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On a New Multi-Dimensional Nonlinear Integral Operator: Theory and Examples

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Abstract
Integral operators play a fundamental role in mathematical analysis and have found numerous applications in various scientific disciplines. In this article, we introduce a new multi-dimensional nonlinear integral operator, named the C operator, which offers a novel approach to transforming functions in a specific domain. It has some relationships between the Box-Cox transformation and a special case of the multi-dimensional version of the Urysohn nonlinear integral operator. We present the theoretical foundations of the C operator and examine its main features. This includes diverse scale and nonlinear properties, manageable series expansions and partial derivatives, lower and upper bounds, and convex properties. The presence of a tuning parameter plays an important role in this regard. In addition, we demonstrate deep connections between the C operator and several important mathematical tools, such as the standard and modified exponential integral functions, beta function, gamma function, Dawson function, error function, etc. The C operator of some precise functions is given, some of them constituting new integral results in the literature. Our theoretical results highlight the advantages and potential of this new multi-dimensional nonlinear integral operator.

Keywords: Integral operators; nonlinear operators; multi-dimensional operator; convexity; special functions.

Introduction

Integral operators have long been recognized as powerful tools for analyzing and solving a wide range of mathematical and scientific problems. The classical integral operators include the Fourier transform and the Laplace transform (see [7]). They have been extensively studied and have proven to be invaluable in diverse fields. However, the continuous advancement of science and technology constantly presents new challenges that may require innovative mathematical approaches.

Basically, we distinguish two types of integral operators: linear integral operators and nonlinear integral operators. Recent developments on the uni-dimensional linear integral operator topic include the Laplace-Carson transform introduced in [19], the natural transform established in [12], the Elzaki transform created in [8], the integral polynomial transform introduced in [1], the integral ratio transform proposed in [20], the Hadamard fractional integral operator established in [9], the generalized Riemann-Liouville-Hadamard fractional integral operator created in [11] and the k-fractional integral of the Riemann-Liouville integral operator developed in [15]. Concerning the nonlinear integral operators, we may mention the Musielak nonlinear integral operator as described in [16], the Urysohn nonlinear integral operator (also called the nonlinear Fredholm

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integral operator) as presented in [21] and [4], and the Hammerstein nonlinear integral operator defined in [17]. Overall, we may refer to [3], [10], and [14].

Creating original nonlinear integral operators is of great importance as they offer a powerful mathematical framework to model and analyze complex, nonlinear phenomena in various scientific and engineering disciplines, enabling more accurate and comprehensive solutions beyond the limitations of linear operators. These novel operators hold the potential to enhance our understanding of intricate systems and provide innovative approaches to problem-solving and optimization tasks.

In this article, we contribute to this direction of research in a theoretical manner: we introduce a new nonlinear integral operator, the C operator, which provides a fresh perspective on transforming functions in a specific domain. It is constructed as the integral on the unit hypercube of a special ratio function; the numerator term depends on an exponential transformation of a general function, and the denominator term is equal to this general function. This ratio form follows the functional spirit of the Box-Cox transformation (see [5]). The C operator can also be viewed as a special case of a multi-dimensional version of the Urysohn nonlinear integral operator (see [4]). The interest behind its development arises from (i) its original multi-dimensional definition, (ii) its attractive mathematical properties, including diverse scale and nonlinear properties, controllable series expansions and partial derivatives, lower and upper bounds, and convex qualities based on a single tuning parameter, and (iii) its deep connections with diverse important mathematical tools, including the Laplace transform, standard and modified exponential integral functions, beta function, standard and incomplete gamma function, Dawson function, error function, logarithmic integral function, and modified Bessel function. By incorporating key principles and integral techniques, we develop a comprehensive framework for the C operator. By considering simple functions, various closed-form expressions of this operator are given. These expressions applied to the established general lower and upper bounds can yield new inequalities, including some involving special functions in particular. As with any integral operator, the findings can be used in various scientific disciplines, including physics, engineering, and signal processing.

The rest of the article contains the following sections: Section presents the C operator as well as some of its general properties. The expression of the C operator for a wide panel of functions is given in Section . A conclusion is given in Section .

Presentation and general properties

Presentation

The mathematical definition of the C operator is presented below.

