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New Properties of Controlled Continuous K-g-Frames in Hilbert C^*-Modules

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Abstract
Frame theory in Hilbert spaces were introduced by Duffin and Schaeffer in 1952 to study some deep problems in non-harmonic Fourier series. This theory has a great revolution for recent years. It has been extended from Hilbert Space to Hilbert C^*-modules. In this paper, we study some New Properties of (C, C')-controlled continuous K-g-Frames in Hilbert C^*-modules and we established some equalities for controlled continuous K-g-Frames.

Keywords: controlled continuous-K-g-frame; continuous K-g-frame; Hilbert C^*-modules.

Introduction and Preliminaries

Frame theory in Hilbert spaces is a new theory which was introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series. This theory was reintroduced and developed by Daubechies, Grossman and Meyer [6].

In 1993, S. T. Ali, J. P. Antoine and J. P. Gazeau [1] introduced the concept of continuous frames in Hilbert spaces. Gabardo and Han in [11] called these kinds of frames, frames associated with measurable spaces.

In 2000, Frank and Larson [10] introduced the notion of frame in Hilbert C^*-modules as a generalization of frame in Hilbert spaces. The theory of continuous frames has been generalized in Hilbert C^*-modules. For more details, see [2, 23, 28].

The aim of this paper is to extend results, given for Hilbert C^*-module in discret case.

Let U and V be tow Hilbert C^*-modules and $\{V_m\}_{m\in M}$ is a sequence of subspaces of V, where M is a subset of Z and $End^*_{\mathcal{A}}(U, V_m)$ is the collection of all adjointable $\mathcal{A}$-linear maps from U into V_m, $\mathcal{GL}(\mathcal{U})$ is the set of all bounded linear operators which have bounded inverses and $\mathcal{GL}^+(\mathcal{U})$ is the set of all positive operators in $\mathcal{GL}(\mathcal{U})$, and $\mathcal{C}, \mathcal{C}' \in \mathcal{GL}^+(\mathcal{U})$.

Let us recall basic definitions and notations of controlled continuous-K-g-frames in Hilbert C^*-modules.

Definition 1.1. [21] Let $\mathcal{A}$ be a unital C^*-algebra and U be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{U}$ are compatible. U is a pre-Hilbert $\mathcal{A}$-module if U is equipped with an $\mathcal{A}$-valued in product $\langle ., .\rangle_{\mathcal{A}}$: $\mathcal{U} \times \mathcal{U} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

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(i) $\langle \alpha f + \beta g, h\rangle_{\mathcal{A}} = \alpha\langle f, h\rangle_{\mathcal{A}} + \beta\langle g, h\rangle_{\mathcal{A}}$ for all $f, g, h, \in \mathcal{U}$ and $\alpha, \beta \in C$,

(ii) $\langle af, g\rangle_{\mathcal{A}} = a\langle f, g\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $f, g \in \mathcal{U}$,

(iii) $\langle f, g\rangle_{\mathcal{A}} = \langle g, f\rangle_{\mathcal{A}}^*$ for all $f, h \in \mathcal{U}$,

(iv) $\langle f, f\rangle_{\mathcal{A}} \geq 0$ for all $f \in \mathcal{U}$ and $\langle f, f\rangle_{\mathcal{A}} = 0$ if and only if $f = 0$.

For $f \in \mathcal{U}$ we definie a norm on $\mathcal{U}$, $\|f\| = \|\langle f, f\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{U}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.
For every a in C^*-algebra $\mathcal{A}$ we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|f| = \langle f, f\rangle^{\frac{1}{2}}$ for $x \in \mathcal{U}$.
Let $\mathcal{U}$ and $\mathcal{K}$ be tow Hilbert $\mathcal{A}$ modules,
A map $T : \mathcal{U} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{U}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $y \in \mathcal{K}$ and $x \in \mathcal{U}$.

Lemma 1.2. *[2] Let A be a C^*-algebra, U and V two Hilbert $\mathcal{A}$ module, and $T \in End_{\mathcal{A}}^*(U, V)$. Then the following statements are equivalent:*

(i) T is surjective,

(ii) T^* is bounded below with respect to norm, that is, $m \geq 0$, such that $\|T^*\| \geqslant m\|f\|$ for all $f \in U$,

(iii) T^* is bounded below with respect to the inner product, that is, $m' \geq 0$ such that $\langle T^*f, T^*f\rangle\| \geqslant m'\|f\|$, for all $f \in U$.

