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On Two-Dimensional Functions With an Integral Equal to Zero Over a Rectangle: Application to a Modified Gaussian Distribution

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Abstract
The study of two-dimensional real functions with an integral over a rectangle equal to zero can lead to a better understanding of integration theory and its applications in various fields such as physics, engineering, and mathematics. Additionally, it is of importance in the development of numerical methods for computing integrals. In this article, we investigate some manageable conditions under which two-dimensional functions satisfy this integral property. They include the standard separable and centered integral conditions, the odd conditions, the antisymmetric conditions, the trigonometric conditions and the composite function difference conditions. The findings are presented as propositions with thorough and self-contained proofs. Various examples are given to illustrate the findings. An application part is devoted to two new modified two-dimensional Gaussian distributions. These modifications are made by the use of flexible trigonometric perturbation functions, which are able to produce two-dimensional multi-modality and various weights on the tails. These new perturbed Gaussian distributions represent promising new avenues for research in probability theory, with potential applications in a wide range of fields.

Keywords: two-dimensional functions; two-dimensional integrals; trigonometric functions; Gaussian distribution.

Introduction

Two-dimensional functions with an integral of zero over a rectangle have numerous applications in mathematics, physics, engineering, and other fields. Here are some areas of interest for these functions:

In vector calculus, they can correspond to vector fields that are divergence-free. This property is important in the study of fluid dynamics, electromagnetism, and other areas of physics. See [1].

In differential equations, they can be used to construct solutions to certain types of differential equations, such as the wave equation and the heat equation. See [2].

In mathematical physics, they correspond to conserved quantities, such as the total charge or energy of a system. These quantities are particularly useful in the study of classical mechanics, quantum mechanics, and other areas of physics. See [3, 4].

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In harmonic analysis, they can be expressed as a linear combination of harmonic functions, which have important properties in analysis and geometry. See [5].

In Fourier analysis, they can be decomposed into Fourier series, which have applications in signal processing, data analysis, and other fields. See [6].

In probability theory, they arise in the study of probability distributions, such as the joint distribution of two random variables. Applications can be found in multivariate Gaussian distributions, regression analysis, and other areas of statistics. See [7].

In numerical methods, they can be approximated numerically using specific methods, such as finite element methods and spectral methods. See [8].

These are just a few examples of the many areas in which two-dimensional functions with an integral of zero over a rectangle are of interest. Further reading on these and related topics can be found in the references provided.

In the overall literature, a lot of different and precise examples exist, but rare are the attempts to exhibit clear general conditions for two-dimensional functions with such properties. This article aims to fill this gap.

The first part determines several different sets of conditions, such as the standard separable and centered integral conditions involving intermediary integral terms, the odd conditions involving odd functions according to a single variable, the antisymmetric (or skew-symmetric) conditions involving antisymmetric functions, i.e., functions $U(x, y)$ such that $U(x, y) = -U(y, x)$ for any $(x, y) \in \mathbb{R}^2$ (see [13]), the trigonometric conditions involving sine and cosine functions, and the composite function difference conditions based on a specific form and a two-dimensional change of variables with one variable on $\mathbb{R}$. They are presented in the form of clear propositions, with detailed proofs. In order to relate to concrete calculus, each proposition is illustrated with concrete examples. This part thus presents the main theoretical results, and they can be taken independently of the researchers' interests.

The second part is devoted to an application of the findings to a very useful probability tool. To communicate its importance, a retrospective on the two-dimensional Gaussian distribution is necessary. To begin, a two-dimensional Gaussian distribution is a multivariate distribution with two dimensions characterized by its elliptical contours and smooth bell shape. It has many applications in fields such as signal processing, statistics, image processing, and machine learning (see [9], [10], [11] and [12]). Various extensions of it have been elaborated (see [14] for the skewed two-dimensional Gaussian distribution, [15] for the wrapped two-dimensional Gaussian distribution, [16] for the mixture two-dimensional Gaussian distribution, and [17] for the two-dimensional Gaussian copula). Nevertheless, trigonometric extensions of it are rare, despite potential applications in the context of image processing and pattern recognition, among others. We make a contribution in this direction; using the trigonometric conditions established in the first part of the article, we create new sine and cosine-modified two-dimensional Gaussian distributions. More precisely, sine and cosine perturbation functions are used to make these modifications, which can result in multi-modality and different tail weights. The findings are illustrated by graphics. Intriguing oscillating shapes are observed, along with a high level of functional versatility on the peaks and tails. The potential applications of these distributions are numerous in mathematics, physics, engineering, and other fields.

The rest of the article considers the following sections: Section describes the main context, and the separable and centered integral conditions. Section is devoted to the analysis of odd-type conditions. Antisymmetric conditions are described in Section . The trigonometric conditions are the subject of Section . Section focuses on the composite function difference conditions. The new trigonometric two-dimensional Gaussian distributions are presented in Section . Conclusions are given in Section .

Context and classical integral conditions

Context

In all the article, we consider four real numbers, α, β, γ and λ, such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). Based on them, we define the rectangle (or rectangular domain): $[\alpha, \beta] \times [\gamma, \lambda]$ as

$$[\alpha, \beta] \times [\gamma, \lambda] = \{(x, y) \in \mathbb{R}^2 \mid \alpha \le x \le \beta, \ \gamma \le y \le \lambda\}.$$

In other words, $[\alpha, \beta] \times [\gamma, \lambda]$ represents the Cartesian product of $[\alpha, \beta]$ and $[\gamma, \lambda]$. When $\gamma = \alpha$ and $\lambda = \beta$, we write $[\alpha, \beta] \times [\gamma, \lambda] = [\alpha, \beta]^2$. In addition, when $\alpha = -\infty$ and $\beta = +\infty$, then $[\alpha, \beta]$ represents the entire real line: $\mathbb{R}$.

In the first part of this article, as developed in the introductory section, we aim to determine general and manageable conditions on two-dimensional integrable (real) functions $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ satisfying

$$\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y) dx dy = 0.$$

The possible multiplicative constant terms are easy to handle and therefore omitted. To this end, it is supposed that $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y) dx dy$ exists (or converge in the integral sense), and that we can switch the order of integration (under the wide conditions of the Fubini theorem, for instance). Overall, the book of Stewart provides a comprehensive introduction to the integration of two-dimensional functions and is a valuable resource (see [18]).

Separable conditions

The most basic conditions that immediately come to mind are those involving separable functions, as described in the following result.

Proposition 2.1 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional integrable function $f(x)$ defined on $[\alpha, \beta]$,*
- *an uni-dimensional integrable function $g(y)$ defined on $[\gamma, \lambda]$,*

*satisfying $\int_{\alpha}^{\beta} f(x) dx = 0$ **or** $\int_{\gamma}^{\lambda} g(y) dy = 0$, and such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have*

$$P(x, y) = f(x) g(y).$$

Then we have $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y) dx dy = 0$.

Proof. Clearly, by the distributive law, we get

$$\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y) dx dy = \int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} f(x) g(y) dx dy = \left(\int_{\alpha}^{\beta} f(x) dx \right) \left(\int_{\gamma}^{\lambda} g(y) dy \right) = 0.$$

The proof ends. □

For illustrative purposes, two scholar examples are given below.

Example 1: Let us consider $P(x, y) = \sin(\pi x) \cos(\pi y)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 2.1, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = f(x) g(y)$ with $f(x) = \sin(\pi x)$ and $g(y) = \cos(\pi y)$. Since $\int_0^1 g(y) dy = (1/\pi) \sin(\pi y)|_{y=0}^{y=1} = 0$, we have $\int_0^1 \int_0^1 P(x, y) dx dy = 0$.

Example 2: Let us consider $P(x,y) = x^3 \exp(y)$ for $(x,y) \in [-1,1]\times[0,1]$. Thus, in relation to the notations of Proposition 2.1, we have $\alpha = -1$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x,y) = f(x)g(y)$ with $f(x) = x^3$ and $g(y) = \exp(y)$. Since $\int_{-1}^{1} f(x)dx = (1/4)x^4\big|_{x=-1}^{x=1} = 0$, we have $\int_0^1 \int_{-1}^1 P(x,y)dxdy = 0$.