Definition 2.1 *Let n be a non-positive integer, f be a function defined on the unit hypercube $[0,1]^n = \underbrace{[0,1]\times\ldots\times[0,1]}_{n \text{ times}}$ and with values into $\mathbb{R}$, and $a \in \mathbb{R}$. Let $x_{(n)} = (x_1,\ldots,x_n)$, where, for any $i = 1,\ldots,n$, x_i is a variable into $[0,1]$, $\int_{[0,1]^n} = \underbrace{\int_{[0,1]}\ldots\int_{[0,1]}}_{n \text{ times}}$ and $dx_{(n)} = \underbrace{dx_1\ldots dx_n}_{n \text{ times}}$. With these notations, we define the C operator as*

$$C_n(f;a) = \int_{[0,1]^n} \frac{\exp[af(x_{(n)})]-1}{f(x_{(n)})} dx_{(n)},$$

provided that it exists in the integral convergence sense. In the case $f(x_{(n)}) = 0$ for any $x_{(n)} \in [0,1]^n$, we set $C_n(f;a) = a$. Eventually, we can name the C operator, the C functional, or the C transform.

For any given function $f(x_{(n)})$, the existence of the C operator can be checked by employing the classical Riemann integral convergence rules. As a specific comment, it is obvious that the C operator exists if f is continuous and $\min_{x_{(n)}\in[0,1]^n} |f(x_{(n)})| > 0$. In the rest of the article, when the C operator or any other quantities are presented, it is implicitly assumed that they exist in the mathematical sense.

As sketched in the introduction, the ratio form of the integrated term is derived from the Box-Cox transformation, which has numerous applications, mainly in statistics and econometric (see [5]). Here, we use it to offer a fresh perspective on transforming multi-dimensional functions. We can also write the C operator as

$$C_n(f;a) = \int_{\mathcal{D}} K[a, x_{(n)}, f(x_{(n)})]dx_{(n)},$$

with $\mathcal{D} = [0,1]^n$ and $K(a, x_{(n)}, y) = [\exp(ay) - 1]/y$ (which is in fact independent of $x_{(n)}$). This form can be assimilated into a particular multi-dimensional version of the Urysohn nonlinear integral operator. We refer to [4] for more detail on this operator.

Eventually, another possible expression of the C operator involving the exponential and hyperbolic sine functions is

$$C_n(f;a) = 2\int_{[0,1]^n} \frac{\exp[(a/2)f(x_{(n)})]\sinh[(a/2)f(x_{(n)})]}{f(x_{(n)})}dx_{(n)}.$$

As an important fact, the C operator enjoys several attractive properties, which are examined in the next section.

General properties

Some basic properties of the C operator are given in the next proposition.

Proposition 2.2 *The following properties hold:*

1. *We have* $C_n(f;0) = 0$.
2. *For any* $b \in \mathbb{R}/\{0\}$, *we have*

$$C_n(bf;a) = \frac{1}{b}C_n(f;ab).$$

As a result, the C operator is obviously nonlinear.

3. *Let* $(1-x)_{(n)} = (1-x_1,\dots,1-x_n)$ *and* f_* *be the function defined by* $f_*(x_{(n)}) = f[(1-x)_{(n)}]$. *Then we have*

$$C_n(f_*;a) = C_n(f;a).$$

More generally, by considering the function f_{**} *defined by* $f_{**}(x_{(n)}) = f(z_{(n)})$, *where* $z_{(n)} = (z_1,\dots,z_n)$ *and, for any* $i = 1,\dots,n$, $z_i = x_i$ *or* $z_i = 1 - x_i$, *we have*

$$C_n(f_{**};a) = C_n(f;a).$$

4. *Under the multi-dimensional interchange of integral and sum rule assumptions, the following expansion holds:*

$$C_n(f;a) = \sum_{i=0}^{+\infty} \frac{a^{i+1}\xi_i}{(i+1)!},$$

where $\xi_i = \int_{[0,1]^n}[f(x_{(n)})]^i dx_{(n)}$.

5. *Under the multi-dimensional Leibniz integral rule assumptions, we have*

$$\frac{\partial}{\partial a}C_n(f;a) = \int_{[0,1]^n} \exp[af(x_{(n)})]dx_{(n)}.$$

Proof. Let us prove the five points in turn.