Lemma 1.3. *[29] Let (Ω, μ) be a measure space, X and Y are tow Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to Y$ measurable function, then*

$$\lambda\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} (\lambda f) d\mu.$$

Lemma 1.4. *[7] Suppose that V_1 and V_2 two Hilbert $\mathcal{A}$-Modules $\mathcal{U}$ and $L_1 \in End_{\mathcal{A}}^*(V_1, U)$, $L_2 \in End_{\mathcal{A}}^*(V_2, U)$. Then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$

(ii) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda > 0$;

(iii) There exists a mapping $T \in End_{\mathcal{A}}^*(V_1, V_2)$ such that $L_1 = L_2T$.

Moreover, if above conditions are valid, then there exists a unique operator U such that

(i) $\|U\|^2 = \inf\{\alpha > 0\ L_1L_1^* \leq \alpha L_2L_2^*\}$,

(ii) $\ker(L_1) = \ker(U)$,

(iii) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

If an operator T has a closed range, then there exists a right-inverse operator $T^\dagger$, (pseudo-inverse of T) in the following sense.

Lemma 1.5. *[28] Let $T \in End_{\mathcal{A}}^*(V_1, V_2)$ be a bounded operator with closed range $\mathcal{R}(T)$. Then there exists a bounded operator $T^\dagger \in End_{\mathcal{A}}^*(V_2, V_1)$ for which*

$$TT^\dagger x = x, \qquad x \in \mathcal{R}(T).$$

Lemma 1.6. *[21] Let $\mathcal{U}$ and $\mathcal{V}$ two Hilbert $\mathcal{A}$-module and $T \in End_{\mathcal{A}}^*(U, V)$. Then, the following assertions are equivalent:*

(i) The operator T is bounded and $\mathcal{A}$-linear.

(ii) There exist $k > 0$ such that $\langle Tx, Tx\rangle_{\mathcal{A}} \leq k\langle x, x\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$.

Lemma 1.7. *[27] Let U be a Hilbert C^*-module. If $P, Q \in End^*_{\mathcal{A}}(U, U)$ are two bounded A-linear operators in H and $P + Q = I_H$, the one has*

$$P - P^*P = Q^* - Q^*Q.$$

Lemma 1.8. *[27] Let U be a Hilbert C^*-module. If $P, Q \in End^*_{\mathcal{A}}(U, U)$ are two bounded, selfadjoint A-linear operators in H and $P + Q = I_U$,the one has*

$$\langle Pf, f\rangle + |Qf|^2 = \langle Qf, f\rangle + |Pf|^2 \geqslant \frac{3}{4}\langle f, f\rangle$$

.

Lemma 1.9. *[27] Let U be a Hilbert C^*-module. If T is a bounded, selfadjoint linear operator and satisfy $\langle Tf, f\rangle = 0$, for all $f \in U$, then $T = 0$.*

Definition 1.10. A family $\Lambda := \{\Lambda_m \in End^*_{\mathcal{A}}(U, V_m)\}_{m\in M}$ is called a Continuous g-frame in Hilbert $\mathcal{A}$ module $\mathcal{U}$ with respect to $\{V_m\}_{m\in M}$ if there exist constants
$0 < A \leq B < +\infty$ such that for each $f \in \mathcal{U}$,

$$A\langle f, f\rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m f, \Lambda_m f\rangle_{\mathcal{A}} d\mu(m) \leq B\langle f, f\rangle_{\mathcal{A}}.$$

For this frames the g frame operator is defined by

$$S_\Lambda(f) = \int_M \Lambda^*_m \Lambda_m f d\mu(m), \qquad f \in \mathcal{U}$$

which is positive and invertible.

Definition 1.11. Let $C \in \mathcal{GL}(\mathcal{U})$. We say that $F := \{f_m\}_{m\in M}$ is a C-controlled frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ if there existe constants $0 < A_c < B_c < +\infty$ such that for each $f \in \mathcal{U}$,

$$A_c\langle f, f\rangle_{\mathcal{A}} \leq \int_M \langle f, f_m\rangle\langle Cf_m, f\rangle_{\mathcal{A}} d\mu(m) \leq B_c\langle f, f\rangle_{\mathcal{A}} \tag{1.1}$$

Definition 1.12. Let $C, C' \in \mathcal{GL}^+(\mathcal{U})$, we say that $\Lambda := \{\Lambda_m \in End^*_{\mathcal{A}}(U, V_m)\}_{m\in M}$ is a (C, C')-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module U if there existe constants $0 < A_{cc'} < B_{cc'} < +\infty$ such that for each $f \in U$,

$$A_{cc'}\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m) \leq B_{cc'}\langle f, f\rangle_{\mathcal{A}} \tag{1.2}$$

If the right hand of holds, Λ is called a (C, C')-controlled-K-g-Bessel sequence in Hilbert $\mathcal{A}$-module $\mathcal{U}$ with bound B_c.
We call Λ a Parseval (C, C')-controlled Continuous K-g-frame if

$$\langle K^*f, K^*f\rangle_{\mathcal{A}} = \int_M \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m).$$

If $K = I_U$. Then Λ is (C, C')-controlled continuous g-frame.

