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On Logarithmic $[p, q]$th Order of Entire Functions

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Abstract
After the works of Chern [2] on logarithmic order of entire functions, in this paper we introduce the concept of logarithmic $[p, q]$th order of entire functions to generalise a result of Long and Qin [7].

Keywords: Entire function, Logarithmic order, Central index, Maximum term.

Introduction, Definitions and Notations

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series in $\mathbf{C}$. Then the maximum term $\mu(r, f)$ and central index $\nu(r, f)$ of f are defined as

$$\mu(r, f) = \max_{n\geq 0}\{|a_n|r^n\}$$

and

$$\nu(r, f) = \max\{n : \mu(r, f) = |a_n|r^n\}.$$

When no other function except f is involved, $\mu(r, f)$ and $\nu(r, f)$ can simply be denoted by $\mu(r)$ and $\nu(r)$ respectively. We note that $\nu(r, f)$ is real, non-decreasing function of r .

Similarly for the functions $f^{(k)}$, $(k = 1, 2, \ldots,)$ the kth derivative of f we define the functions

$$\mu_k(r) = \mu(r, f^{(k)})$$

and

$$\nu_k(r) = \nu(r, f^{(k)}) \ .$$

Further we write $\mu_0(r) = \mu(r, f)$ and $\nu_0(r) = \nu(r, f)$.

For a nonconstant meromorphic function f in the complex plane $\mathbf{C}$, the order and the lower order of f are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna's characteristic function of $f(z)$.

If f is entire, then $T(r, f)$ is replaced with $\log M(r, f)$ in above equalities, where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

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For entire functions of order zero, Chern [2] introduced the concept of logarithmic order as follows.

Let f be an entire function of zero order. Then the logarithmic order of f is defined by

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

In 1995, Lahiri and Banerjee [5] introduced the concept of $[p,q]$th order of meromorphic functions as follows.

If f be a non-constant meromorphic function then the $[p,q]$th order of f is defined by

$$\rho_f[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$$

where p,q are positive integers with $p \geq q$.

If f is entire then

$$\rho_f[p,q] = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}.$$

For $p = 1$ and $q = 1$,

$$\rho_f[1,1] = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

which is the order of growth of the entire function f.

The techniques which are used for functions with finite positive $[p,q]$th order often are not used for functions of $[p,q]$th order zero. In order to make some progress with functions of $[p,q]$th order zero, it is therefore reasonable to introduce the concept of logarithmic $[p,q]$th order as follows.

The logarithmic $[p,q]$th order of an entire function f with its $[p,q]$th order being zero is defined as

$$\rho_{log}[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q+1]} r} = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q+1]} r}$$

where p and q are positive integers with $p \geq q$.

For $p = 1$ and $q = 1$

$$\rho_{log}[1,1] = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}$$

which is the liogarithmic order of the entire function f.

In 2020, some new results on the growth rates of composite entire functions depending on generalized relative logarithmic order were developed by Ghosh et al.[4]. Also in 2021, Biswas et al.[1] investigated certain results associated with the comparative growth properties of composite entire functions using generalized order (α,β) and generalized lower order (α,β).

Here the main purpose of this paper is to establish a relation of logarithmic $[p,q]$th order of an entire function with the maximum term of that function.

Preliminary Results

To reach the main result of this paper we first need the following results.

Theorem 2.1. Let f be a transcendental entire series with finite logarithmic $[p, q]$th order. Then the following statements are equivalent.

1. $T(r, f)$ has logarithmic $[p, q]$th order $\rho_{log}[p, q]$;
2. $\log M(r, f)$ has logarithmic $[p, q]$th order $\rho_{log}[p, q]$;
3. $\log \mu(r, f)$ has logarithmic $[p, q]$th order $\rho_{log}[p, q]$;
4. $\nu(r, f)$ has logarithmic $[p, q]$th order $\rho_{log}[p, q]$.

