mathscr{A}}$.

The set of all continuous C^*-seminorms on $\mathscr{A}$ is denoted by $S(\mathscr{A})$. If $\mathscr{A}^+$ denotes the set of all positive elements of $\mathscr{A}$, then $\mathscr{A}^+$ is a closed convex C^*-seminorms on $\mathscr{A}$.

Example 1.1. Every C^*-algebra is a pro-C^*-algebra.

Proposition 1.2. *[6]. Let $\mathscr{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathscr{A}}$. Then for any $p \in S(\mathscr{A})$, we have:*

(1) $p(a) = p(a^*)$ for all $a \in A$

(2) $p(1_{\mathscr{A}}) = 1$

(3) If $a, b \in \mathscr{A}^+$ and $a \leq b$, then $p(a) \leq p(b)$

(4) If $1_{\mathscr{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathscr{A}}$

(5) If $a, b \in \mathscr{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$

(6) If $a, b, c \in \mathscr{A}$ and $a \leq b$ then $c^*ac \leq c^*bc$

(7) If $a, b \in \mathscr{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$

Definition 1.3. [9]. A pre-Hilbert module over pro-C^*-algebra $\mathscr{A}$, is a complex vector space E which is also a left $\mathscr{A}$-module compatible with the complex algebra structure, equipped with an $\mathscr{A}$-valued inner product $\langle ., . \rangle$ $E \times E \to \mathscr{A}$ which is $\mathbb{C}$-and $\mathscr{A}$-linear in its first variable and satisfies the following conditions:

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$

2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$

3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$

for every $\xi, \eta \in E$. We say E is a Hilbert $\mathscr{A}$-module (or Hilbert pro-C^*-module over $\mathscr{A}$). If E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)} \quad \xi \in E, p \in S(\mathscr{A})$$

Let $\mathscr{A}$ be a pro-C^*-algebra and let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathscr{A}$ -modules.A bounded $\mathscr{A}$-module map from $\mathcal{X}$ to $\mathcal{Y}$ is called an operators from $\mathcal{X}$ to $\mathcal{Y}$. We denote the set of all operator from $\mathcal{X}$ to $\mathcal{Y}$ by $Hom_{\mathscr{A}}(\mathcal{X}, \mathcal{Y})$.

Definition 1.4. An $\mathscr{A}$-module map $T : \mathfrak{X} \longrightarrow \mathcal{Y}$ is adjointable if there is a map $T^* : \mathcal{Y} \longrightarrow \mathfrak{X}$ such that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi \in \mathfrak{X}, \eta \in \mathcal{Y}$, and is called bounded if for all $p \in S(\mathscr{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(T\xi) \leq M_p \bar{p}_{\mathfrak{X}}(\xi)$ for all $\xi \in \mathfrak{X}$.

It is clear that every adjointable map is a bounded $\mathscr{A}$-module map. The set of all adjointable maps from $\mathfrak{X}$ into $\mathcal{Y}$ is denoted by $Hom^*_{\mathscr{A}}(\mathfrak{X}, \mathcal{Y})$ and we write $Hom^*_{\mathscr{A}}(\mathfrak{X}) = Hom^*_{\mathscr{A}}(\mathfrak{X}, \mathfrak{X})$ The vector space $Hom^*_{\mathscr{A}}(\mathfrak{X}, \mathcal{Y})$ is a complete locally convex space.

The Hilbert $M(\mathscr{A})$-module $\mathscr{L}(\mathscr{A}, \mathfrak{X})$ is called the multiplier module of $\mathfrak{X}$ and it is denoted by $M(\mathfrak{X})$. For all $h \in M(\mathfrak{X})$ and $\xi \in \mathfrak{X}$, we have $\langle h, \xi\rangle_{M(\mathfrak{X})} = h^*(\xi)$. Moreover, if $a \in \mathscr{A}$ and $h \in M(\mathfrak{X})$, then $h.a$ can be identified by $h(a)$.