Let $\Lambda := \{\Lambda_m \in End^*_{\mathcal{A}}(U, V_m)\}_{m\in M}$ is a $\mathcal{C}, \mathcal{C}'$-controlled continuous-$g$-frame in Hilbert $\mathcal{A}$-module U with respect to $\{V_m\}_{m\in M}$.
Define the controlled continuous-g-frame operator $S_{CC'}$ on U by:

$$S_{CC'}(f) := \int_M C^*\Lambda^*_m \Lambda_m C' f d\mu(m)$$

Proposition 1.13. *The frames operator $S_{CC'}$ is a positve, bounded and invertible.*

Proof. For any $f \in U$ we have

$$\begin{aligned}\langle S_{CC'}f, f\rangle &= \langle \int_M C^*\Lambda_m^*\Lambda_m C' f d\mu(m), f\rangle, \\ &= \int_M \langle C^*\Lambda_m^*\Lambda_m C' f, f\rangle d\mu(m), \\ &= \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle d\mu(m).\end{aligned}$$

So

$$A\langle f, f\rangle_{\mathcal{A}} \leq \langle S_{CC'}f, f\rangle \leq B\langle f, f\rangle_{\mathcal{A}}.$$

Or in the notation from operator theory $AI \leq S_{CC'} \leq BI$, thus $S_{CC'}$ is a positive operator. Furthermore, $0 \leq A^{-1}S_\Lambda - I \leq (\frac{B-A}{A})I$, and consequently

$$\|A^{-1}S_\Lambda - I\| \leq \|(\frac{B-A}{A})I\| \leq 1.$$

This shows that $S_{CC'}$ is invertible. Now we show that $S_{CC'}$ is a bounded operator

$$\|S_{CC'}\| = \sup_{\|f\|\leq 1} \langle S_{CC'}f, f\rangle \| = \sup_{\|f\|\leq 1} \|\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle d\mu(m)\| \leq B$$

□

Some characterizations of controlled continuous K-g-frames in Hilbert C^*-modules

In this section, we will characterize the equivalencies of controlled continuous K-g-frames in Hilbert C^*-modules from several aspects. Now, we can defne the synthesis and analysis operators of the CC'-controlled continuous g-frames.

Theorem 1.14. *Let $\{\Lambda_m \in End_{\mathcal{A}}^*(U, V_m)\}_{m\in M}$ for any $\{m \in M\}$. Then $\{\Lambda_m : m \in M\}$ is a $\mathcal{C}, \mathcal{C}'$-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module U with respect to $\{V_m\}_{m\in M}$ if and only if there existe constants $A, B \geqslant 0$ such that for any $f \in \mathcal{U}$,*

$$A\|K^*f\|^2 \leq \|\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m)\| \leq B\|f\|^2. \tag{1.3}$$

Proof. Let $\{\Lambda_m : m \in M\}$ is a $\mathcal{C}, \mathcal{C}'$-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module U with respect to $\{V_m\}_{m\in M}$. Then

$$A\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m) \leq B\langle f, f\rangle_{\mathcal{A}},$$

$$A\|\langle K^*f, K^*f\rangle_{\mathcal{A}}\| \leq \|\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m)\| \leq B\|\langle f, f\rangle_{\mathcal{A}}\|$$

If inequality (1.3) holds, then by Proposition 1.13

$$\langle S^{1/2}f, S^{1/2}f\rangle = \langle Sf, f\rangle = \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m).$$

Hence

$$\sqrt{A}\|K^*f\| \leq \|S^{1/2}f\| \leq \sqrt{B}\|f\|$$

$$A_1\langle K^*f, K^*f\rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m) \leq B_1\langle f, f\rangle_{\mathcal{A}}.$$