1. Since $\exp(0) = 1$, it is clear that

$$C_n(f;0) = \int_{[0,1]^n} \frac{\exp[0 \times f(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = \int_{[0,1]^n} 0 dx_{(n)} = 0.$$

2. We have

$$C_n(bf;a) = \int_{[0,1]^n} \frac{\exp\{a[bf(x_{(n)})]\} - 1}{bf(x_{(n)})} dx_{(n)} = \frac{1}{b} \int_{[0,1]^n} \frac{\exp[abf(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)}$$
$$= \frac{1}{b} C_n(f;ab).$$

3. By applying the multi-dimensional change of variables $y_{(n)} = (1-x)_{(n)}$, with $y_{(n)} \in [0,1]^n$ and the corresponding Jacobian equals to $(-1)^n$, we get

$$C_n(f_*;a) = \int_{[0,1]^n} \frac{\exp[af_*(x_{(n)})] - 1}{f_*(x_{(n)})} dx_{(n)} = |(-1)^n| \int_{[0,1]^n} \frac{\exp[af(y_{(n)})] - 1}{f(y_{(n)})} dy_{(n)}$$
$$= C_n(f;a).$$

For the general case, the proof follows immediately from the change of variables $y_{(n)} = z_{(n)}$, with $y_{(n)} \in [0,1]^n$ and the absolute value of the corresponding Jacobian equals to 1 in all circumstances.

4. Using the exponential series expansion, we obtain

$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} = \frac{1}{f(x_{(n)})} \left\{ \left[\sum_{i=0}^{+\infty} \frac{[af(x_{(n)})]^i}{i!} \right] - 1 \right\} = \sum_{i=1}^{+\infty} \frac{a^i [f(x_{(n)})]^{i-1}}{i!}$$
$$= \sum_{i=0}^{+\infty} \frac{a^{i+1} [f(x_{(n)})]^i}{(i+1)!}.$$

It follows from the supposed multi-dimensional interchange of integral and sum rule assumptions that

$$C_n(f;a) = \int_{[0,1]^n} \sum_{i=0}^{+\infty} \frac{a^{i+1} [f(x_{(n)})]^i}{(i+1)!} dx_{(n)} = \sum_{i=0}^{+\infty} \frac{a^{i+1}}{(i+1)!} \int_{[0,1]^n} [f(x_{(n)})]^i dx_{(n)}$$
$$= \sum_{i=0}^{+\infty} \frac{a^{i+1} \xi_i}{(i+1)!}.$$

5. Using the multi-dimensional Leibniz integral rule assumptions, we obtain

$$\frac{\partial}{\partial a} C_n(f;a) = \frac{\partial}{\partial a} \left\{ \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \right\} = \int_{[0,1]^n} \frac{\partial}{\partial a} \left\{ \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \right\} dx_{(n)}$$
$$= \int_{[0,1]^n} \frac{f(x_{(n)})}{f(x_{(n)})} \exp[af(x_{(n)})] dx_{(n)} = \int_{[0,1]^n} \exp[af(x_{(n)})] dx_{(n)}.$$

This ends the proof. □

Proposition 2.2 shows that the C operator has notable properties and is exploitable from an analytical viewpoint. In particular, the second and third points show that, for any function f such that $C_n(f;a)$ is manageable, we can define a family of functions of the form bf_* with a manageable C operator. Furthermore, based on the fourth point, a series expansion involving simple integral terms is established, opening the door to some approximation aspects. Owing to the first and last points, we can calculate $C_n(f;a)$ as

$$C_n(f;a) = \int_{[0,a]} \frac{\partial}{\partial a} C_n(f;y)dy = \int_{[0,a]} \int_{[0,1]^n} \exp[yf(x_{(n)})]dx_{(n)}dy.$$

Some examples of calculus based on this integral-differentiation scheme will be discussed throughout the article.

The following lemma will be useful in some of our proofs. On the other hand, it can also be viewed as an independent interest.

Lemma 2.3 *For any $a \in \mathbb{R}$, let us set*

$$g(x) = \frac{\exp(ax) - 1}{x}, \quad x \in \mathbb{R}, \tag{1}$$

(with the natural extension for continuity $g(0) = a$). Then, for $a \geq 0$, g is convex, and for $a \leq 0$, g is concave.