Equivalence of each of these statements are obvious except the portion (3) implying (4). For this, we need the following two consequences of G. Valiron [9] in the form of Lemmas.

Lemma 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function in **C**. Then for any r_0 with $0 < r_0 < r$

$$\log \mu(r, f) = \log \mu(r_0, f) + \int_{r_0}^{r} \frac{\nu(t, f)}{t} dt.$$

Lemma 2.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function in **C**. Then for all $r < R$

$$M(r, f) < \mu(r, f)\{\nu(R, f) + \frac{R}{R - r}\}.$$

Lemma 2.3 [8]. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function in **C**. Then for sufficiently large values of r

$$\mu(r, f) \leq M(r, f) \leq 2\mu(2r, f).$$

Proof of the theorem : Without loss of generality we may assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $|a_0| \neq 0$. Then from Lemma 2.1 letting $r_0 \to 0+$ we have

$$\log \mu(r, f) = \log |a_0| + \int_{0}^{r} \frac{\nu(t, f)}{t} dt. \tag{2.1}$$

Replacing r by $2r$ in (2.1), we have

$$\log \mu(2r, f) = \log |a_0| + \int_{0}^{2r} \frac{\nu(t, f)}{t} dt. \tag{2.2}$$

Since $\nu(r, f)$ is a positive non-decreasing function of r

$$\begin{aligned} \int_{0}^{2r} \frac{\nu(t, f)}{t} dt &\geq \int_{r}^{2r} \frac{\nu(t, f)}{t} dt \\ &\geq \nu(r, f) \int_{r}^{2r} \frac{1}{t} dt \\ &= \nu(r, f) \log 2 \end{aligned}$$

So from (2.2)

$$\log \mu(2r, f) \geq \log |a_0| + \nu(r, f) \log 2\ . \tag{2.3}$$

Also from Lemma 2.3 we have for sufficiently large values of r

$$\mu(2r, f) \leq M(2r, f).$$

So from (2.3)

$$\log|a_0| + \nu(r, f)\log 2 \leq \log M(2r, f).$$

Taking logarithm p times, we have

$$\log^{[p]} \nu(r, f) \leq \log^{[p+1]} M(2r, f) + O(1).$$

Hence

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, f)}{\log^{[q+1]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q+1]} r} = \rho_{log}[p, q]. \tag{2.4}$$

On the other hand in Lemma 2.2 replacing R by 2r we have

$$M(r, f) < |a_{\nu(r,f)}| r^{\nu(r,f)}\{\nu(2r, f) + 2\}, \quad since \;\; \mu(r, f) = |a_{\nu(r,f)}| r^{\nu(r,f)}.$$

$$So, \;\; \log M(r, f) < \log|a_{\nu(r,f)}| + \nu(r, f)\log r + \log\nu(2r, f) + \log\{1 + \frac{2}{\nu(2r, f)}\}.$$

Again since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function, hence $\{|a_n|\}$ is bounded. Thus above inequality leads to

$$\begin{aligned}
\log^{[p]} M(r, f) &\leq \log^{[p-1]} \nu(r, f) + \log^{[p]} \nu(2r, f) + \log^{[p]} r + O(1)\\
&\leq \log^{[p-1]} \nu(2r, f) + \log^{[p]} \nu(2r, f) + \log^{[p]} r + O(1)\\
&\leq 2\log^{[p-1]} \nu(2r, f) + \log^{[p]} r + O(1).
\end{aligned}$$

$$So, \;\; \log^{[p+1]} M(r, f) \leq \log^{[p]} \nu(2r, f) + \log^{[p+1]} r + O(1)$$

$$i.e., \;\; \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q+1]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \nu(2r, f)}{\log^{[q+1]} r} + \limsup_{r\to\infty} \frac{\log^{[p+1]} r}{\log^{[q+1]} r}$$

$$So, \;\; \rho_{log}[p, q] = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q+1]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, f)}{\log^{[q+1]} r}. \tag{2.5}$$

Thus combining (2.4) and (2.5) we get

$$\rho_{log}[p, q] = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q+1]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, f)}{\log^{[q+1]} r}$$

which means

$$\rho_{log}[p, q] = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, f)}{\log^{[q+1]} r}. \tag{2.6}$$

Thus if $\log\mu(r, f)$ has logarithmic $[p, q]$th order $\rho_{log}[p, q]$ then $\nu(r, f)$ also has logarithmic $[p, q]$th order $\rho_{log}[p, q]$.