Which implies that $\{\Lambda_m : m \in M\}$ is a $\mathscr{C},\mathscr{C}'$-controlled continuous K-g-frame in Hilbert $\mathscr{A}$-module U with respect to $\{V_m\}_{m\in M}$.
Now, let

$$\mathscr{R} := \{(C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f : f \in \mathscr{U}\}_{m\in M} \subset (\sum_{m\in M} \oplus H)_{\ell^2}.$$

It is easy to check that $\mathscr{R}$ is a closed of $(\sum_{m\in M} \oplus H)_{\ell^2}$.
For any $f = \{f_m : m \in M\}$ and $g = \{g_m : m \in M\}$, if the A-valued inner product is defined by $\langle f, g\rangle = \int_M \langle f_m, g_m\rangle d\mu(m)$, the norme is defined by $\|f\|^2 = \|\langle f, f\rangle\|$. Now, we can define the synthesis and analysis operators of the CC'-controlled g-frames as

$$T_{\mathscr{C}\mathscr{C}'} : \mathscr{R} \to \mathscr{U},$$

$$T_{\mathscr{C}\mathscr{C}'}((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f)_{m\in M} = \int_m (C^*\Lambda_m^*\Lambda_m C' f) d\mu(m),$$

and

$$T^*_{\mathscr{C}\mathscr{C}'} : \mathscr{U} \to \mathscr{R},$$

$$T^*_{\mathscr{C}\mathscr{C}'}(f) = ((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f)_{m\in M}.$$

Thus, the CC'-controlled g-frame operator is given by

$$\begin{aligned} S_{\mathscr{C}\mathscr{C}'}(f) &= T_{\mathscr{C}\mathscr{C}'} T^*_{\mathscr{C}\mathscr{C}'}(f) \\ &= \int_M (C^*\Lambda_m^*\Lambda_m C' f) d\mu(m). \end{aligned}$$

□

Lemma 1.15. *Let $C, C' \in \mathscr{GL}^+(\mathscr{H})$. A sequence Λ is a CC'-controlled continuous g-Bessel sequence in Hilbert $\mathscr{A}$-module with bound $B_{\mathscr{C}\mathscr{C}'}$ if and only if the operator*

$$T_{\mathscr{C}\mathscr{C}'} : \mathscr{R} \to \mathscr{H},$$

$$T_{\mathscr{C}\mathscr{C}'}((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f)_{m\in M} = \int_M (C^*\Lambda_m^*\Lambda_m C' f) d\mu(m)$$

is well-defined and bounded with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$.

Proof. We only need to prove the sufficient condition. Let $T_{\mathscr{C}\mathscr{C}'}$ be a well-defined and bounded operator with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$. For each $f \in H$, we have

$$\begin{aligned} \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathscr{A}} d\mu(m) &= \int_M \langle C^*\Lambda_m^*\Lambda_m C' f, f\rangle_{\mathscr{A}} d\mu(m) \\ &= \langle \int_M C^*\Lambda_m^*\Lambda_m C' f, f\rangle_{\mathscr{A}} d\mu(m) \\ &= \langle T_{CC'}((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f))_{m\in M}, f\rangle_{\mathscr{A}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\langle T_{CC'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}, f\rangle_{\mathscr{A}}\| &\leq \|T_{CC'}((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}\| \|f\| \\ &\leq \|T_{CC'}\| \|((C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f))_{i\in I}\| \|f\|. \end{aligned}$$

But

$$\|((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}} f)\|^2 = \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle d\mu(m)$$

$$\|((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}}f)\| \le \|T_{CC'}\|\|f\|,$$
$$\|((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}}f)\|^2 \le \|T_{CC'}\|^2\|f\|^2.$$

It follows that

$$\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m) \le Bcc'\|\langle f, f\rangle_{\mathcal{A}}\|.$$

and this means that Λ is aCC'-controlled continuous g-Bessel sequence. □

Lemma 1.16. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. A sequence Λ is a CC'-controlled continuous g-frame sequence in Hilbert $\mathcal{A}$-module if and only if the operator*

$$T_{\mathcal{C}\mathcal{C}'} : \mathcal{R} \to \mathcal{H},$$
$$T_{\mathcal{C}\mathcal{C}'}((C^*\Lambda_m^*\Lambda_m C')^{\frac{1}{2}}f) = \int_M (C^*\Lambda_m^*\Lambda_m C' f) d\mu(m)$$

is well-defined, bounded and surjective.