Proof. By applying standard differentiation rules, we obtain

$$g''(x) = \frac{\exp(ax)(a^2x^2 - 2ax + 2) - 2}{x^3}.$$

Let us set $h(x) = \exp(ax)(a^2x^2 - 2ax + 2) - 2$, which corresponds to the numerator of $g''(x)$. Then we have $h'(x) = a^3x^2 \exp(ax)$. Let us now distinguish the cases $a \geq 0$ and $a \leq 0$.

- For the case $a \geq 0$, we have $h'(x) \geq 0$ and $h(x)$ is non-decreasing. Hence, for any $x \geq 0$, we have $h(x) \geq h(0) = 0$ and, since $x^3 \geq 0$, we have $g''(x) \geq 0$. On the other hand, for $x \leq 0$, we have $h(x) \leq h(0) = 0$ and, since $x^3 \leq 0$, we also have $g''(x) \geq 0$. As a result, for any $x \in \mathbb{R}$, we have $g''(x) \geq 0$, thus g is convex.

- On the other hand, for the case $a \leq 0$, since $a^3 \leq 0$ and $\exp(ax) \geq 0$, let us remark that $h'(x) = a^3x^2 \exp(ax) \leq 0$, so $h(x)$ is non-increasing. Hence, for any $x \geq 0$, we have $h(x) \leq h(0) = 0$ and, since $x^3 \geq 0$, we have $g''(x) \leq 0$. On the other hand, for $x \leq 0$, we have $h(x) \geq h(0) = 0$ and, since $x^3 \leq 0$, we also have $g''(x) \leq 0$. As a result, for any $x \in \mathbb{R}$, we have $g''(x) \leq 0$, thus g is concave.

The proof is now complete. □

Lemma 2.3 is the key result to study the convex property of the C operator, as formulated in the next result.

Proposition 2.4 *For any $a \geq 0$, the C operator is convex, i.e., for any functions h and k defined on $[0,1]^n$, and $\lambda \in [0,1]$, we have*

$$C_n[\lambda h + (1-\lambda)k; a] \leq \lambda C_n(h;a) + (1-\lambda)C_n(k;a).$$

On the other hand, for any $a \leq 0$, the C operator is concave, i.e., for any functions h and k defined on $[0,1]^n$, and $\lambda \in [0,1]$, we have

$$\lambda C_n(h;a) + (1-\lambda)C_n(k;a) \leq C_n[\lambda h + (1-\lambda)k; a].$$

Proof. We can write the C operator as

$$C_n(f;a) = \int_{[0,1]^n} g[f(x_{(n)})]dx_{(n)},$$

where g is the function defined in Equation (1). Let us now distinguish the cases $a \geq 0$ and $a \leq 0$.

- For the case $a \geq 0$, it is established in Lemma 2.3 that g is convex. Therefore, for $a \geq 0$, by using the standard integral properties, we have

$$\begin{aligned}
C_n[\lambda h + (1-\lambda)k; a] &= \int_{[0,1]^n} g[\lambda h(x_{(n)}) + (1-\lambda)k(x_{(n)})]dx_{(n)} \\
&\leq \int_{[0,1]^n} \left\{\lambda g[h(x_{(n)})] + (1-\lambda)g[k(x_{(n)})]\right\} dx_{(n)} \\
&= \lambda \int_{[0,1]^n} g[h(x_{(n)})]dx_{(n)} + (1-\lambda) \int_{[0,1]^n} g[k(x_{(n)})]dx_{(n)} \\
&= \lambda C_n(h; a) + (1-\lambda)C_n(k; a).
\end{aligned}$$

- On the other hand, for the case $a \leq 0$, it is established in Lemma 2.3 that g is concave. Hence, with similar developments as above, we have

$$\begin{aligned}
C_n[\lambda h + (1-\lambda)k; a] &= \int_{[0,1]^n} g[\lambda h(x_{(n)}) + (1-\lambda)k(x_{(n)})]dx_{(n)} \\
&\geq \int_{[0,1]^n} \left\{\lambda g[h(x_{(n)})] + (1-\lambda)g[k(x_{(n)})]\right\} dx_{(n)} \\
&= \lambda C_n(h; a) + (1-\lambda)C_n(k; a).
\end{aligned}$$

This ends the proof. □

Thus, for $a \geq 0$, the C operator belongs to the family of convex operators. Such operators are of great interest due to their ability to preserve the convexity of sets, enabling efficient optimization and guaranteeing convergence to global optima. Their mathematical properties offer powerful tools for a wide range of applications in fields such as machine learning, signal processing, and operations research. The concave property of the C operator for $a \leq 0$ can also be used for various mathematical purposes.