Of course, in many situations, the separable assumption on $P(x,y)$ is very optimistic and can be immediately refuted. More sophisticated conditions are investigated below.

Centered integral conditions

Other conditions for two-dimensional functions to have an integral equal to zero over a rectangle are the centered integral conditions consisting of subtracting with an intermediary integral term. These conditions are presented in several propositions below.

Proposition 2.2 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x,y)$ defined on $[\alpha,\beta]\times[\gamma,\lambda]$ under the assumption below.*

Suppose that there exists a two-dimensional integrable function $Q(x,y)$ defined on $[\alpha,\beta]\times[\gamma,\lambda]$ such that, for any $(x,y) \in [\alpha,\beta]\times[\gamma,\lambda]$, we have

$$P(x,y) = Q(x,y) - \frac{1}{(\beta-\alpha)(\lambda-\gamma)} \int_\gamma^\lambda \int_\alpha^\beta Q(x,y)dxdy.$$

Then we have $\int_\gamma^\lambda \int_\alpha^\beta P(x,y)dxdy = 0$.

Proof. By setting $\theta = \int_\gamma^\lambda \int_\alpha^\beta Q(x,y)dxdy$, we obtain

$$\int_\gamma^\lambda \int_\alpha^\beta P(x,y)dxdy = \int_\gamma^\lambda \int_\alpha^\beta \left[Q(x,y) - \frac{\theta}{(\beta-\alpha)(\lambda-\gamma)}\right] dxdy$$
$$= \int_\gamma^\lambda \int_\alpha^\beta Q(x,y)dxdy - \frac{\theta}{(\beta-\alpha)(\lambda-\gamma)} \int_\gamma^\lambda \int_\alpha^\beta dxdy$$
$$= \theta - \frac{\theta}{(\beta-\alpha)(\lambda-\gamma)}(\beta-\alpha)(\lambda-\gamma) = \theta - \theta = 0.$$

This ends the proof. □

Two examples are provided below for explanatory purposes.

Example 1: Let us consider $P(x,y) = (1-xy)^2 - 11/18$ for $(x,y) \in [0,1]^2$. Thus, in relation to the notations of Proposition 2.2, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x,y) = Q(x,y) - \int_0^1\int_0^1 Q(x,y)dxdy$ with $Q(x,y) = (1-xy)^2$, by noticing that $\int_0^1\int_0^1 Q(x,y)dxdy = \int_0^1 -(1/(3y))\,(1-xy)^3\big|_{x=0}^{x=1} dy = (1/3)\int_0^1 (y^2 - 3y + 3)dy = (1/3) \times \left((1/3)y^3 - (3/2)y^2 + 3y\right)\big|_{y=0}^{y=1} = 11/18$. Hence we have $\int_0^1\int_0^1 P(x,y)dxdy = 0$.

Example 2: Let us consider $P(x,y) = x\cos(xy) - 2/\pi$ for $(x,y) \in [0,\pi]\times[0,1]$. Thus, in relation to the notations of Proposition 2.2, we have $\alpha = 0$, $\beta = \pi$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x,y) = Q(x,y) - (1/\pi)\int_0^1\int_0^\pi Q(x,y)dxdy$ with $Q(x,y) = x\cos(xy)$, by noticing that $\int_0^1\int_0^\pi Q(x,y)dxdy = \int_0^\pi \sin(xy)\big|_{y=0}^{y=1} dx = \int_0^\pi \sin(x)dx = -\cos(x)\big|_{x=0}^{x=\pi} = 2$. Hence we have $\int_0^1\int_0^\pi P(x,y)dxdy = 0$.

Another kind of centered integral conditions are presented below.

Proposition 2.3 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x,y)$ defined on $[\alpha,\beta]\times[\gamma,\lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(y)$ defined on $[\gamma, \lambda]$ such that $T(\gamma) = \gamma$ and $T(\lambda) = \lambda$, and with $t(y) = T'(y)$,*
- *a two-dimensional integrable function $Q(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$,*

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = Q(x, y) - \frac{1}{\beta - \alpha} t(y) \int_{\alpha}^{\beta} Q(x, T(y))dx.$$

Then we have $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y)dxdy = 0$.

Proof. Let us set $\tau(y) = t(y) \int_{\alpha}^{\beta} Q(x, T(y))dx$. By applying the change of variables $v = T(y)$, so that $T'(y) = t(y)$, $T(\gamma) = \gamma$ and $T(\lambda) = \lambda$, we get

$$\int_{\gamma}^{\lambda} \tau(y)dy = \int_{\gamma}^{\lambda} t(y) \int_{\alpha}^{\beta} Q(x, T(y))dxdy = \int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} Q(x, v)dxdv.$$

By setting $\theta = \int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} Q(x, y)dxdy$, we have

$$\begin{aligned}
&\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y)dxdy = \int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} \left[Q(x, y) - \frac{1}{\beta - \alpha} \tau(y)\right] dxdy \\
&= \int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} Q(x, y)dxdy - \frac{1}{\beta - \alpha} \int_{\gamma}^{\lambda} \tau(y)dy \int_{\alpha}^{\beta} dx \\
&= \theta - \frac{\theta}{\beta - \alpha}(\beta - \alpha) = \theta - \theta = 0.
\end{aligned}$$

This ends the proof. □

The next example is provided for illustration only.

Example: Let us consider $P(x, y) = xy^2 - y^5$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 2.3, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = Q(x, y) - t(y) \int_0^1 Q(x, T(y))dx$ with $Q(x, y) = xy^2$ and $T(y) = y^2$ satisfying $T(0) = 0$ and $T(1) = 1$, and $t(y) = T'(y) = 2y$, by noticing that $t(y) \int_0^1 Q(x, T(y))dx = 2y\ (1/2)x^2y^4|_{x=0}^{x=1} = y^5$. Hence we have $\int_0^1 \int_0^1 P(x, y)dxdy = 0$.

Proposition 2.4 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(x)$ defined on $[\alpha, \beta]$ such that $T(\alpha) = \alpha$ and $T(\beta) = \beta$, and with $t(x) = T'(x)$,*
- *a two-dimensional integrable function $Q(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$,*

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = Q(x, y) - \frac{1}{\lambda - \gamma} t(x) \int_{\gamma}^{\lambda} Q(T(x), y)dy.$$

Then we have $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y)dxdy = 0$.

The proof is almost identical to that of Proposition 2.3. For this reason, it is omitted.

The next example is provided for illustration only.

Example: Let us consider $P(x,y) = xy^2 - (2/3)x^3$ for $(x,y) \in [0,1]^2$. Thus, in relation to the notations of Proposition 2.4, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x,y) = Q(x,y) - t(x)\int_0^1 Q(T(x),y)dy$ with $Q(x,y) = xy^2$ and $T(x) = x^2$ satisfying $T(0) = 0$ and $T(1) = 1$, and $t(x) = T'(x) = 2x$, by noticing that $t(x)\int_0^1 Q(T(x),y)dy = 2x\,(1/3)x^2y^3\big|_{y=0}^{y=1} = (2/3)x^3$. Hence we have $\int_0^1\int_0^1 P(x,y)dxdy = 0$.

A limitation in the conditions presented in this section is that the involved integral terms on $Q(x,y)$, i.e., $\int_\gamma^\lambda\int_\alpha^\beta Q(x,y)dxdy$ or $\int_\alpha^\beta Q(x,y)dx$ or $\int_\gamma^\lambda Q(x,y)dy$, have not always a closed form or can be complicated to calculate. Furthermore, they are a bit artificial in the sense that they are rarely encountered in natural phenomena. Alternative conditions are developed in the next sections.

Odd conditions

It is common to deal with two-dimensional functions involving odd functions with respect to a single variable. In this case, the corresponding integral can be zero under precise conditions, as formulated in the next proposition.