Theorem 2.2. Let f be a transcendental meromorphic function in **C** with its $[p, q]$th order being zero. Then f and $f^{(k)}, k = 1, 2, ...,$ have the same logarithmic $[p, q]$th order where $f^{(k)}, k = 1, 2, ...,$ denotes the kth derivative of f.

The following two lemmas will be needed for the purpose to prove this theorem which will help to establish the main result.

Lemma 2.4 [6]. If $f(z)$ is a transcendental meromorphic function then

$$T(r,f^{'}) \leq 2T(2r,f)) + o\{T(2r,f)\}$$

for all large values of r.

Lemma 2.5 [3]. Let f(z) be a meromorphic function. Then for all large r

$$T(r,f) < C\{T(2r,f^{'}) + \log r\}$$

where C is a constant which is only dependent on $f(0)$.

Proof of the theorem : We have from Lemma 2.4 by taking logarithm on both sides

$$\log T(r,f^{'}) \leq \log[T(2r,f)\{2+o(1)\}]$$

$$So, \quad \log T(r,f^{'}) \leq \log T(2r,f) + O(1).$$

Taking repeated logarithm $(p-1)$ times on both sides of the above inequality

$$\log^{[p]} T(r,f^{'}) \leq \log^{[p]} T(2r,f) + o(1). \tag{2.7}$$

Assuming $\rho_{log}[p,q]$ and $\rho^{'}_{log}[p,q]$ to be the logarithmic $[p,q]$th orders of f and $f^{'}$ respectively and using (2.7)

$$\rho^{'}_{log}[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f^{'})}{\log^{[q+1]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(2r,f)}{\log^{[q+1]} 2r} = \rho_{log}[p,q]. \tag{2.8}$$

On the other hand, from Lemma 2.5

$$\log T(r,f) \leq \log T(2r,f^{'}) + O(1) \quad since, \log r = o\{T(r,f)\}\ .$$

Taking repeated logarithm $(p-1)$ times on both sides of the above inequality

$$\log^{[p]} T(r,f) \leq \log^{[p]} T(2r,f^{'}) + O(1). \tag{2.9}$$

So from (2.9)

$$\rho_{log}[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q+1]} r} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(2r,f^{'})}{\log^{[q+1]} 2r} = \rho^{'}_{log}[p,q]. \tag{2.10}$$

Combining (2.8) and (2.10)

$$\rho_{log}[p,q] = \rho^{'}_{log}[p,q].$$

Thus f and $f^{'}$ have same logarithmic $[p,q]$th order. Hence by induction it can be concluded that f and $f^{(k)}$, $k = 1, 2, \ldots$, will also have same logarithmic $[p,q]$th order.

Main Result

In this section we present the main result of this paper.

Theorem 3.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series with finite logarithmic $[p,q]$th order $\rho_{log}[p,q]$. Then

$$\rho_{log}[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]}[r\{\mu_k(r)/\mu(r)\}^{1/k}]}{\log^{[q+1]} r}, \quad k = 1, 2, \ldots .$$

Proof : From the conclusion obtained in Theorem 2.2 that the logarithmic $[p, q]$th order of a transcendental entire function and its derivative are the same and in view of (2.6), it follows that

$$\rho_{log}[p,q] = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q+1]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} \nu_k(r,f)}{\log^{[q+1]} r}, \; k = 0, 1, 2, \dots . \tag{3.11}$$

Successive differentiation on both sides of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ gives

$f^{(1)}(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$f^{(2)}(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$,

$f^{(3)}(z) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n z^{n-3}$,

$\dots \; \dots \; \dots \; \dots \; \dots \; \dots$

$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1) a_n z^{n-k}$.