Proof. Suppose that Λ is a CC'-controlled continuous g-frame in Hilbert $\mathcal{A}$-module. Since, $S_{\mathcal{C}\mathcal{C}'}$ is surjective operator, so $T_{\mathcal{C}\mathcal{C}'}$. For the opposite implication, by Lemma 1.15; $T_{\mathcal{C}\mathcal{C}'}$ is a well-defined and bounded operator. So Λ is a CC'-controlled continuous g-Bessel sequence. Now, for each $f \in H$, we have $f = T_{CC'}T_{CC'}^{\dagger}f$. Hence

$$\begin{aligned}
\|f\|^4 &= \|\langle f, f\rangle\|^2 \\
&= \|\langle T_{CC'}T_{CC'}^{\dagger}f, f\rangle\|^2 \\
&= \|\langle T_{CC'}^{\dagger}f, T_{CC'}^{*}f\rangle\|^2 \\
&\le \|\langle T_{CC'}^{\dagger}f, T_{CC'}^{\dagger}f\rangle\|^2\|\langle T_{CC'}^{*}f, T_{CC'}^{*}f\rangle\|^2 \\
&\le \|T_{CC'}^{\dagger}f\|^2\|\|T_{CC'}^{*}f\|^2 \\
&\le \|T_{CC'}^{\dagger}\|^2\|f\|^2\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m).
\end{aligned}$$

We conclude that

$$(\|T_{CC'}^{\dagger}\|^2)^{-1}\|\langle f, f\rangle\| \le \int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m).$$

□

Operators preserving controlled K-g-frames

In this section, for the CC'-controlled continuous K-g-frames $\{\Lambda_m\}_{m\in M}$. We consider some proper relations between the operators $W, K \in B(U)$ and $C, C' \in \mathcal{GL}^+(\mathcal{U})$ and investigate the cases that $\{\Lambda_m W\}_{m\in M}$, $\{\Lambda_m W^*\}_{m\in M}$ can also CC'-controlled continuous K-g- frame. Next, by putting connections between the operators $\{S_\Lambda\}, K, C$ and C' we reach to necessary and sufficient conditions that $\{\Lambda_m\}_{m\in M}$ can be a Parseval CC'-controlled continuous K-g-frames.

Theorem 1.17. *Let Λ be a CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ and $W \in B(U)$ be a co-isometry (i.e. $WW^* = Id_U$) such that $WK = KW$ and W^* commutes with C and C'. Then $(\Lambda_m W^*)_{m\in M}$ is a CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$.*

Proof. Suppose that Λ is a CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ with frame bounds $A_{CC'}$. And $B_{CC'}$ for each $f \in \mathcal{U}$, we have

$$\begin{aligned}
\int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f\rangle_{\mathcal{A}} d\mu(m) &= \int_M \langle \Lambda_m C' W^* f, \Lambda_m C W^* f\rangle_{\mathcal{A}} d\mu(m) \\
&\le B_{CC'}\langle W^* f, W^* f\rangle_{\mathcal{A}}
\end{aligned}$$

hence,

$$\int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f \rangle_{\mathcal{A}} d\mu(m) \leq B_{CC'} \|W^*\|^2 \langle f, f \rangle_{\mathcal{A}}$$

So, $(\Lambda_m^* W*)_{m\in M}$ is a CC'-controlled continuous g-Bessel sequence. For the lower bound, we can write

$$\begin{aligned}\int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f \rangle_{\mathcal{A}} d\mu(m) &= \int_M \langle \Lambda_m C' W^* f, \Lambda_m C W^* f \rangle_{\mathcal{A}} d\mu(m)\\ &\geqslant A_{CC'} \langle K^* W^* f, K^* W^* f \rangle_{\mathcal{A}}\\ &= A_{CC'} \langle (WK)^* f, (WK)^* f \rangle_{\mathcal{A}}\\ &= A_{CC'} \langle (KW)^* f, (KW)^* f \rangle_{\mathcal{A}}\\ &= A_{CC'} \langle W^* K^* f, W^* K^* f \rangle_{\mathcal{A}}\\ &= A_{CC'} \langle K^* f, K^* f \rangle_{\mathcal{A}}\end{aligned}$$

□

Theorem 1.18. *Let $\Lambda := \{\Lambda_m \in B(U, V_m)_{m\in M}\}$ and $\ominus := \{\ominus_m \in B(U, V_m)_{m\in M}\}$ be tow CC'-controlled continuous K-g-Bessel sequences in Hilbert $\mathcal{A}$-module $\mathcal{U}$ with bounds B_Λ and $B_\ominus$ respectively. suppose that $T_{\Lambda,C,C'}$ and $T_{\ominus,CC'}$ are their synthesis operators such that $T_{\ominus,C,C'} T^*_{\Lambda,C,C'} = K^*$. Then Λ and $\ominus$ are CC'-controlled continuous K and K^*-g-frames, respectively.*