Proposition 3.5 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x,y)$ defined on $[\alpha,\beta]\times[\gamma,\lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(y)$ defined on $[\gamma,\lambda]$, and with $t(y) = T'(y)$,*
- *a real number ξ satisfying $T(\gamma) = -\xi$ and $T(\lambda) = \xi$,*
- *a two-dimensional function $U(x,y)$ defined on $[\alpha,\beta]\times[-\xi,\xi]$ satisfying, for any $(x,y) \in [\alpha,\beta]\times[-\xi,\xi]$,*

$$U(x,-y) = -U(x,y),$$

such that, for any $(x,y) \in [\alpha,\beta]\times[\gamma,\lambda]$, we have

$$P(x,y) = t(y)U(x,T(y)).$$

Then we have $\int_\gamma^\lambda\int_\alpha^\beta P(x,y)dxdy = 0$.

Proof. By the change of variables $v = T(y)$, so that $T'(y) = t(y)$, $T(\gamma) = -\xi$ and $T(\lambda) = \xi$, (the switch the order of integration, assumed as possible in all the article), the change of variables $w = -v$, the property $U(x,-w) = -U(x,w)$, and the change of variables $w = T(z)$, we obtain

$$\begin{aligned}
I &= \int_\gamma^\lambda\int_\alpha^\beta P(x,y)dxdy = \int_\gamma^\lambda\int_\alpha^\beta t(y)U(x,T(y))dxdy \\
&= \int_{-\xi}^{\xi}\int_\alpha^\beta U(x,v)dxdv = \int_\alpha^\beta\int_{-\xi}^{\xi} U(x,v)dvdx \\
&= \int_\alpha^\beta\int_{-\xi}^{\xi} U(x,-w)dwdx = -\int_\alpha^\beta\int_{-\xi}^{\xi} U(x,w)dwdx \\
&= -\int_\alpha^\beta\int_\gamma^\lambda t(z)U(x,T(z))dzdx = -\int_\gamma^\lambda\int_\alpha^\beta P(x,z)dxdz = -I.
\end{aligned}$$

As a result, we have $I = 0$. This ends the proof. □

Of course, Proposition 3.5 can be applied to a function $U(x,y)$ with a high level of complexity. This is one of the interests of the determination of general conditions.

Example 1: Let us consider $P(x, y) = \exp(x+y) - \exp(x-y) - \exp(-x+y) + \exp(-x-y)$ for $(x, y) \in [0, 1] \times [-1, 1]$. Thus, in relation to the notations of Proposition 3.5, we have $\alpha = 0$, $\beta = 1$, $\gamma = -1$, $\lambda = 1$, and we can write $P(x, y) = t(y)U(x, T(y))$ with $U(x, y) = \exp(x+y) - \exp(x-y) - \exp(-x+y) + \exp(-x-y)$ and $T(y) = y$ with $t(y) = T'(y) = 1$. We have $T(-1) = -1$ and $T(1) = 1$, so $\xi = 1$. Since $U(x, -y) = \exp(x-y) - \exp(x+y) - \exp(-x-y) + \exp(-x+y) = -U(x, y)$, we have $\int_{-1}^{1} \int_{0}^{1} P(x, y)dxdy = 0$.

Example 2: Let us consider $P(x, y) = \exp(x(y - 1/2)) - \exp(-x(y - 1/2))$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 3.5, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(y)U(x, T(y))$ with $U(x, y) = \exp(xy) - \exp(-xy)$ and $T(y) = y - 1/2$ with $t(y) = T'(y) = 1$. We have $T(0) = -1/2$ and $T(1) = 1/2$, so $\xi = 1/2$. Since $U(x, -y) = \exp(-xy) - \exp(xy) = -U(x, y)$, we have $\int_{0}^{1} \int_{0}^{1} P(x, y)dxdy = 0$.

A twin result to Proposition 3.5 is given below.

Proposition 3.6 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(x)$ defined on $[\alpha, \beta]$, and with $t(x) = T'(x)$,*
- *a real number ξ satisfying $T(\alpha) = -\xi$ and $T(\beta) = \xi$,*
- *a two-dimensional function $U(x, y)$ defined on $[-\xi, \xi] \times [\gamma, \lambda]$, satisfying, for any $(x, y) \in [-\xi, \xi] \times [\gamma, \lambda]$,*

$$U(-x, y) = -U(x, y)$$

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = t(x)U(T(x), y).$$

Then we have $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P(x, y)dxdy = 0$.

The proof is almost identical to that of Proposition 3.5; only the domain of integration and the change of variables differ. For this reason, it is omitted.

The next two examples are provided for illustration only.

Example 1: Let us consider $P(x, y) = \sin(xy^2)$ for $(x, y) \in [-1, 1] \times [0, 1]$. Thus, in relation to the notations of Proposition 3.6, we have $\alpha = -1$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(x)U(T(x), y)$ with $U(x, y) = \sin(xy^2)$ and $T(x) = x$ with $t(x) = T'(x) = 1$. We have $T(-1) = -1$ and $T(1) = 1$, so $\xi = 1$. Since $U(-x, y) = -\sin(xy^2) = -U(x, y)$, we have $\int_{0}^{1} \int_{-1}^{1} P(x, y)dxdy = 0$.

Example 2: Let us consider $P(x, y) = (x - 1/2)\cos(y) - y\sin(x - 1/2)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 3.6, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(x)U(T(x), y)$ with $U(x, y) = x\cos(y) - y\sin(x)$ and $T(x) = x - 1/2$ with $t(x) = T'(x) = 1$. We have $T(0) = -1/2$ and $T(1) = 1/2$, so $\xi = 1/2$. Since $U(-x, y) = -x\cos(y) + y\sin(x) = -U(x, y)$, we have $\int_{0}^{1} \int_{0}^{1} P(x, y)dxdy = 0$.

Antisymmetric conditions

We recall that an antisymmetric two-dimensional function is a function $U(x, y)$ that satisfies the condition $U(x, y) = -U(y, x)$ for any $(x, y) \in \mathbb{R}^2$. This property is also known as skew symmetry. More information on this notion can be found in [13]. Antisymmetric functions have important applications in physics, particularly in the study of electromagnetism and quantum mechanics. They also play a role in signal processing and image analysis, where they are used to represent certain types of data with symmetries.

In the result below, we show how antisymmetric two-dimensional functions are involved in functions with an integral over a rectangle equal to zero.

Proposition 4.7 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(y)$ defined on $[\gamma, \lambda]$ such that $T(\gamma) = \alpha$ and $T(\lambda) = \beta$, and with $t(y) = T'(y)$,*
- *a two-dimensional function $U(x, y)$ defined on $[\alpha, \beta]^2$ satisfying, for any $(x, y) \in [\alpha, \beta]^2$,*

$$U(x, y) = -U(y, x),$$

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = t(y)U(x, T(y)).$$

Then we have $\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = 0$.

Proof. By the change of variables $v = T(y)$, so that $T'(y) = t(y)$, $T(\gamma) = \alpha$ and $T(\lambda) = \beta$, the property $U(x, v) = -U(v, x)$, and the change of variable $x = T(w)$, we have

$$\begin{aligned} I &= \int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = \int_\gamma^\lambda \int_\alpha^\beta t(y)U(x, T(y))dxdy \\ &= \int_\alpha^\beta \int_\alpha^\beta U(x, v)dxdv = -\int_\alpha^\beta \int_\alpha^\beta U(v, x)dxdv \\ &= -\int_\alpha^\beta \int_\alpha^\beta U(v, x)dvdx = -\int_\gamma^\lambda \int_\alpha^\beta t(w)U(v, T(w))dvdw = -I. \end{aligned}$$

As a result, we have $I = 0$. This ends the proof. □

The interest of Proposition 4.7 is that, despite the degree of complexity of the antisymmetric function $U(x, y)$, the resulting integral is always 0 under the precise stated conditions. Most of the symbolic software does not detect the antisymmetric property, which can lead to an exorbitantly long time of computation to finally obtain zero (if they are able to finish).

The next two examples are provided for illustration only.

Example 1: Let us consider $P(x, y) = (x - y)\sin(x + y)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 4.7, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(y)U(x, T(y))$ with $U(x, y) = (x - y)\sin(x + y)$ and $T(y) = y$ with $t(y) = T'(y) = 1$. We have $T(0) = 0$ and $T(1) = 1$. Since $U(x, y) = -(y - x)\sin(y + x) = -U(y, x)$, we have $\int_0^1 \int_0^1 P(x, y)dxdy = 0$.