Now let

$$f^{(k)}(z) = \sum_{n=k}^{\infty} A_n z^{n-k}, \quad k = 1, 2, \dots, where \;\; A_n = n(n-1)(n-2)\dots(n-k+1)a_n.$$

Also let $\nu_k(r) = N$ and $\nu_{k+1}(r) = N_1$. Then $\mu_k(r) = |A_N| r^{N-k}$.

Now, $f^{(k+1)}(z) = \sum_{n=k+1}^{\infty} n(n-1)(n-2)\dots(n-k+1)(n-k) a_n z^{n-k-1}$

$= \sum_{n=k+1}^{\infty} B_n z^{n-k-1}, say,$

where $B_n = n(n-1)(n-2)\dots(n-k+1)(n-k)a_n = (n-k)A_n$.

Therefore

$$\begin{aligned} \mu_{k+1}(r) &= |B_{N_1}| r^{N_1-k-1}, \quad since \;\; \nu_{k+1}(r) = N_1 \\ &= (N_1 - k)|A_{N_1}| r^{N_1-k-1}, \quad since \;\; B_{N_1} = (N_1-k)A_{N_1} \\ &= \frac{(N_1-k)}{r} |A_{N_1}| r^{N_1-k} \\ &\le \frac{\nu_{k+1}(r) - k}{r} |A_N| r^{N-k}, \end{aligned}$$

since $\nu_{k+1}(r) = N_1$ and $|A_{N_1}| r^{N_1-k} \le |A_N| r^{N-k}$ as $\mu_k(r) = |A_N| r^{N-k}$ is the maximum of all such terms viz. $|A_n| r^{n-k}$ for $n = k, k+1, \dots$.

Hence

$$\mu_{k+1}(r) \le \frac{\nu_{k+1}(r) - k}{r} \mu_k(r)$$

$$So, \quad r\frac{\mu_{k+1}(r)}{\mu_k(r)} \le \nu_{k+1}(r) - k, \quad k = 0, 1, 2, \dots . \tag{3.12}$$

On the other hand, for $k = 0, 1, 2, \dots,$

$$\mu_k(r) = |A_N| r^{N-k}$$

$$= \frac{r}{N-k}(N-k)|A_N|r^{N-k-1}$$
$$\leq \frac{r}{\nu_k(r)-k}\mu_{k+1}(r) \ ,$$

since $\mu_{k+1}(r) = (N_1 - k)|A_{N_1}|r^{N_1-k-1}$ is the maximum of all such terms viz. $(n-k)|A_n|r^{n-k-1}$ for $n = k+1, k+2, \ldots$.

This implies

$$r\frac{\mu_{k+1}(r)}{\mu_k(r)} \geq \nu_k(r) - k, \quad k = 0, 1, 2, \ldots \ . \tag{3.13}$$

Combining (3.12) and (3.13), it can be written that

$$\nu_k(r) - k \leq r\frac{\mu_{k+1}(r)}{\mu_k(r)} \leq \nu_{k+1}(r) - k, \quad k = 0, 1, 2, \ldots \ . \tag{3.14}$$

Putting $k = 0, 1, 2, \ldots$ in (3.14), the following inequalities are obtained

$$\nu_0(r) - 0 \leq r\frac{\mu_1(r)}{\mu_0(r)} \leq \nu_1(r) - 0,$$

$$\nu_1(r) - 1 \leq r\frac{\mu_2(r)}{\mu_1(r)} \leq \nu_2(r) - 1,$$

$$\nu_2(r) - 2 \leq r\frac{\mu_3(r)}{\mu_2(r)} \leq \nu_3(r) - 2,$$

$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$

and $\quad \nu_{k-1}(r) - (k-1) \leq r\frac{\mu_k(r)}{\mu_{k-1}(r)} \leq \nu_k(r) - (k-1).$