Definition 1.5. Let $\mathscr{A}$ be a pro-C^*-algebra and $\mathfrak{X}, \mathcal{Y}$ be two Hilbert $\mathscr{A}$-modules. The operator $T : \mathfrak{X} \to \mathcal{Y}$ is called uniformly bounded below, if there exists $C > 0$ such that for each $p \in S(\mathscr{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \leqslant C\bar{p}_{\mathfrak{X}}(\xi), \qquad \text{for all } \xi \in \mathfrak{X}$$

and is called uniformly bounded above if there exists $C' > 0$ such that for each $p \in S(\mathscr{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \geqslant C'\bar{p}_{\mathfrak{X}}(\xi), \qquad \text{for all } \xi \in \mathfrak{X}$$

$$\|T\|_\infty = \inf\{M : M \text{ is an upper bound for } T\}$$

$$\hat{p}_{\mathcal{Y}}(T) = \sup\left\{\bar{p}_{\mathcal{Y}}(T(x)) : \xi \in \mathfrak{X}, \quad \bar{p}_{\mathfrak{X}}(\xi) \leqslant 1\right\}$$

It's clear to see that, $\hat{p}(T) \leqslant \|T\|_\infty$ for all $p \in S(\mathscr{A})$.

Proposition 1.6. *[5]. Let T be an uniformly bounded below operator in $Hom^*_{\mathscr{A}}(\mathfrak{X}, \mathcal{Y})$. then T is closed and injective.*

Proposition 1.7. *[2]. Let $\mathfrak{X}$ be a Hilbert module over pro-C^*-algebra $\mathscr{A}$ and T be an invertible element in $Hom^*_A(\mathfrak{X})$ such that both are uniformly bounded. Then for each $\xi \in \mathfrak{X}$,*

$$\left\|T^{-1}\right\|_\infty^{-2} \langle \xi, \xi\rangle \leq \langle T\xi, T\xi\rangle \leq \|T\|^2_\infty \langle \xi, \xi\rangle.$$

Let $\mathfrak{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules, and $\{\mathcal{Y}_i\}_{i\in I}$ be a countable sequence of closed submodules of $\mathcal{Y}$.

Definition 1.8. [8]. We call a sequence $\Lambda = \{\Lambda_i \in Hom_{\mathscr{A}}(\mathfrak{X}, \mathcal{Y}_i)\}_{i\in I}$ a $*$-g-frame for $\mathfrak{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I} \langle \Lambda_i \xi, \Lambda_i \xi\rangle \leq B\langle \xi, \xi\rangle B^* \tag{1.1}$$

for all $\xi \in \mathfrak{X}$ and strictly nonzero elements $A, B \in \mathscr{A}$. The number A and B are called $*$-g-frame bounds for Λ. The $*$-g-frame is called tight if $A = B$ and a Parseval if $A = B = 1$. If in the above we only have the upper bound, then Λ is called a $*$-g-Bessel sequence. Also if for each $i \in I$, $\mathcal{Y}_i = \mathcal{Y}$, we call Λ a $*$-g-frame for $\mathfrak{X}$ with respect to $\mathcal{Y}$.

$*$-K-g-frames in Hilbert pro-C^*-modules

Let $\mathscr{A}$ be a pro-C^*-algebra, $\mathfrak{X}$ and $\mathcal{Y}$ two Hilbert $\mathscr{A}$-modules, and $\{\mathcal{Y}_i\}_{i\in J}$ is a countable sequence of closed submodules of $\mathcal{Y}$.

Definition 1.9. Let $K \in Hom_{\mathcal{A}}^{*}(\mathcal{X})$. We say that $\{\Lambda_i \in Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}_i)\}_{i\in I}$ is $*$-K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if there exist nonzero elements $A, B \in \mathcal{A}$ such that for all $\xi \in \mathcal{X}$,

$$A\langle K^*\xi, K^*\xi\rangle A^* \leq \sum_{i\in I} \langle \Lambda_i\xi, \Lambda_i\xi\rangle \leq B\langle \xi, \xi\rangle B^* \tag{1.2}$$

The numbers A and B are called lower and upper bound of the $*$-K-g-frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle A^* = \sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle, \forall\xi \in \mathcal{X}. \tag{1.3}$$

The $*$-K-g-frame is A-tight.