Proof.

$$\begin{aligned}\|K^* f\|^4 &= \|\langle K^* f, K^* f \rangle\|^2\\ &= \|\langle T_{\ominus,C,C'} T^*_{\Lambda,C,C'} f, K^* f \rangle\|^2\\ &\leq \|T^*_{\Lambda,C,C'} f\|^2 \|T^*_{\ominus,C,C'} K^* f\|^2\\ &= \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \int_M \langle \ominus_m C' K^* f, \ominus_m C' K^* f \rangle d\mu(m)\\ &\leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) B_\ominus \|\langle K^* f, K^* f \rangle_{\mathcal{A}}\|\end{aligned}$$

So,

$$\|\langle K^* f, K^* f \rangle\| \leq B_\ominus \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m)$$

Thus

$$B_\ominus^{-1} \|\langle K^* f, K^* f \rangle_{\mathcal{A}}\| \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m)$$

This that Λ is a CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ with frame operator S_Λ. For ech $f \in \mathcal{U}$, we have $T_{\Lambda,C,C'} T^*_{\ominus,C,C'} = K$. Thus

$$B_\Lambda^{-1} \|\langle K f, K f \rangle_{\mathcal{A}}\| \leq \int_M \langle \ominus_m C' f, \ominus_m C f \rangle_{\mathcal{A}} d\mu(m)$$

This that $\ominus$ is a CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$. □

Theorem 1.19. *Let Λ be a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ with frame operator S_Λ. Also assume that Λ is a CC'-controlled continuous g- Bessel sequence with frame operator $S_{CC'}$. Then Λ is a Parseval CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$ if and only if $C = (S_\Lambda^{-p})^* \Phi$ and $C' = (S_\Lambda^{-q}) \Psi$ where Φ, Ψ are two operators in Hilbert $\mathcal{A}$- module $\mathcal{U}$ such that $\Phi^* \Psi = KK^*$ and $p + q = 1$ where $p, q \in \mathcal{R}$.*

Proof. Assume that Λ is a Parseval CC'-controlled continuous K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{U}$

$$\begin{aligned}\int_M \langle \Lambda_m C' f, \Lambda_m C f\rangle_{\mathcal{A}} d\mu(m) &= \langle K^* f, K^* f\rangle_{\mathcal{A}} \\ &= \int_M \langle f, C^* \Lambda_m^* \Lambda_m C' f\rangle_{\mathcal{A}} d\mu(m) \\ &= \langle f, \int_M C^* \Lambda_m^* \Lambda_m C' f\rangle_{\mathcal{A}} d\mu(m) \\ &= \langle f, S_{CC'} f\rangle_{\mathcal{A}} \\ &= \langle f, KK^* f\rangle_{\mathcal{A}}\end{aligned}$$

$$\begin{aligned}S_{CC'}(f) &= \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \\ &= C^* \int_M \Lambda_m^* \Lambda_m C'(f) d\mu(m) \\ &= C^* S_\Lambda C'(f).\end{aligned}$$

Hence $S_{CC'} = C^* S_\Lambda C'$ and $S_{CC'} = KK^*$. Therefore, for each $p, q \in \mathcal{R}$ such that $p + q = 1$, we obtain

$$KK^* = C^* S_\Lambda^p S_\Lambda^q C'.$$

We define $\Phi = (S_\Lambda^p)^* C$ and $\Psi = (S_\Lambda^q)^* C'$ So

$$\Phi^* \Psi = C^* S_\Lambda^p S_\Lambda^q C' = KK^*.$$

Conversely, let Φ and Ψ be tow operators in Hilbert $\mathcal{A}$- module $\mathcal{U}$ such that $\Phi^*\Psi = KK^*$. Suppose that $C = (S_\Lambda^{-p})^* \Phi$ and $C' = (S_\Lambda^{-q})^* \Psi$ are tow operators on Hilbert $\mathcal{A}$-module $\mathcal{H}$ where$p, q \in R$ and $p + q = 1$, Since

$$KK^* = \Phi^* \Psi = C^* S_\Lambda^p S_\Lambda^q C' = C^* S_\Lambda C' = S_{CC'}.$$