Example 2: Let us consider $P(x, y) = 8y\sinh(3(x - 4y^2))$ for $(x, y) \in [0, 1] \times [0, 1/2]$. Thus, in relation to the notations of Proposition 4.7, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1/2$, and we can write $P(x, y) = t(y)U(x, T(y))$ with $U(x, y) = \sinh(3(x - y))$ and $T(y) = 4y^2$ with $t(y) = T'(y) = 8y$. We have $T(0) = 0$ and $T(1/2) = 1$. Since $U(x, y) = -\sinh(3(y - x)) = -U(y, x)$, we have $\int_0^{1/2} \int_0^1 P(x, y)dxdy = 0$.

A twin result can be presented as follows:

Proposition 4.8 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional bijective and differentiable function $T(x)$ defined on $[\alpha, \beta]$ such that $T(\alpha) = \gamma$ and $T(\beta) = \lambda$, and with $t(x) = T'(x)$,*
- *a two-dimensional function $U(x, y)$ defined on $[\gamma, \lambda]^2$ satisfying, for any $(x, y) \in [\gamma, \lambda]^2$,*

$$U(x, y) = -U(y, x),$$

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = t(x)U(T(x), y).$$

Then we have $\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = 0$.

The proof is almost identical to that of Proposition 4.7; only the domain of integration and the change of variables differ. For this reason, it is omitted.

The next two examples are provided for illustration only.

Example 1: Let us consider $P(x, y) = (x^2 - y^2)\exp(xy)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 4.8, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(x)U(T(x), y)$ with $U(x, y) = (x^2 - y^2)\exp(xy)$ and $T(x) = x$ with $t(x) = T'(x) = 1$. We have $T(0) = 0$ and $T(1) = 1$. Since $U(x, y) = -(y^2 - x^2)\exp(yx) = -U(y, x)$, we have $\int_0^1 \int_0^1 P(x, y)dxdy = 0$.

Example 2: Let us consider $P(x, y) = 2(4x^2 - y^2)/(4x^2 + y^2)$ for $(x, y) \in [0, 1/2] \times [0, 1]$. Thus, in relation to the notations of Proposition 4.8, we have $\alpha = 0$, $\beta = 1/2$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(x)U(T(x), y)$ with $U(x, y) = (x^2-y^2)/(x^2+y^2)$ and $T(x) = 2x$ with $t(x) = T'(x) = 2$. We have $T(0) = 0$ and $T(1/2) = 1$. Since $U(x, y) = -(y^2 - x^2)/(y^2 + x^2) = -U(y, x)$, we have $\int_0^1 \int_0^{1/2} P(x, y)dxdy = 0$.

Our results thus contribute to understanding the properties and behavior of the antisymmetric functions, which is essential for mastering the techniques and applications of integration in mathematics and its diverse scientific and engineering applications.

In the next section, we focus on a different approach based on trigonometric conditions.

Trigonometric conditions

The next proposition presents conditions on two-dimensional functions with an integral equal to zero over a rectangle based on trigonometric functions.

Proposition 5.9 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an integer m (it can be non-positive),*
- *a real number ϵ,*
- *an uni-dimensional function $S(x)$ defined on $[\alpha, \beta]$,*
- *an uni-dimensional integrable function $k(x)$ defined on $[\alpha, \beta]$,*

- *a bijective and differentiable function $T(y)$ defined on $[\gamma, \lambda]$ such that $T(\gamma) = 0$ and $T(\lambda) = 1$, and with $t(y) = T'(y)$,*

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = k(x)t(y) \sin [2\pi (S(x) + mT(y)) + \epsilon]$$

or

$$P(x, y) = k(x)t(y) \cos [2\pi (S(x) + mT(y)) + \epsilon] .$$

Then we have $\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = 0$.

Proof. Let us consider the sine definition of $P(x, y)$. By using the complex formula $\sin(u) = \mathrm{Imag}(\exp(iu))$, where $i^2 = -1$ and Imag denotes the imaginary part operator, we obtain

$$\begin{aligned}\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy &= \int_\gamma^\lambda \int_\alpha^\beta k(x)t(y) \sin [2\pi (S(x) + mT(y)) + \epsilon] \, dxdy \\ &= \int_\gamma^\lambda \int_\alpha^\beta k(x)t(y) \,\mathrm{Imag} \{\exp [i2\pi (S(x) + mT(y)) + i\epsilon]\} \, dxdy \\ &= \mathrm{Imag} (\exp(i\epsilon) \times I \times J) , \end{aligned} \tag{1}$$

where

$$I = \int_\alpha^\beta k(x) \exp (i2\pi S(x)) \, dx, \quad J = \int_\gamma^\lambda t(y) \exp (i2\pi mT(y)) \, dy.$$

Let us prove that $J = 0$. By the change of variables $v = T(y)$, so that $t(y) = T'(y)$, $T(\gamma) = 0$ and $T(\lambda) = 1$, and since m is an integer, implying that $\exp(i2\pi m) = \cos(2\pi m) + i \sin(2\pi m) = 1 + i \times 0 = 1$, we have

$$J = \int_\gamma^\lambda t(y) \exp (i2\pi mT(y)) \, dy = \int_0^1 \exp(i2\pi mv)dv = \frac{1}{i2\pi m} (\exp(i2\pi m) - 1) = 0.$$

Therefore, we have

$$\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = \mathrm{Imag} (\exp(i\epsilon) \times I \times 0) = 0.$$

For the cosine definition of $P(x, y)$, by using the complex formula $\cos(u) = \mathrm{Real}(\exp(iu))$, where Real denotes the real part operator, and proceeding in a similar way than for the sine definition, we get

$$\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = \mathrm{Real} (\exp(i\epsilon) \times I \times J) = \mathrm{Real} (\exp(i\epsilon) \times I \times 0) = 0.$$

This ends the proof. □

The next two examples are provided for illustration only.

Example 1: Let us consider $P(x, y) = \sin \left(2\pi(2x^2 - 3y) + 4\right)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 5.9, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = k(x)t(y) \sin [2\pi (S(x) + mT(y)) + \epsilon]$, with $m = -3$, $\epsilon = 4$, $S(x) = 2x^2$, $k(x) = 1$, and $T(y) = y$ with $t(y) = T'(y) = 1$. We have $T(0) = 0$ and $T(1) = 1$. Therefore, we have $\int_0^1 \int_0^1 P(x, y)dxdy = 0$.

Example 2: Let us consider $P(x, y) = \pi y^{\pi-1} e^x \cos (2\pi(\cos(x) + y^\pi) - 1/2)$ for $(x, y) \in [0, 1]^2$. Thus, in relation to the notations of Proposition 5.9, we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = k(x)t(y) \cos [2\pi (S(x) + mT(y)) + \epsilon]$, with $m = 1$, $\epsilon = -1/2$, $S(x) = \cos(x)$, $k(x) = e^x$, and $T(y) = y^\pi$ with $t(y) = T'(y) = \pi y^{\pi-1}$. We have $T(0) = 0$ and $T(1) = 1$. Therefore, we have $\int_0^1 \int_0^1 P(x, y)dxdy = 0$.

A twin result is presented below.

Proposition 5.10 *Let α, β, γ and λ be real numbers such that $\alpha < \beta$ and $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exists

- *an integer m (it can be non-positive),*
- *a real number ϵ,*
- *an uni-dimensional function $S(y)$ defined on $[\gamma, \lambda]$,*
- *an uni-dimensional integrable function $k(y)$ defined on $[\gamma, \lambda]$,*
- *a bijective and differentiable function $T(x)$ defined on $[\alpha, \beta]$ such that $T(\alpha) = 0$ and $T(\beta) = 1$, and with $t(x) = T'(x)$,*

such that, for any $(x, y) \in [\alpha, \beta] \times [\gamma, \lambda]$, we have

$$P(x, y) = t(x)k(y) \sin [2\pi (mT(x) + S(y)) + \epsilon]$$

or

$$P(x, y) = t(x)k(y) \cos [2\pi (mT(x) + S(y)) + \epsilon] .$$

Then we have $\int_\gamma^\lambda \int_\alpha^\beta P(x, y)dxdy = 0$.