Example 1.10. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbf{N}}, v = \{v_j\}_{j\in\mathbf{N}} \in l^\infty$, we define

$$uv = \{u_jv_j\}_{j\in\mathbf{N}}, u^* = \{\bar{u_j}\}_{j\in\mathbf{N}}, \|u\| = \sup_{j\in\mathbf{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a C^*-algebra. Consequently $\mathcal{A} = \{l^\infty, \|.\|\}$ is pro-C^*-algebra.

Let $\mathcal{X} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j\bar{u_j}\}_{j\in\mathbf{N}}.$$

Then $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module.

Define $f_j = \{f_i^j\}_{i\in\mathbf{N}^*}$ by $f_i^j = \frac{1}{2} + \frac{1}{i}$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ $\forall j \in \mathbf{N}^*$. Now define the adjointable operator $\Lambda_j : \mathcal{X} \to \mathcal{A}$, $\Lambda_j\xi = \langle \xi, f_j\rangle$.

then for every $\xi \in \mathcal{X}$ we have

$$\sum_{j\in\mathbf{N}}\langle \Lambda_j\xi, \Lambda_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbf{N}^*}\langle \xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbf{N}^*}.$$

Let $K : \mathcal{X} \to \mathcal{X}$ defined by $K\xi = \{\frac{\xi_i}{i}\}_{i\in\mathbf{N}^*}$.

Then for every $\xi \in \mathcal{X}$ we have

$$\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} \leq \sum_{j\in\mathbf{N}}\langle \Lambda_j\xi, \Lambda_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbf{N}^*}\langle \xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbf{N}^*}.$$

Which shows that $\{\Lambda_j\}_{j\in\mathbf{N}}$ is a $*$-K-g-frame for $\mathcal{X}$ with bounds 1 and $\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbf{N}^*}$.

Remark 1.11.
1. Every $*$-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$ is an $*$-K-g-frame, for any $K \in Hom_{\mathcal{A}}^{*}(\mathcal{X})$: $K \neq 0$.
2. If $K \in Hom_{\mathcal{A}}(\mathcal{X})$ is an bounded surjective operator, then every $*$-K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$ is a $*$-g-frame.

Example 1.12. Let $\mathcal{X}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module. $Hom_A^*(\mathcal{X})$ Let $K \in Hom_{\mathcal{A}}^{*}(\mathcal{X})$ an invertible element such that both are uniformly bounded and $K \neq 0$. Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself with the inner product $\langle a, b\rangle = ab^*$. Let $\{x_i\}_{i\in I}$ be an $*$-frame for $\mathcal{X}$ with bounds A and B, respectively. For each $i \in I$, we define $\Lambda_i : \mathcal{X} \to \mathcal{A}$ by $\Lambda_i\xi = \langle \xi, x_i\rangle$, $\forall\xi \in \mathcal{X}$. Λ_i is adjointable and $\Lambda_i^*a = ax_i$ for each $a \in \mathcal{A}$. And we have

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle \xi, x_i\rangle\langle x_i, \xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall\xi \in \mathcal{X}.$$

Or

$$\langle K^*\xi, K^*\xi\rangle \leq \|K\|_\infty^2\langle \xi, \xi\rangle, \forall\xi \in \mathcal{X}.$$

Then

$$\|K\|_\infty^{-1}A\langle K^*\xi, K^*\xi\rangle(\|K\|_\infty^{-1}A)^* \leq \sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall\xi \in \mathcal{X}.$$

So $\{\Lambda_i\}_{i\in I}$ is $*$-K-g-frame for $\mathcal{X}$ with bounds $\|K\|_\infty^{-1}A$ and B, respectively.

Definition 1.13. Let $\{\Lambda_i\}_{i\in I}$ be an $*$-K-g-frame in $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$. We define the analysis operator as follows

$$T : \mathcal{X} \to \oplus_{i\in I}\mathcal{Y}_i \quad \text{by} \quad T\xi = \{\Lambda_i\xi\}_i, \forall \xi \in \mathcal{X}$$

So the synthesis operator is

$$T^* : \oplus_{i\in I}\mathcal{Y}_i \to \mathcal{X} \quad \text{given by} \quad T^*(\{\xi_i\}_i) = \sum_{i\in I}\Lambda_i^*\xi_i, \quad \forall\{\xi_i\}_i \in \oplus_{i\in I}\mathcal{Y}_i.$$

The combination of T and T^*, gives the frame operator $S : \mathcal{X} \to \mathcal{X}$ such that $S\xi = T^*T\xi = \sum_{i\in I}\Lambda_i^*\Lambda_i\xi$.