So, for each $f \in \mathcal{H}$,

$$\langle KK^* f, f\rangle_{\mathcal{A}} = \langle K^* f, K^* f\rangle_{\mathcal{A}} = \langle \int_M C^* \Lambda_m^* \Lambda_m C' f, f\rangle_{\mathcal{A}} d\mu(m).$$

Thus Λ is Parseval CC'-controlled continuous k-g-frame on Hilbert $\mathcal{A}$- module $\mathcal{U}$. □

Some Equalities for CC'-controlled Continuous K-g-frames in Hilbert C^*-Modules

Some equalities for frames involving the real parts of some complex numbers have been established in [12]. These inequalities generalized in [27] for g-frame in Hilbert C^*-modules. In this section, we generalize the equalities to a more general form which generalized before equalities and we deduce some equalities for Controlled Continuous K-g-frames in Hilbert C^*− modules.

Proposition 1.20. *Let $\{\Lambda_m : m \in M\}$ be a CC' controlled continuous g-frame for U with respect to $\{V_m : m \in M\}$ with controlled continuous g-frame operator S_Λ with bounds A and B. Then $\{\Lambda_m : m \in M\}$ defined by $\widetilde{\Lambda_m} = \Lambda_m S^{-1}$ is a controlled continuous g-frame for U with respect to $\{V_m : m \in M\}$ with controlled continuous g-frame operator S_Λ^{-1} with bounds B^{-1} and A^{-1}. That is called controlled continuous canonical dual g-frame of $\{\Lambda_m : m \in M\}$*

Proof. Let $\widetilde{S}$ be the controlled continuous g-frame operator associated with $\{\widetilde{\Lambda_m : m \in M}\}$ that is $\widetilde{S}f = \int_M C^*\widetilde{\Lambda_m^*}\widetilde{\Lambda_m}C'f d\mu(m)$. Then for $f \in U$,

$$\begin{aligned} S\widetilde{S}f &= \int_M C^*\widetilde{\Lambda_m^*}\widetilde{\Lambda_m}C'f d\mu(m) \\ &= \int_M C^*SS^{-1}\Lambda_m^*\Lambda_m C'S^{-1}f d\mu(m) \\ &= \int_M C^*\Lambda_m^*\Lambda_m C'S^{-1}f d\mu(m) \\ &= SS^{-1}f = f. \end{aligned}$$

Hence $\widetilde{S} = S^{-1}$.
Since $\{\Lambda_m : m \in M\}$ is a controlled continuous g-frame for U, then $AI \leq S \leq BI$. On other hand since I and S are selfadjoint and S^{-1} commutative with I and S, $AIS^{-1} \leq SS^{-1} \leq BIS^{-1}$, and hence $B^{-1}I \leq S^{-1} \leq A^{-1}I$. □

Remark 1.21. We have $\widetilde{\Lambda_m}\widetilde{S^{-1}} = \Lambda_m S^{-1}S = \Lambda_m$. In other words $\{\Lambda_m : m \in M\}$ and $\{\widetilde{\Lambda_m} : m \in M\}$ are dual controlled continuous g-frame with respect to each other.

Theorem 1.22. *Let $\{\Lambda_m \in End_{\mathcal{A}}^*(U, V_m) : m \in M\}$ be a CC' controlled continuous K-g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ and let $\{\widetilde{\Lambda_m} : m \in M\}$ be the CC' canonical dual controlled continuous k-g frame of $\{\Lambda_m : m \in M\}$, then for any measurable subset $J \subset M$ and $f \in U$, one has*

$$\begin{aligned} &\int_J \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m) + \int_M \langle \widetilde{\Lambda_m} S_{J^c} C'f, \widetilde{\Lambda_m} S_{J^c} C'f\rangle_{\mathcal{A}} d\mu(m) = \\ &\int_{J^c} \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m) + \int_M \langle \widetilde{\Lambda_m} S_J C'f, \widetilde{\Lambda_m} S_J C'f\rangle_{\mathcal{A}} d\mu(m) \geq \\ &\qquad \int_M \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m) \geq \frac{3}{4}\langle K^*f, K^*f\rangle_{\mathcal{A}} \end{aligned}$$

Proof. Since S is an invertible, positive operator on U, and $S_J + S_{J^c} = S$, then $S^{-1/2}S_J S^{-1/2} + S^{-1/2}S_{J^c}S^{-1/2} = I_{\mathcal{U}}$. Let $P = S^{-1/2}S_J S^{-1/2}$, $Q = S^{-1/2}S_{J^c}S^{-1/2}$. By Lemma 1.8, we obtain

$$\langle S^{-1/2}S_J S^{-1/2}f, f\rangle + |S^{-1/2}S_{J^c}S^{-1/2}f|^2 = \langle S^{-1/2}S_{J^c}S^{-1/2}f, f\rangle + |S^{-1/2}S_{J^c}S^{-1/2}f|^2 \geq \frac{3}{4}\langle f, f\rangle.$$