The proof is almost identical to the one of Proposition 5.9, so it is omitted.

The next sophisticated example is provided for illustration only.

Example: Let us consider $P(x, y) = 2e^{-2x} \sinh(y) \sin[2\pi(2(1 - e^{-2x}) - 3y^2) + 5]$ for $(x, y) \in [0, +\infty) \times [0, 1]$. Thus, in relation to the notations of Proposition 5.10, we have $\alpha = 0$, $\beta = +\infty$, $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = t(x)k(y) \sin [2\pi (mT(x) + S(y)) + \epsilon]$, with $m = 2$, $\epsilon = 5$, $S(y) = -3y^2$, $k(y) = \sinh(y)$, and $T(x) = 1 - e^{-2x}$ with $t(x) = T'(x) = 2e^{-2x}$. We have $T(0) = 0$ and $\lim_{x\to+\infty} T(x) = 1$. Therefore, we have $\int_0^1 \int_0^{+\infty} P(x, y)dxdy = 0$.

It is interesting to note that, for this example among others, symbolic software failed to give the result 0 because of the high complexity of the integrated function; thanks to the proposed condition, the result follows immediately.

Composite function difference conditions

This section investigates more marginal conditions based on composite function differences. They can be applied only if one domain of integration is $\mathbb{R}$ and the functions of interest have imposed forms involving differences of functions with different variables.

Proposition 6.11 *Let γ and λ be real numbers such that $\gamma < \lambda$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $\mathbb{R} \times [\gamma, \lambda]$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional differentiable function $T(y)$ defined on $[\gamma, \lambda]$,*
- *a two-dimensional function $U(x, y)$ defined on $\mathbb{R} \times [\gamma, \lambda]$ satisfying*

$$\int_\gamma^\lambda \int_{-\infty}^{+\infty} U(x, y)dxdy = 0,$$

such that, for any $(x, y) \in \mathbb{R} \times [\gamma, \lambda]$, *we have*

$$P(x, y) = U(x - T(y), y).$$

Then we have $\int_{\gamma}^{\lambda} \int_{-\infty}^{+\infty} P(x, y)dxdy = 0$.

Proof. By the two-dimensional change of variables $(v, y) = (x - T(y), y)$ which has the Jacobian equals to 1, with $\lim_{x\to-\infty}[x - T(y)] = -\infty$ and $\lim_{x\to+\infty}[x - T(y)] = +\infty$, we have

$$\int_{\gamma}^{\lambda} \int_{-\infty}^{+\infty} P(x, y)dxdy = \int_{\gamma}^{\lambda} \int_{-\infty}^{+\infty} U(x - T(y), y)dxdy = \int_{\gamma}^{\lambda} \int_{-\infty}^{+\infty} U(v, y)dvdy = 0.$$

This ends the proof. □

Such composite function difference conditions are classical in probability theory when dealing with the two-dimensional Gaussian distribution in particular.

An example is detailed below.

Example: Let us consider $P(x, y) = \exp(-|x - e^y|)\cos(\pi y)$ for $(x, y) \in \mathbb{R} \times [0, 1]$. Thus, in relation to the notations of Proposition 6.11, we have $\gamma = 0$, $\lambda = 1$, and we can write $P(x, y) = U(x - T(y), y)$, with $U(x, y) = \exp(-|x|)\cos(\pi y)$ and $T(y) = e^y$. Since $\int_0^1 \cos(\pi y)dy = 0$, due to the separability of $U(x, y)$, we have $\int_{\gamma}^{\lambda} \int_{-\infty}^{+\infty} U(x, y)dxdy = 0$. Hence, we have $\int_0^1 \int_{-\infty}^{+\infty} P(x, y)dxdy = 0$.

The twin version of Proposition 6.11 is presented below.

Proposition 6.12 *Let α and β be real numbers such that $\alpha < \beta$ (possibly equal to $-\infty$ or $+\infty$). We consider an integrable two-dimensional function $P(x, y)$ defined on $[\alpha, \beta] \times \mathbb{R}$ under the assumption below.*

Suppose that there exist

- *an uni-dimensional differentiable function $T(x)$ defined on $[\alpha, \beta]$,*
- *a two-dimensional function $U(x, y)$ defined on $[\alpha, \beta] \times \mathbb{R}$ satisfying*

$$\int_{-\infty}^{+\infty} \int_{\alpha}^{\beta} U(x, y)dxdy = 0,$$

such that, for any $(x, y) \in [\alpha, \beta] \times \mathbb{R}$, *we have*

$$P(x, y) = U(x, y - T(x)).$$

Then we have $\int_{-\infty}^{+\infty} \int_{\alpha}^{\beta} P(x, y)dxdy = 0$.

The proof is similar to the one in Proposition 6.11, so it is omitted.

A proposition of example is presented below.

Example: Let us consider $P(x, y) = x\exp(-x^2 - (y - x)^4)$ for $(x, y) \in \mathbb{R}^2$. Thus, in relation to the notations of Proposition 6.12, we have $\alpha = -\infty$, $\beta = +\infty$, and we can write $P(x, y) = U(x, y - T(x))$, with $U(x, y) = x\exp(-x^2)\exp(-y^4)$ and $T(x) = x$. Since $\int_{-\infty}^{+\infty} x\exp(-x^2)dx = 0$, due to the separability of $U(x, y)$, we have $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U(x, y)dxdy = 0$. Hence, we have $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(x, y)dxdy = 0$.

A general remark about all the results presented in the above sections is formulated below.

Remark 6.13 *All the conditions presented in the previous propositions can be combined under a mixture form in the following sense: Let m be a positive integer and $P_1(x, y), \ldots, P_m(x, y)$ be two-dimensional functions satisfying the conditions in Propositions 2.1, 2.2, 2.3, 2.4, 3.5, 3.6, 4.7, 4.8, 5.9, 5.10, 6.11, or 6.12, and $a_1, \ldots, a_m$ be m real numbers. Then the function $P_{mix}(x, y) = \sum_{i=1}^{m} a_i P_i(x, y)$ satisfies $\int_{\gamma}^{\lambda} \int_{\alpha}^{\beta} P_{mix}(x, y)dxdy = 0$. However, in practice, it can be complicated to identify the main functional components of $P_{mix}(x, y)$.*

The remainder of the article focuses on how the findings can be applied to the two-dimensional Gaussian distribution, a crucial probability tool used in many different fields.

Application: Perturbed two-dimensional Gaussian distributions

The creation of some modified Gaussian distributions is presented in this section. They are based on the trigonometric conditions described in Section .

A general result

A general result on perturbed two-dimensional Gaussian distributions is presented below.

Proposition 7.14 *Let*

- $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 > 0$ *and* $\sigma_2 > 0$,
- $\phi_1(x)$ *be the probability density function associated with a Gaussian distribution with parameters* μ_1 *and* σ_1*, i.e.,*

$$\phi_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[-\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right], \quad x \in \mathbb{R},$$

- $\Phi_1(x)$ *be the cumulative distribution function associated with* $\phi_1(x)$*, i.e.,* $\Phi_1(x) = \int_{-\infty}^{x} \phi_1(t)dt$,
- $\phi_2(y)$ *be the probability density function associated with a Gaussian distribution with parameters* μ_2 *and* σ_2*, i.e.,*

$$\phi_2(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[-\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right], \quad y \in \mathbb{R},$$

- $\Phi_2(y)$ *be the cumulative distribution function associated with* $\phi_2(y)$*, i.e.,* $\Phi_2(y) = \int_{-\infty}^{y} \phi_2(t)dt$.

Let us set

$$f(x,y) = \phi_1(x)\phi_2(y) + \zeta P(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{2}$$

where

- $P(x,y)$ *is a two-dimensional function defined on* $\mathbb{R}^2$ *such that* $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,y)dxdy = 0$ *(the support of* $P(x,y)$ *can be included into* $\mathbb{R}^2$*),*
- ξ *is a real number satisfying*

$$|\zeta| \le \left[\sup_{(x,y)\in\mathbb{R}^2}\left|\frac{P(x,y)}{\phi_1(x)\phi_2(y)}\right|\right]^{-1}. \tag{3}$$

Then $f(x,y)$ *is a valid probability density function on* $\mathbb{R}^2$*.*

Proof. To prove that $f(x,y)$ is a valid probability density function on $\mathbb{R}^2$, we must demonstrate that, under the mentioned conditions, for any $(x,y) \in \mathbb{R}^2$, we have $f(x,y) \ge 0$, and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$.