Theorem 1.14. *Let $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$ be an bounded surjective operator. If $\{\Lambda_i\}_{i\in I}$ is an $*$-K-g-frame in $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$, then the frame operator S is invertible, positive and it is self-adjoint such that :*

$$AI_{\mathcal{X}}A^* \leq S \leq BI_{\mathcal{X}}B^*$$

Where $I_{\mathcal{X}}$ is the identity function on $\mathcal{X}$.

Proof. Result of (2) in Remark 1.11 and Theorem 3.1 in [8]. □

Let $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$, in the following theorem we construct an $*$-K-g-frame using an $*$-g-frame.

Theorem 1.15. *Let $K \in Hom_{\mathcal{A}}^*(\mathcal{X})$ an invertible element such that both are uniformly bounded and $\{\Lambda_i\}_{i\in I}$ be an $*$-g-frame in $\mathcal{X}$ with respect to $\{\mathcal{X}_i : i \in I\}$ with bounds A, B. Then $\{\Lambda_iK\}_{i\in I}$ is an $*$-K^*-g-frame in $\mathcal{X}$ with respect to $\{\mathcal{X}_i : i \in I\}$ with bounds A, $\|K\|_\infty B$. The frame operator of $\{\Lambda_iK\}_{i\in I}$ is $S' = K^*SK$, where S is the frame operator of $\{\Lambda_i\}_{i\in I}$.*

Proof. From

$$A\langle\xi,\xi\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle\Lambda_i\xi,\Lambda_i\xi\rangle_{\mathcal{A}} \leq B\langle\xi,\xi\rangle_{\mathcal{A}}B^*, \forall\xi \in \mathcal{X}.$$

We get for all $\xi \in \mathcal{X}$,

$$A\langle K\xi,K\xi\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle\Lambda_iK\xi,\Lambda_iK\xi\rangle_{\mathcal{A}} \leq B\langle K\xi,K\xi\rangle_{\mathcal{A}}B^* \leq \|K\|_\infty B\langle\xi,\xi\rangle_{\mathcal{A}}(\|K\|_\infty B)^*.$$

Then $\{\Lambda_iK\}_{i\in I}$ is an $*$-K^*-g-frame in $\mathcal{X}$ with respect to $\{\mathcal{Y}_i : i \in I\}$ with bounds A, $\|K\|_\infty B$.

By definition of S,we have $SK\xi = \sum_{i\in I}\Lambda_i^*\Lambda_iK\xi$. Then

$$K^*SK = K^*\sum_{i\in I}\Lambda_i^*\Lambda_iK\xi = \sum_{i\in I}K^*\Lambda_i^*\Lambda_iK\xi.$$

Hence $S' = K^*SK$. □

Corollary 1.16. *Let $K \in Hom_{\mathcal{A}}(\mathcal{X})$ and $\{\Lambda_i\}_{i\in I}$ be an $*$-g-frame. Then $\{\Lambda_iS^{-1}K\}_{i\in I}$ is an $*$-K^*-g-frame, where S is the frame operator of $\{\Lambda_i\}_{i\in I}$.*

Proof. Result of the Theorem 1.15 for the $*$-g-frame $\{\Lambda_iS^{-1}\}_{i\in I}$. □