Replacing f by $S^{1/2}f$, then one has

$$\langle S_J f, f\rangle + \langle S^{-1}S_{J^c}f, S^{J^c}f\rangle = \langle S_{J^c}f, f\rangle + \langle S^{-1}S_J f, S^J f\rangle \geq \frac{3}{4}\langle Sf, f\rangle.$$

On the other hand, we have

$$\langle S_J f, f\rangle = \langle \int_J C^*\Lambda_m^*\Lambda_m C'f d\mu(m), f\rangle_{\mathcal{A}} = \int_J \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m),$$

$$\int_J \langle \widetilde{\Lambda_m} C'f, \widetilde{\Lambda_m} C'f\rangle_{\mathcal{A}} d\mu(m) = \int_J \langle \Lambda_m S^{-1}C'f, \Lambda_m S^{-1}Cf\rangle_{\mathcal{A}} d\mu(m),$$

$$= \int_J \langle C^*\Lambda_m^*\Lambda_m S^{-1}C'f, S^{-1}f\rangle_{\mathcal{A}} d\mu(m) = \langle SS^{-1}f, S^{-1}f\rangle_{\mathcal{A}} = \langle f, S^{-1}f\rangle_{\mathcal{A}} = \langle S^{-1}f, f\rangle_{\mathcal{A}}.$$

□

Theorem 1.23. *Let $\{\Lambda_m \in End_{\mathcal{A}}^*(U, V_m) : m \in M\}$ be a CC' controlled continuous parseval K-g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ then for any measurable subset $J \subset M$ and $f \in U$, one has*

$$\int_J \langle \Lambda_m C'f, \Lambda_m Cf\rangle_{\mathcal{A}} d\mu(m) + |\int_{J^c} C^*\Lambda_m^*\Lambda_m C'f d\mu(m)|^2$$

$$= \int_{J^c} \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) + | \int_J C^* \Lambda_m^* \Lambda_m C' f d\mu(m)|^2$$
$$\geq \frac{3}{4} \langle K^* f, K^* f \rangle_{\mathcal{A}}$$

Proof. Let $\{\Lambda_m \in End_{\mathcal{A}}^*(U, V_m) : m \in M\}$ is a CC' controlled continuous Parseval K-g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ then for any $f \in U$, we have

$$\int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) = \langle K^* f, K^* f \rangle_{\mathcal{A}} \tag{1.4}$$

So

$$\langle Sf, f \rangle = \langle \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m), f \rangle \qquad = \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) = \langle K^* f, K^* f \rangle_{\mathcal{A}}$$

Hence for any $f \in \mathcal{U}$, we have$\langle (S - I_U), f \rangle_{\mathcal{A}} = 0$. Let $T = S - I_U$. Since S is bounded, selfadjoint, then $T^* = (S - I_U)^* = S^* - I_U^* = S - I_U = T$, So T is also bounded, selfadjoint. By Lemma 1.9, we have $T = 0$, namely, $S = I_U$, so $\widetilde{\Lambda_m} = \Lambda_m S^{-1} = \Lambda_m$. From (1.4), then we have that for any measurable subset $J \subset M$ and $f \in \mathcal{U}$,

$$\int_M \langle \widetilde{\Lambda_m} S_J C' f, \widetilde{\Lambda_m} S_J C' f \rangle_{\mathcal{A}} d\mu(m) = \int_M \langle \Lambda_m S_J C' f, \Lambda_m S_J C' f \rangle_{\mathcal{A}} d\mu(m)$$
$$= \langle S_J f, S_J f \rangle_{\mathcal{A}} = \left| \int_J C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2 .$$

$$\int_M \langle \widetilde{\Lambda_m} S_{J^c} C' f, \widetilde{\Lambda_m} S_{J^c} C' f \rangle_{\mathcal{A}} d\mu(m) = \int_M \langle \Lambda_m S_{J^c} C' f, \Lambda_m S_{J^c} C' f \rangle_{\mathcal{A}} d\mu(m)$$
$$= \langle S_{J^c} f, S_{J^c} f \rangle_{\mathcal{A}} = \left| \int_{J^c} C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2 ,$$

Combining (1.4) and Theorem 1.22, we get the result. □

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