Proof of $f(x,y) \ge 0$: It is clear that, for any $(x,y) \in \mathbb{R}^2$, we have $\phi_1(x)\phi_2(y) \ge 0$. Moreover, thanks to Equation (3), with the use of the absolute value, for any $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned} f(x,y) &= \phi_1(x)\phi_2(y) + \zeta P(x,y) \ge \phi_1(x)\phi_2(y) - |\zeta||P(x,y)| \\ &= \phi_1(x)\phi_2(y)\left[1 - |\zeta|\frac{|P(x,y)|}{\phi_1(x)\phi_2(y)}\right] \\ &\ge \phi_1(x)\phi_2(y)\left[1 - |\zeta|\sup_{(x,y)\in\mathbb{R}^2}\frac{|P(x,y)|}{\phi_1(x)\phi_2(y)}\right] \ge 0. \end{aligned}$$

Proof of $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$: Since $\int_{-\infty}^{+\infty} \phi_1(x)dx = 1$, $\int_{-\infty}^{+\infty} \phi_2(y)dy = 1$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,y)dxdy = 0$, by the additive and multiplicative laws, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \phi_1(x)\phi_2(y)dxdy + \zeta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,y)dxdy \\ &= \left(\int_{-\infty}^{+\infty} \phi_1(x)dx\right)\left(\int_{-\infty}^{+\infty} \phi_2(y)dy\right) + \zeta \times 0 = 1.\end{aligned}$$

The desired results are obtained. □

The distribution associated with the probability density function $f(x,y)$ as defined in Equation (2) is called a perturbed Gaussian distribution, "perturbed" in the sense that $P(x,y)$ has an effect on the independent Gaussian probability density function defined by $g(x,y) = \phi_1(x)\phi_2(y)$, and ζ modulates this effect; if $\zeta \neq 0$ or $P(x,y) \neq 0$, the independence is broken. This functional perturbation of the independence principle is derived from the idea of the Farlie-Gumbel-Morgenstern copula (see [19], [20] and [21]).

In this setting and based on the trigonometric conditions presented in Propositions 5.9 and 5.10, we propose two trigonometric perturbed Gaussian distributions.

Sine perturbed Gaussian distribution

In the setting of Proposition 7.14, for any integers m and n, and any real numbers ϵ, let us set

$$P(x,y) = \phi_1(x)\phi_2(y)\sin\left\{2\pi\left[m\Phi_1(x) + n\Phi_2(y)\right] + \epsilon\right\}, \quad (x,y) \in \mathbb{R}^2.$$

Then, by Proposition 5.9 (or Proposition 5.10), we have $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,y)dxdy = 0$, and

$$\left|\frac{P(x,y)}{\phi_1(x)\phi_2(y)}\right| = \left|\sin\left\{2\pi\left[m\Phi_1(x) + n\Phi_2(y)\right] + \epsilon\right\}\right| \leq 1.$$

Hence, we define the sine perturbed Gaussian (SPG) distribution by the following probability density function:

$$f(x,y) = \phi_1(x)\phi_2(y)\left[1 + \zeta\sin\left\{2\pi\left[m\Phi_1(x) + n\Phi_2(y)\right] + \epsilon\right\}\right], \quad (x,y) \in \mathbb{R}^2,$$

where $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\epsilon \in \mathbb{R}$, m and n are two integers (possibly non-positive), and ζ satisfies $|\zeta| \leq 1$.

Thus, the parameter ζ modulates the sine term, and ϵ plays the role of an additional angle parameter. To the best of our knowledge, the SPG distribution has not been studied in the literature. It can be viewed as a trigonometric modification of the two-dimensional independent Gaussian distribution, which has received rare attention. In particular, the sine term introduces oscillations into the distribution, which leads to interesting and non-trivial behavior. Especially, thanks to its action,

- multiple peaks can be produced in the shapes of the probability density function; the SPG distribution can be multi-modal, unlike the two-dimensional Gaussian distribution, which is ideal for dealing with clustering phenomena,
- wider tails than the two-dimensional independent distribution can be produced, which is ideal to capture extreme values.

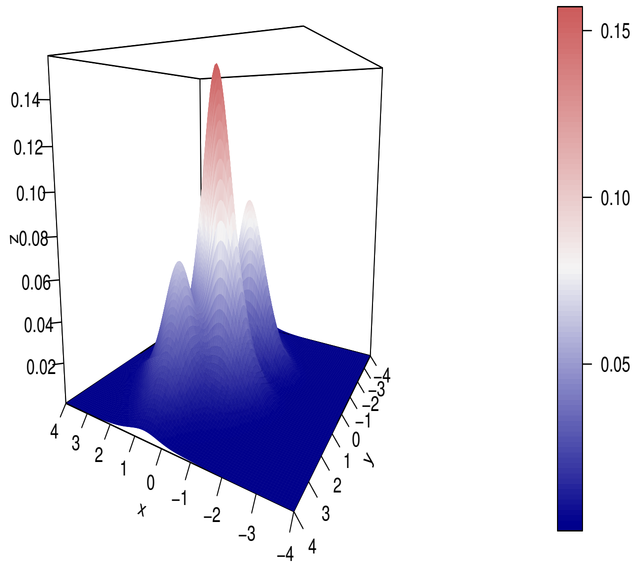
For an illustration of the SPG distribution, Figures 1, 2 and 3 display the standard and intensity plots of the associated probability density function under the following parameter configurations:

- $\mu_1 = 1/2$, $\mu_2 = 0$, $\sigma_1 = 1$, $\sigma_2 = 3/2$, $\epsilon = 1$, $m = 1$, $n = 2$, and $\zeta = -1/2$,

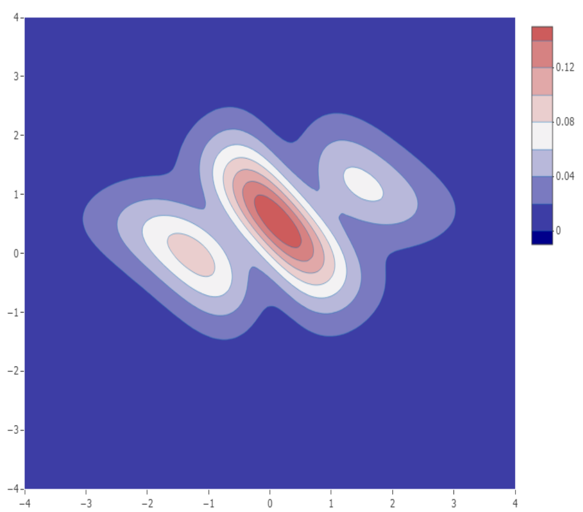
- $\mu_1 = 1/2$, $\mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 1/2$, $\epsilon = -3/2$, $m = -1$, $n = 2$, and $\zeta = 1$,

- $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1 = 1/2$, $\sigma_2 = 2$, $\epsilon = -2$, $m = 5$, $n = 3$, and $\zeta = 1/2$,

respectively. These configurations are chosen to produce different visual results. The plots are made with the use of the free software R, and the libraries `plot3D` and `plotly` in particular (see [22]).