Theorem 1.17. *Let $K \in Hom_{\mathcal{A}}(\mathcal{X})$ bounded and surjective such that $K = K^*$, $\{\Lambda_i\}_{i\in I} \in Hom_{\mathcal{A}}(\mathcal{X},\mathcal{Y}_i)$ and $\sum_{i\in I}\langle\Lambda_i\xi,\Lambda_i\xi\rangle$ converge in the semi-norm for $\xi \in \mathcal{X}$. Then $\Lambda = \{\Lambda_j\}_{i\in I}$ is a $*$-K-g-frame for $\mathcal{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ if and only if there are two strictly nonzero elements $C, D \in \mathcal{A}$ and two constants $m, M > 0$ such that for every $\xi \in \mathcal{X}$,*

$$p\left((Cm^{\frac{1}{2}})^{-1}\right)^{-1}p(\langle\xi,\xi\rangle)p\left((Cm^{\frac{1}{2}})^{*-1}\right)^{-1} \leq p\left(\sum_{i\in I}\langle\Lambda_i\xi,\Lambda_i\xi\rangle\right) \leq p(D)p(\langle\xi,\xi\rangle)p(D^*). \tag{1.4}$$

Proof. Suppose that $\{\Lambda_i\}_{i\in I} \in Hom_{\mathscr{A}}(\mathfrak{X}, \mathcal{Y}_i)$ is a $*$-K-g-frame for $\mathfrak{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$, then by Corollary 2.3 in [1], there exist $m > 0$ such that $m\langle \xi, \xi\rangle \leq \langle K^*\xi, K^*\xi\rangle$. Then

$$\langle \xi, \xi\rangle \leq (Cm^{\frac{1}{2}})^{-1}\left(\sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle\right)\left((Cm^{\frac{1}{2}})^*\right)^{-1}$$

and

$$\left(\sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle\right) \leq D\langle \xi, \xi\rangle D^*$$

Hence, by Proposition 1.2

$$p(Cm^{\frac{1}{2}})^{-1})^{-1}p(\langle \xi, \xi\rangle)p(Cm^{\frac{1}{2}})^{*-1})^{-1} \leq p\left(\sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle\right) \leq p(D)p(\langle \xi, \xi\rangle)p\left(D^*\right)$$

Conversely, if we suppose that hold.Then we can define :

$$T : \mathfrak{X} \to \bigoplus_{i\in I}\mathcal{Y}_i, \quad T(\xi) = \{\Lambda_i\xi\}_{i\in I}, \quad \forall \xi \in \mathfrak{X}.$$

as a linear operator, such that

$$\langle T\xi, T\xi\rangle = \sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle, \forall \xi \in \mathfrak{X}$$

We have $\bar{p}_{\mathfrak{X}}(T(\xi)) = \sqrt{\langle T\xi, T\xi\rangle}$,(3.3) implies

$$\bar{p}_{\mathfrak{X}}(T(\xi)) \leq p(D)^{\frac{1}{2}}\bar{p}_{\mathfrak{X}}(\xi)p\left(D^*\right)^{\frac{1}{2}}$$

which implies that T is uniformly bounded. We write $T^*T = U$. Then $\langle T(\xi), T(\xi)\rangle = \langle T^*T(\xi), \xi\rangle = \langle U(\xi), \xi\rangle$. Therefore, U is positive. On the one hand we have, $U^* = T^*T$, then U is self-adjoint.
On the other hand,

$$\left\langle U^{\frac{1}{2}}\xi, U^{\frac{1}{2}}\xi\right\rangle = \langle U\xi, \xi\rangle = \sum_{i\in I}\langle \Lambda_i\xi, \Lambda_i\xi\rangle$$

Then by Proposition 1.6 and (3.3), U is invertible and uniformily bounded. Hence by Proposition 1.6, we get:

$$\|U^{-\frac{1}{2}}\|_{\infty}^{-1}\langle \xi, \xi\rangle\|U^{-\frac{1}{2}}\|_{\infty}^{-1^*} \leq \langle U^{\frac{1}{2}}(\xi), U^{\frac{1}{2}}(\xi)\rangle \leq \|U^{\frac{1}{2}}\|_{\infty}\langle \xi, \xi\rangle\|U^{\frac{1}{2}}\|_{\infty}$$