Bounds

The C operator can be bounded in diverse ways. In this part, some of them are examined.

We begin with an ordering property of the C operator with respect to the parameter a.

Proposition 2.5 *For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $a \leq b$, we have*

$$C_n(f; a) \leq C_n(f; b).$$

Proof. The exponential function is increasing. With ths in mind, let us prove the result by distinguishing the cases $f(x_{(n)}) \geq 0$ and $f(x_{(n)}) \leq 0$, where $x_{(n)} \in [0, 1]^n$ is fixed.

- For the case $f(x_{(n)}) \geq 0$, the inequality $a \leq b$ implies that $af(x_{(n)}) \leq bf(x_{(n)})$, so $\exp[af(x_{(n)})] - 1 \leq \exp[bf(x_{(n)})] - 1$, and we obtain

$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \leq \frac{\exp[bf(x_{(n)})] - 1}{f(x_{(n)})}.$$

- On the other hand, for the case $f(x_{(n)}) \leq 0$, the inequality $a \leq b$ implies that $bf(x_{(n)}) \leq af(x_{(n)})$, so $\exp[bf(x_{(n)})] - 1 \leq \exp[af(x_{(n)})] - 1$, and we still obtain

$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \leq \frac{\exp[bf(x_{(n)})] - 1}{f(x_{(n)})}.$$

For the two cases, upon integration for $x_{(n)} \in [0, 1]^n$, we get

$$C_n(f; a) = \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \leq \int_{[0,1]^n} \frac{\exp[bf(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = C_n(f; b).$$

The stated ordering property is proved.

Remark: An alternative proof involving some additional assumptions is as follows: using the multi-dimensional Leibniz integral rule assumptions, the last point of Proposition 2.2 and $\exp[af(x_{(n)})] \geq 0$, we obtain

$$\frac{\partial}{\partial a} C_n(f; a) = \int_{[0,1]^n} \exp[af(x_{(n)})] dx_{(n)} \geq 0.$$

Hence, $C_n(f; a)$ is non-decreasing with respect to a, implying that, for $a \leq b$, we have $C_n(f; a) \leq C_n(f; b)$. □

The comparison of $C_n(f; a)$ and $C_n(-f; a)$ is made in the next result.

Proposition 2.6 *For any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \geq 0$, we have*

$$C_n(f; a) \geq C_n(-f; a).$$

On the other hand, for any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \leq 0$, we have

$$C_n(f; a) \leq C_n(-f; a).$$

We recall that the following relation holds: $C_n(-f; a) = -C_n(f; -a)$ (by the second point of Proposition 2.2 with $b = -1$).

Proof. First of all, let us notice that

$$C_n(f; a) = \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = \int_{[0,1]^n} \{-\exp[af(x_{(n)})]\} \frac{\exp[-af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)}. \quad (2)$$

- First, let us consider the case $f(x_{(n)}) \geq 0$ for any $x_{(n)} \in [0, 1]^n$, and let us distinguish the cases $a \leq 0$ and $a \geq 0$.
  - For the case $a \leq 0$, we have $af(x_{(n)}) \leq 0$, implying that $-\exp[af(x_{(n)})] \geq -1$ and $\exp[-af(x_{(n)})] - 1 \geq 0$, so $\{\exp[-af(x_{(n)})] - 1\}/f(x_{(n)}) \geq 0$. Therefore, based on Equation (2), we have

    $$C_n(f; a) \geq -\int_{[0,1]^n} \frac{\exp[-af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = C_n(-f; a).$$