On multiplication

$$\{\nu_0(r) - 0\}\{\nu_1(r) - 1\}\{\nu_2(r) - 2\}\ldots\{\nu_{k-1}(r) - (k-1)\}$$
$$\leq \{r\frac{\mu_1(r)}{\mu_0(r)}\}\{r\frac{\mu_2(r)}{\mu_1(r)}\}\{r\frac{\mu_3(r)}{\mu_2(r)}\}\ldots\{r\frac{\mu_k(r)}{\mu_{k-1}(r)}\}$$
$$\leq \{\nu_1(r) - 0\}\{\nu_2(r) - 1\}\{\nu_3(r) - 2\}\ldots\{\nu_k(r) - (k-1)\}.$$

Now since $\nu_{k-1}(r) \leq \nu_k(r)$ for $k = 1, 2, 3, \ldots$, it is quite evident that

$$\nu_0(r) - (k-1) \leq \nu_0(r) - 0, \quad \nu_0(r) - (k-1) \leq \nu_1(r) - 1,$$

$$\nu_0(r) - (k-1) \leq \nu_2(r) - 2, \ldots, \nu_0(r) - (k-1) \leq \nu_{k-1}(r) - (k-1).$$

And similarly

$$\nu_1(r) - 0 \leq \nu_k(r) - 0, \quad \nu_2(r) - 1 \leq \nu_k(r) - 0,$$

$$\nu_3(r) - 2 \leq \nu_k(r) - 0, \ldots, \nu_k(r) - (k-1) \leq \nu_k(r) - 0.$$

So,

$$\{\nu_0(r) - (k-1)\}^k \leq r^k\frac{\mu_k(r)}{\mu_0(r)} \leq \{\nu_k(r) - 0\}^k,$$

which gives

$$\frac{\{\nu_0(r) - (k-1)\}^k}{r^k} \leq \frac{\mu_k(r)}{\mu_0(r)} \leq \frac{\{\nu_k(r) - 0\}^k}{r^k}, \quad k = 1, 2, \ldots.$$

$$i.e., \quad \frac{\{\nu_0(r) - (k-1)\}}{r} \leq \{\frac{\mu_k(r)}{\mu_0(r)}\}^{1/k} \leq \frac{\nu_k(r)}{r}, \quad k = 1, 2, \ldots.$$

$$i.e., \quad \{\nu(r) - (k-1)\} \leq r\{\frac{\mu_k(r)}{\mu_0(r)}\}^{1/k} \leq \nu_k(r), \quad k = 1, 2, \ldots, \quad since \;\; \nu_0(r) = \nu(r).$$

So,

$$\frac{\log^{[p]}[\nu(r) - (k-1)]}{\log^{[q+1]} r} \leq \frac{\log^{[p]}[r\{\frac{\mu_k(r)}{\mu(r)}\}^{1/k}]}{\log^{[q+1]} r} \leq \frac{\log^{[p]} \nu_k(r)}{\log^{[q+1]} r}, k = 1, 2, \ldots \;. \tag{3.15}$$

Clearly $\nu(r)$ and $\nu(r) - (k-1)$ will have same logarithmic $[p, q]$th order. Hence from (3.11)

$$\rho_{log}[p, q] = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r)}{\log^{[q+1]} r} = \limsup_{r\to\infty} \frac{\log^{[p]}[\nu(r) - (k-1)]}{\log^{[q+1]} r} \;. \tag{3.16}$$

From (3.15) and (3.16) we have

$$\rho_{log}[p, q] = \limsup_{r\to\infty} \frac{\log^{[p]}[r\{\frac{\mu_k(r)}{\mu(r)}\}^{1/k}]}{\log^{[q+1]} r}, \quad k = 1, 2, \ldots \;.$$

Hence the result.

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