For all $K \in Hom_A^*(\mathfrak{X})$ bounded and surjective such that $K = K^*$, we have

$$\langle K^*\xi, K^*\xi\rangle \leq \|K\|_{\infty}^2\langle \xi, \xi\rangle$$

Then

$$M^{-1}\|U^{-\frac{1}{2}}\|_{\infty}^{-1}\langle K^*\xi, K^*\xi\rangle(M^{-1}\|U^{-\frac{1}{2}}\|_{\infty}^{-1})^* \leq \|U^{-\frac{1}{2}}\|_{\infty}^{-1}\langle \xi, \xi\rangle\|U^{-\frac{1}{2}}\|_{\infty}^{-1^*}$$

Therefore $\{\Lambda_i K\}_{i\in I}$ is an $*$-K^*-g-frame in $\mathfrak{X}$ with respect to $\{\mathcal{Y}_i\}_{i\in I}$ □

Tensor Product

The minimal or injective tensor product of the pro-C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A}\otimes\mathscr{B}$, is the completion of the algebraic tensor product $\mathscr{A}\otimes_{\text{alg}}\mathscr{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathfrak{X}$ is a Hilbert module over a pro-C^*-algebra $\mathscr{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathscr{B}$. The algebraic tensor product $\mathfrak{X}\otimes_{\text{alg}}\mathcal{Y}$ of $\mathfrak{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathscr{A}\otimes\mathscr{B}$-module with the action of $\mathscr{A}\otimes\mathscr{B}$ on $\mathfrak{X}\otimes_{\text{alg}}\mathcal{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b) = \xi a\otimes\eta b \text{ for all } \xi\in\mathfrak{X}, \eta\in\mathcal{Y}, a\in\mathscr{A} \text{ and } b\in\mathscr{B}$$

and the inner product

$$\langle \cdot, \cdot \rangle : (\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}) \times (\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}) \to \mathcal{A} \otimes_{\text{alg}} \mathcal{B}. \text{ defined by}$$

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle$$

We also know that for $z = \sum_{i=1}^{n} \xi_i \otimes \eta_i$ in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ we have $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j} \langle \xi_i, \xi_j \rangle_{\mathcal{A}} \otimes \langle \eta_i, \eta_j \rangle_{\mathcal{B}} \geq 0$ and $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z = 0$.
The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X} \otimes \mathcal{Y}$ over $\mathcal{A} \otimes \mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$.

If $P \in M(\mathcal{X})$ and $Q \in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P \otimes Q : \mathcal{A} \otimes \mathcal{B} \to \mathcal{X} \otimes \mathcal{Y}$ such that $(P \otimes Q)(a \otimes b) = P(a) \otimes Q(b)$ and $(P \otimes Q)^*(a \otimes b) = P^*(a) \otimes Q^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, [7]).

Let I and J be countable index sets.

Theorem 1.18. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over unitary pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{\Lambda_i\}_{i\in I} \subset Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}_i)$ be an $*$-K-g-frame for $\mathcal{X}$ with bounds A and B and frame operators S_Λ and $\{\Gamma_j\}_{j\in J} \subset Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Z}_i)$ be an $*$-L-g-frame for $\mathcal{Y}$ with bounds C and D and frame operators S_Γ. Then $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is an $*$-K⊗L-g-frame for Hibert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with frame operator $S_\Lambda \otimes S_\Gamma$ and bounds $A \otimes C$ and $B \otimes D$.*

Proof. The definition of $*$-K-g-frame $\{\Lambda_i\}_{i\in I}$ and $*$-L-g-frame $\{\Gamma_j\}_{j\in J}$ gives

$$A\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle \Lambda_i \xi, \Lambda_i \xi \rangle_{\mathcal{A}} \leq B\langle \xi, \xi \rangle_{\mathcal{A}} B^*, \forall \xi \in \mathcal{X}.$$

$$C\langle L^*\eta, L^*\eta\rangle_{\mathcal{B}} C^* \leq \sum_{j\in J} \langle \Gamma_j \eta, \Gamma_j \eta \rangle_{\mathcal{B}} \leq D\langle \eta, \eta \rangle_{\mathcal{B}} D^*, \forall \eta \in \mathcal{Y}.$$