  - On the other hand, for the case $a \geq 0$, we have $af(x_{(n)}) \geq 0$, implying that $-\exp[af(x_{(n)})] \leq -1$ and $\exp[-af(x_{(n)})] - 1 \leq 0$, so $\{\exp[-af(x_{(n)})] - 1\}/f(x_{(n)}) \leq 0$. Therefore, we have

    $$C_n(f; a) \geq -\int_{[0,1]^n} \frac{\exp[-af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = C_n(-f; a).$$

- Now, let us consider the case $f(x_{(n)}) \leq 0$ for any $x_{(n)} \in [0, 1]^n$, and let us distinguish the cases $a \leq 0$ and $a \geq 0$.
  - For the case $a \leq 0$, we have $af(x_{(n)}) \geq 0$, implying that $-\exp[af(x_{(n)})] \leq -1$ and $\exp[-af(x_{(n)})] - 1 \leq 0$, so $\{\exp[-af(x_{(n)})] - 1\}/f(x_{(n)}) \geq 0$. Therefore, we have

    $$C_n(f; a) \leq -\int_{[0,1]^n} \frac{\exp[-af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = C_n(-f; a).$$

– On the other hand, for the case $a \geq 0$, we have $af(x_{(n)}) \leq 0$, implying that $-\exp[af(x_{(n)})] \geq -1$ and $\exp[-af(x_{(n)})] - 1 \geq 0$, so $\left\{\exp[-af(x_{(n)})] - 1\right\}/f(x_{(n)}) \leq 0$. Therefore, we have

$$C_n(f; a) \leq -\int_{[0,1]^n} \frac{\exp[-af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} = C_n(-f; a).$$

The results are proved. □

The results in Proposition 2.6 can give original inequalities, with use beyond the scope of this article. The precise examples emphasized in Section can be useful in this regard.

Below, we determine simple bounds for $C_n(f; a)$ for non-negative or non-positive functions f.

Proposition 2.7 *For any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \geq 0$, we have*

$$C_n(f; a) \geq a.$$

On the other hand, for any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \leq 0$, we have

$$C_n(f; a) \leq a.$$

Proof. The proof is based on the following well-known exponential inequality: for any $x \in \mathbb{R}$, we have $\exp(x) \geq 1 + x$. Therefore, for any $a \in \mathbb{R}$, we have $\exp[af(x_{(n)})] \geq 1 + af(x_{(n)})$. Let us now distinguish the cases $f(x_{(n)}) \geq 0$ for any $x_{(n)} \in [0, 1]^n$ and $f(x_{(n)}) \leq 0$ for any $x_{(n)} \in [0, 1]^n$.

- For the case $f(x_{(n)}) \geq 0$ for any $x_{(n)} \in [0, 1]^n$, upon dividing with $f(x_{(n)})$, we get

$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \geq a.$$

  Upon integration for $x_{(n)} \in [0, 1]^n$, we obtain

$$C_n(f; a) = \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \geq a \int_{[0,1]^n} dx_{(n)} = a.$$

- On the other hand, for the case $f(x_{(n)}) \leq 0$ for any $x_{(n)} \in [0, 1]^n$, upon dividing with $f(x_{(n)})$, we get

$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \leq a.$$

  Upon integration for $x_{(n)} \in [0, 1]^n$, we obtain

$$C_n(f; a) = \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \leq a \int_{[0,1]^n} dx_{(n)} = a.$$

The proof is completed. □

The next proposition is devoted to some bound involving the function g in Equation (1) and the integral $\int_{[0,1]^n} f(x_{(n)}) dx_{(n)}$ beyond the non-negative or non-positive assumptions on f.

Proposition 2.8 *For any $a \geq 0$, we have*

$$C_n(f; a) \geq \frac{\exp(a\xi) - 1}{\xi},$$

where $\xi = \int_{[0,1]^n} f(x_{(n)}) dx_{(n)}$, provided that it exists in the integral convergence sense.

On the other hand, for any $a \leq 0$, we have

$$C_n(f; a) \leq \frac{\exp(a\xi) - 1}{\xi}.$$

Proof.