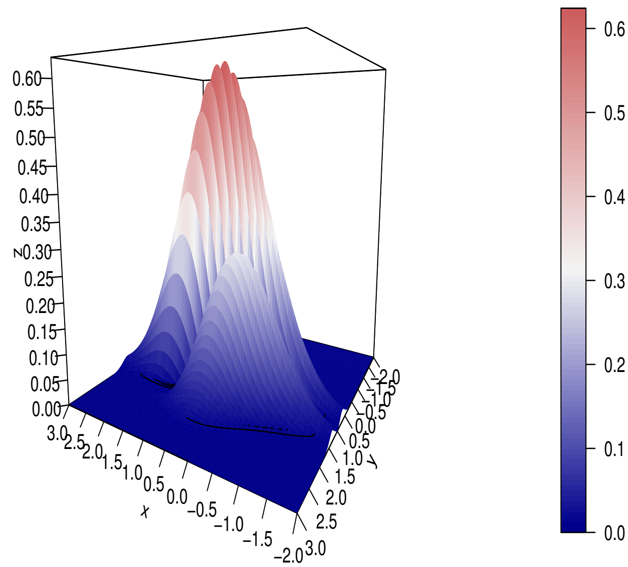


(a)

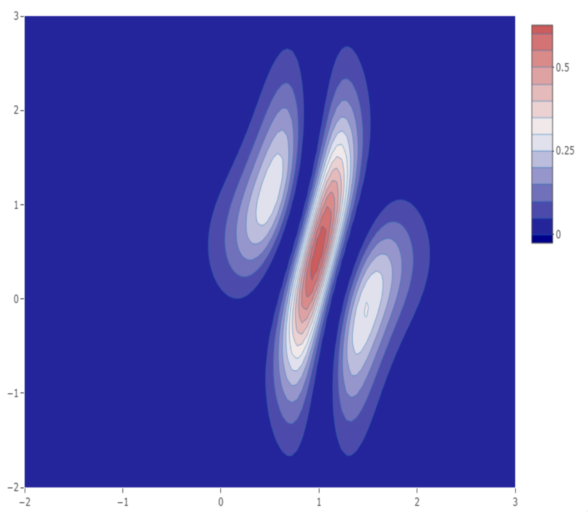


(b)

Figure 1: Plots of the probability density function of the SPG distribution with $\mu_1 = 1/2$, $\mu_2 = 0$, $\sigma_1 = 1$, $\sigma_2 = 3/2$, $\epsilon = 1$, $m = 1$, $n = 2$, and $\zeta = -1/2$: (a) standard plot and (b) intensity plot

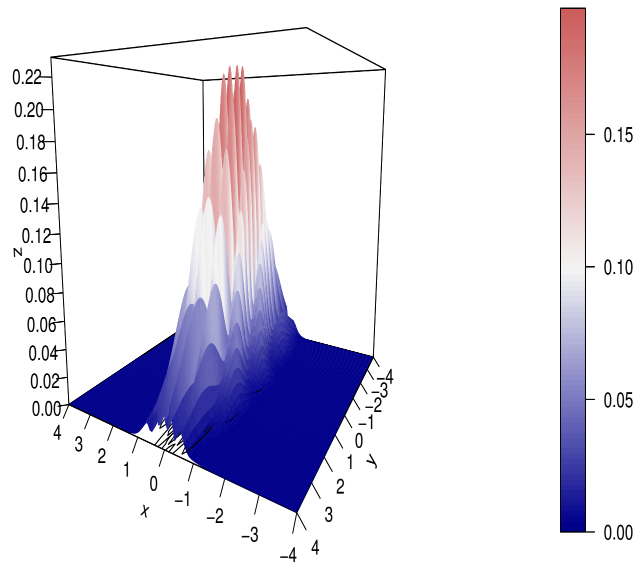


(a)

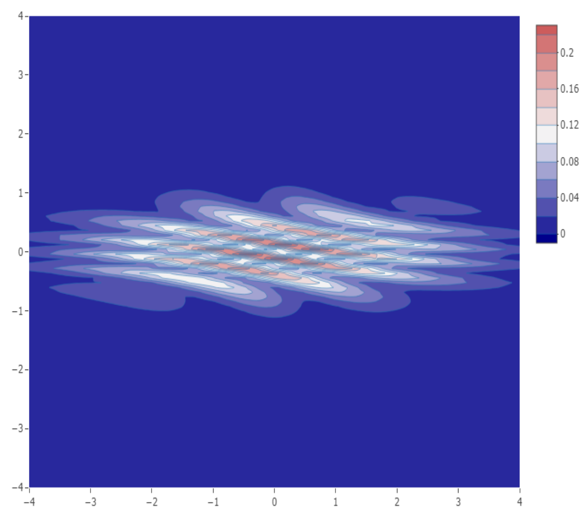


(b)

Figure 2: Plots of the probability density function of the SPG distribution with $\mu_1 = 1/2$, $\mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 1/2$, $\epsilon = -3/2$, $m = -1$, $n = 2$, and $\zeta = 1$: (a) standard plot and (b) intensity plot



(a)



(b)

Figure 3: Plots of the probability density function of the SPG distribution with $\mu_1 = 0$, $\mu_2 = 0$, $\sigma_1 = 1/2$, $\sigma_2 = 2$, $\epsilon = -2$, $m = 5$, $n = 3$, and $\zeta = 1/2$: (a) standard plot and (b) intensity plot

From these figures, we observe the high level of functional versatility of the SPG distribution, with multiple modes, skew shapes, and various weights in the tails. These properties are ideal for modeling a wide variety of phenomena in all fields of applied science.

Cosine perturbed Gaussian distribution

Similarly, in the setting of Proposition 7.14, for any integers m and n, and any real numbers ϵ, let us set

$$P(x, y) = \phi_1(x)\phi_2(y) \cos\{2\pi [m\Phi_1(x) + n\Phi_2(y)] + \epsilon\}, \quad (x, y) \in \mathbb{R}^2.$$

Then, by Proposition 5.9 (or Proposition 5.10), we have $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x, y)dxdy = 0$, and

$$\left|\frac{P(x, y)}{\phi_1(x)\phi_2(y)}\right| = |\cos\{2\pi [m\Phi_1(x) + n\Phi_2(y)] + \epsilon\}| \leq 1.$$

Hence, we define the cosine perturbed Gaussian (CPG) distribution by the following probability density function:

$$f(x, y) = \phi_1(x)\phi_2(y) [1 + \zeta \cos\{2\pi [m\Phi_1(x) + n\Phi_2(y)] + \epsilon\}], \quad (x, y) \in \mathbb{R}^2,$$

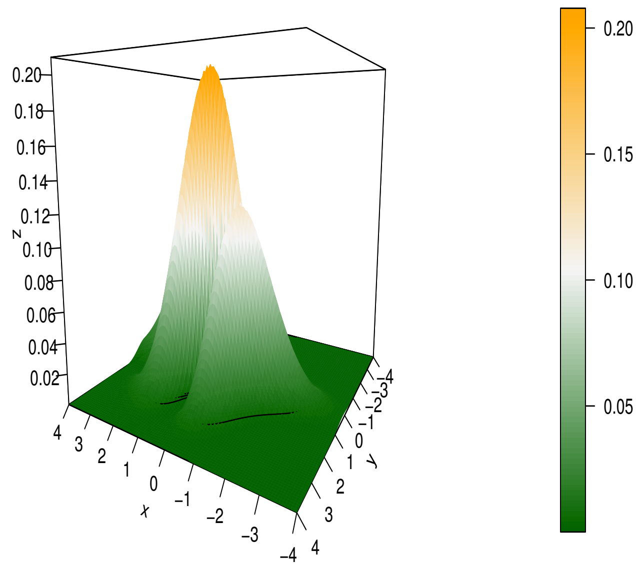
where $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\epsilon \in \mathbb{R}$, m and n are two integers (possibly non-positive), and ζ satisfies $|\zeta| \leq 1$.

As for the SPG distribution, the parameter ζ modulates the cosine term, and ϵ play the role of an angle parameter. To the best of our knowledge, the CPG distribution has not been studied in the literature.