Therefore

$$\begin{aligned}
&(A\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} A^*) \otimes (C\langle L^*\eta, L^*\eta\rangle_{\mathcal{B}} C^*) \\
&\leq \sum_{i\in I} \langle \Lambda_i \xi, \Lambda_i \xi \rangle_{\mathcal{A}} \otimes \sum_{j\in J} \langle \Gamma_j \eta, \Gamma_j \eta \rangle_{\mathcal{B}} \\
&\leq (B\langle \xi, \xi \rangle_{\mathcal{A}} B^*) \otimes (D\langle \eta, \eta \rangle_{\mathcal{B}} D^*), \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Then

$$\begin{aligned}
&(A \otimes C)(\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} \otimes \langle L^*\eta, L^*\eta\rangle_{\mathcal{B}})(A^* \otimes C^*) \\
&\leq \sum_{i\in I, j\in J} \langle \Lambda_i \xi, \Lambda_i \xi \rangle_{\mathcal{A}} \otimes \langle \Gamma_j \eta, \Gamma_j \eta \rangle_{\mathcal{B}} \\
&\leq (B \otimes D)(\langle \xi, \xi \rangle_{\mathcal{A}} \otimes \langle \eta, \eta \rangle_{\mathcal{B}})(B^* \otimes D^*), \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&(A \otimes C)\langle K^*\xi \otimes L^*\eta, K^*\xi \otimes L^*\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(A \otimes C)^* \\
&\leq \sum_{i\in I, j\in J} \langle \Lambda_i \xi \otimes \Gamma_j \eta, \Lambda_i \xi \otimes \Gamma_j \eta \rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq (B \otimes D)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A}\otimes\mathcal{B}}(B \otimes D)^*, \forall \xi \in \mathcal{X}, \forall \eta \in \mathcal{Y}.
\end{aligned}$$

Then for all $\xi \otimes \eta$ in $\mathcal{X} \otimes \mathcal{Y}$ we have

$$\begin{aligned}
&(A \otimes C)\langle (K \otimes L)^*(\xi \otimes \eta), (K \otimes L)^*(\xi \otimes \eta)\rangle_{\mathcal{A}\otimes\mathcal{B}}(A \otimes C)^* \\
&\leq \sum_{i\in I, j\in J} \langle (\Lambda_i \otimes \Gamma_j)(\xi \otimes \eta), (\Lambda_i \otimes \Gamma_j)(\xi \otimes \eta) \rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq (B \otimes D)\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\mathcal{A}\otimes\mathcal{B}}(B \otimes D)^*.
\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is an $*$-$K \otimes L$-g-frame for Hilbert $\mathscr{A} \otimes \mathscr{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with lower and upper bounds $A \otimes C$ and $B \otimes D$, respectively.
By the definition of frame operator S_Λ and S_Γ we have

$$S_\Lambda \xi = \sum_{i\in I} \Lambda_i^* \Lambda_i \xi, \forall \xi \in \mathcal{X}.$$

$$S_\Gamma \eta = \sum_{j\in J} \Gamma_j^* \Gamma_j \eta, \forall \eta \in \mathcal{Y}.$$

Therefore

$$\begin{aligned}
(S_\Lambda \otimes S_\Gamma)(\xi \otimes \eta) &= S_\Lambda \xi \otimes S_\Gamma \eta \\
&= \sum_{i\in I} \Lambda_i^* \Lambda_i \xi \otimes \sum_{j\in J} \Gamma_j^* \Gamma_j \eta \\
&= \sum_{i\in I, j\in J} \Lambda_i^* \Lambda_i \xi \otimes \Gamma_j^* \Gamma_j \eta \\
&= \sum_{i\in I, j\in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \xi \otimes \Gamma_j \eta) \\
&= \sum_{i\in I, j\in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j)(\xi \otimes \eta) \\
&= \sum_{i\in I, j\in J} (\Lambda_i \otimes \Gamma_j)^*(\Lambda_i \otimes \Gamma_j)(\xi \otimes \eta).
\end{aligned}$$

Now by the uniqueness of frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(\xi \otimes \eta)$. Consequently we have $(S_\Lambda \otimes S_\Gamma)(\xi \otimes \eta) = S_{\Lambda\otimes\Gamma}(\xi \otimes \eta)$. The last equality is satisfied for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's satisfied for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It shows that $(S_\Lambda \otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$. □

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