- Owing to Lemma 2.3, for $a \geq 0$, the function g in Equation (1) is convex. It follows from the Jensen inequality that $\int_{[0,1]^n} g[f(x_{(n)})]dx_{(n)} \geq g(\xi)$, i.e.,
$$C_n(f; a) \geq \frac{\exp(a\xi) - 1}{\xi}.$$
- On the other hand, still applying Lemma 2.3, for $a \leq 0$, g is concave. It follows from the Jensen inequality that $g(\xi) \geq \int_{[0,1]^n} g[f(x_{(n)})]dx_{(n)}$, i.e.,
$$C_n(f; a) \leq \frac{\exp(a\xi) - 1}{\xi}.$$

The desired results are obtained. □

Owing to the following exponential inequality: for any $x \in \mathbb{R}$, we have $\exp(x) \geq 1 + x$, one can prove that the bounds obtained in Proposition 2.8 are sharper than those obtained in Proposition 2.7, but under the non-negative or non-positive assumptions on f only.

The result below can be viewed as an alternative result to Proposition 2.8; polynomial bounds depending on the integral $\int_{[0,1]^n} f(x_{(n)})dx_{(n)}$ are established under some assumptions on a and f.

Proposition 2.9 *For any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \geq 0$ and $af(x_{(n)}) < 1.79$, we have*
$$C_n(f; a) \leq a + a^2\xi.$$
On the other hand, for any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $f(x_{(n)}) \leq 0$ and $af(x_{(n)}) < 1.79$, we have
$$C_n(f; a) \geq a + a^2\xi.$$

Proof. The proof is based on the following well-known exponential inequality: for any $x < 1.79$, we have $\exp(x) \leq 1 + x + x^2$. Therefore, for any $a \in \mathbb{R}$ and f such that, for any $x_{(n)} \in [0, 1]^n$, $af(x_{(n)}) < 1.79$, we have $\exp[af(x_{(n)})] \leq 1 + af(x_{(n)}) + [af(x_{(n)})]^2$. Let us now distinguish the cases $f(x_{(n)}) \geq 0$ for any $x_{(n)} \in [0, 1]^n$ and $f(x_{(n)}) \leq 0$ for any $x_{(n)} \in [0, 1]^n$.

- For the case $f(x_{(n)}) \geq 0$ for any $x_{(n)} \in [0, 1]^n$, upon dividing with $f(x_{(n)})$, we get
$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \leq a + a^2 f(x_{(n)}).$$
Upon integration for $x_{(n)} \in [0, 1]^n$, we obtain
$$\begin{aligned} C_n(f; a) &= \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \leq \int_{[0,1]^n} \left[a + a^2 f(x_{(n)})\right] dx_{(n)} \\ &= a + a^2 \int_{[0,1]^n} f(x_{(n)})dx_{(n)} = a + a^2\xi. \end{aligned}$$
- On the other hand, for the case $f(x_{(n)}) \leq 0$ for any $x_{(n)} \in [0, 1]^n$, upon dividing with $f(x_{(n)})$, we get
$$\frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} \geq a + a^2 f(x_{(n)}).$$
Upon integration for $x_{(n)} \in [0, 1]^n$, we get
$$\begin{aligned} C_n(f; a) &= \int_{[0,1]^n} \frac{\exp[af(x_{(n)})] - 1}{f(x_{(n)})} dx_{(n)} \geq \int_{[0,1]^n} \left[a + a^2 f(x_{(n)})\right] dx_{(n)} \\ &= a + a^2 \int_{[0,1]^n} f(x_{(n)})dx_{(n)} = a + a^2\xi. \end{aligned}$$

This ends the proof. □

Thus, the interest in Proposition 2.9 is to complete Propositions 2.7 and 2.8 by providing alternative or complementary bounds under some assumptions. In particular, if we focus on Propositions 2.8 and 2.9, for any $a \geq 0$ and f such that, for any $x_{(n)} \in [0,1]^n$, $f(x_{(n)}) \geq 0$ and $af(x_{(n)}) < 1.79$, we have

$$\frac{\exp(a\xi) - 1}{\xi} \leq C_n(f; a) \leq a + a^2\xi.$$

In addition, for any $a \leq 0$ and f such that, for any $x_{(n)} \in [0,1]^n$, $f(x_{(n)}) \leq 0$ and $af(x_{(n)}) < 1.79$, we have

$$a + a^2\xi \leq C_n(f; a) \leq \frac{\exp(a\xi) - 1}{\xi}.$$

The result below presents a bound based on the product of two functions.