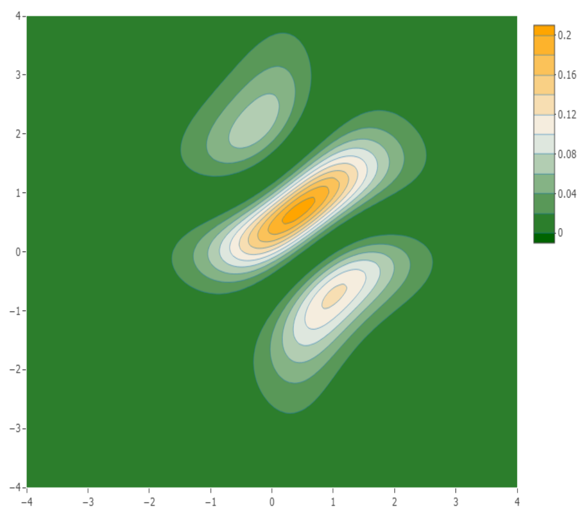
For an illustration of the CPG distribution, Figures 4, 5 and 6 display the standard and intensity plots of the associated probability density function under the following parameter configurations:

- $\mu_1 = 1/2$, $\mu_2 = 1/2$, $\sigma_1 = 3/2$, $\sigma_2 = 1$, $\epsilon = -4$, $m = 2$, $n = -1$, and $\zeta = 1$,
- $\mu_1 = 1/3$, $\mu_2 = 1/3$, $\sigma_1 = 3$, $\sigma_2 = 1$, $\epsilon = \pi$, $m = -2$, $n = 2$, and $\zeta = 3/4$,
- $\mu_1 = 1$, $\mu_2 = 1$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\epsilon = \pi/4$, $m = 3$, $n = -2$, and $\zeta = -1$,

respectively.

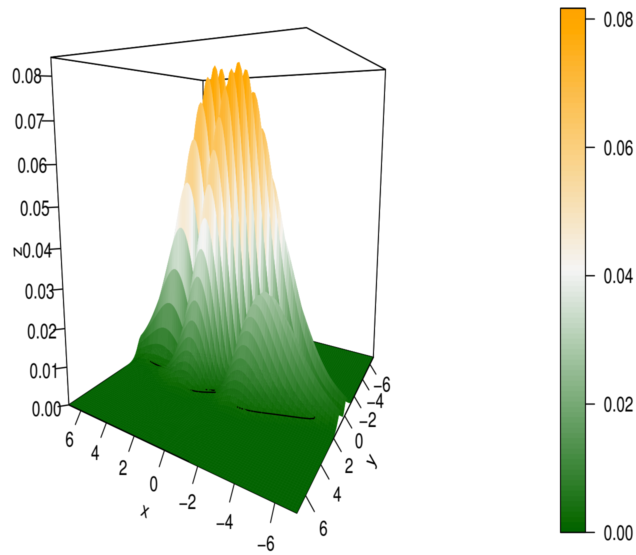


(a)

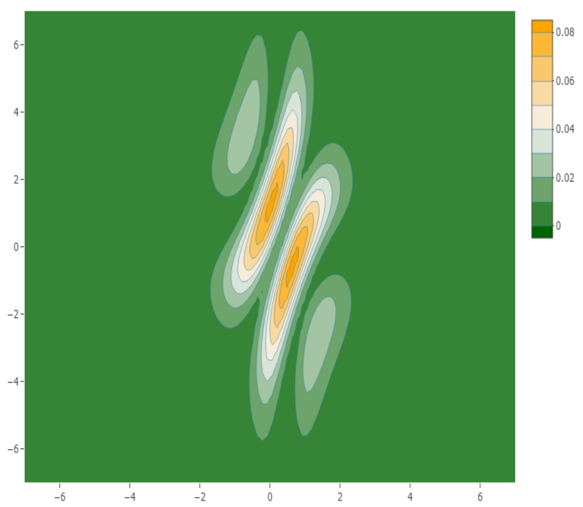


(b)

Figure 4: Plots of the probability density function of the CPG distribution with $\mu_1 = 1/2$, $\mu_2 = 1/2$, $\sigma_1 = 3/2$, $\sigma_2 = 1$, $\epsilon = -4$, $m = 2$, $n = -1$, and $\zeta = 1$: (a) standard plot and (b) intensity plot

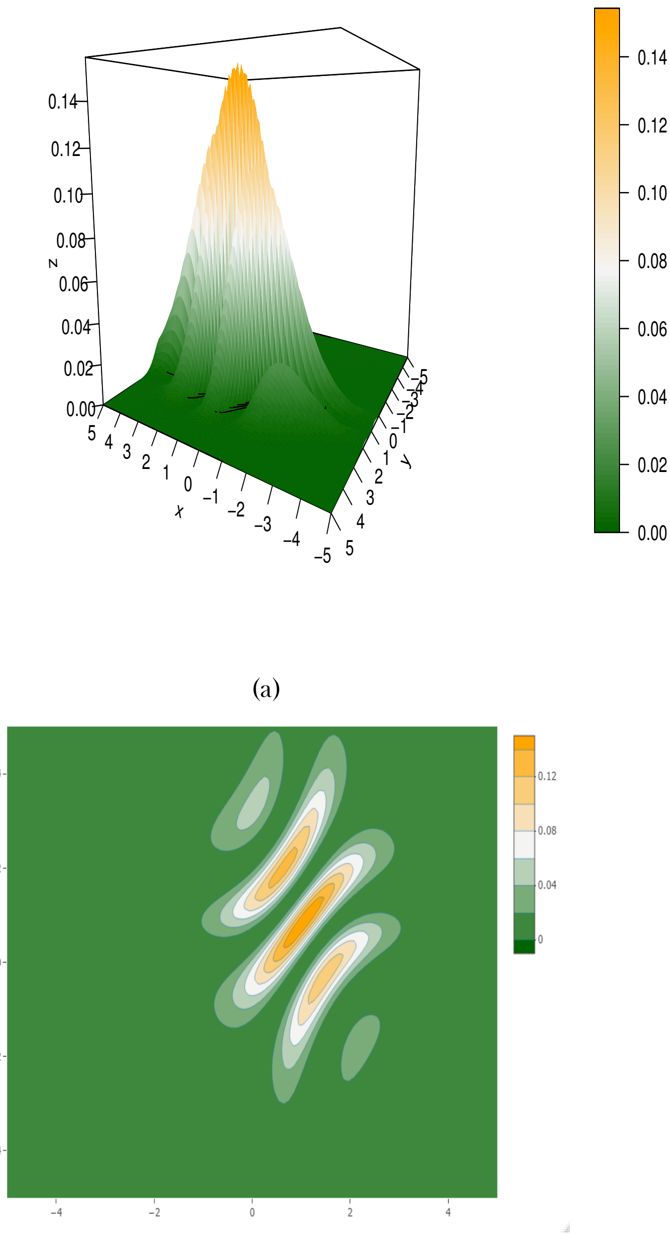


(a)



(b)

Figure 5: Plots of the probability density function of the CPG distribution with $\mu_1 = 1/3$, $\mu_2 = 1/3$, $\sigma_1 = 3$, $\sigma_2 = 1$, $\epsilon = \pi$, $m = -2$, $n = 2$, and $\zeta = 3/4$: (a) standard plot and (b) intensity plot



(a)

(b)

Figure 6: Plots of the probability density function of the CPG distribution with $\mu_1 = 1$, $\mu_2 = 1$, $\sigma_1 = 2$, $\sigma_2 = 1$, $\epsilon = \pi/4$, $m = 3$, $n = -2$, and $\zeta = -1$: (a) standard plot and (b) intensity plot

In light of the aforementioned figures, the CPG distribution can be viewed as an alternative to the SPG distribution with the same general characteristics. With multiple modes, different skew shapes, and different weights in the tails, the CPG distribution exhibits a high degree of functional versatility. Thus, it is perfect for modeling a wide range of phenomena in all branches of applied sciences.

Conclusion

In this article, we determined the main general conditions for a two-dimensional function to have an integral over a rectangle equal to zero. In particular, nontrivial conditions were highlighted, including antisymmetric conditions and trigonometric conditions. Numerous concrete examples were provided to support the findings. The exhibited conditions are important because they allow us to immediately give the zero values of a two-dimensional integral without extra calculus, whatever the complexity of the involved function. The gain in terms of computation costs can be considerable. In addition, these conditions are important in the development of numerical methods for computing integrals and can be involved in numerous applications in various fields such as physics, engineering, and mathematics. In addition, we produced two new modified two-dimensional Gaussian distributions as an implementation of the findings. These modifications were performed by utilizing adaptable trigonometric perturbation functions, which can result in two-dimensional multi-modality and different tail weights. Numerous graphics illustrated this claim. With possible applications in a wide range of domains, these new perturbed Gaussian distributions provide interesting directions for the modeling of diverse natural phenomena.

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