Proposition 2.10 *For any functions h and k defined on $[0,1]^n$, under the multi-dimensional interchange of integral and sum rule assumptions for both of them, the two following inequalities hold:*

1. *For the sum function $h + k$, we have*

$$|C_n(h+k; a)| \leq \frac{1}{4}\left[C_n(|h|; 2|a|) + C_n(|k|; 2|a|)\right].$$

2. *For the product function hk, we have*

$$|C_n(hk; a)| \leq \sqrt{C_n(h^2; |a|)}\sqrt{C_n(k^2; |a|)}.$$

Proof. The proof is based on the fourth point of Proposition 2.2, under the multi-dimensional interchange of integral and sum rule assumptions.

1. We have

$$C_n(h+k; a) = \sum_{i=0}^{+\infty} \frac{a^{i+1}\xi_i}{(i+1)!},$$

where $\xi_i = \int_{[0,1]^n} [h(x_{(n)}) + k(x_{(n)})]^i dx_{(n)}$. By virtue of the inequality $|x+y|^i \leq 2^{i-1}(|x|^i + |y|^i)$ for any $(x, y) \in \mathbb{R}^2$ and non-negative integer i (including $i = 0$), we have

$$|\xi_i| \leq \int_{[0,1]^n} |h(x_{(n)}) + k(x_{(n)})|^i dx_{(n)} \leq 2^{i-1}\left[\int_{[0,1]^n} |h(x_{(n)})|^i dx_{(n)} + \int_{[0,1]^n} |k(x_{(n)})|^i dx_{(n)}\right].$$

Therefore, we have

$$\begin{aligned}
|C_n(h+k; a)| &\leq \sum_{i=0}^{+\infty} \frac{|a|^{i+1}|\xi_i|}{(i+1)!} \\
&\leq \sum_{i=0}^{+\infty} \frac{|a|^{i+1}}{(i+1)!} 2^{i-1}\left[\int_{[0,1]^n} |h(x_{(n)})|^i dx_{(n)} + \int_{[0,1]^n} |k(x_{(n)})|^i dx_{(n)}\right] \\
&= \frac{1}{4}\left[\sum_{i=0}^{+\infty} \frac{(2|a|)^{i+1}}{(i+1)!} \int_{[0,1]^n} |h(x_{(n)})|^i dx_{(n)} + \sum_{i=0}^{+\infty} \frac{(2|a|)^{i+1}}{(i+1)!} \int_{[0,1]^n} |k(x_{(n)})|^i dx_{(n)}\right] \\
&= \frac{1}{4}\left[C_n(|h|; 2|a|) + C_n(|k|; 2|a|)\right].
\end{aligned}$$

2. We have

$$C_n(hk;a) = \sum_{i=0}^{+\infty} \frac{a^{i+1}\xi_i}{(i+1)!},$$

where $\xi_i = \int_{[0,1]^n} [h(x_{(n)})k(x_{(n)})]^i dx_{(n)}$. The Cauchy-Schwarz inequality (integral version) gives

$$|\xi_i| \le \int_{[0,1]^n} |h(x_{(n)})k(x_{(n)})|^i dx_{(n)} \le \sqrt{\int_{[0,1]^n} |h(x_{(n)})|^{2i} dx_{(n)}}\sqrt{\int_{[0,1]^n} |k(x_{(n)})|^{2i} dx_{(n)}}.$$

Therefore, we have

$$\begin{aligned}|C_n(hk;a)| &\le \sum_{i=0}^{+\infty} \frac{|a|^{i+1}|\xi_i|}{(i+1)!} \\ &\le \sum_{i=0}^{+\infty} \frac{|a|^{i+1}}{(i+1)!}\sqrt{\int_{[0,1]^n} |h(x_{(n)})|^{2i} dx_{(n)}}\sqrt{\int_{[0,1]^n} |k(x_{(n)})|^{2i} dx_{(n)}} \\ &= \sum_{i=0}^{+\infty} \sqrt{\frac{|a|^{i+1}}{(i+1)!}\int_{[0,1]^n} |h(x_{(n)})|^{2i} dx_{(n)}}\sqrt{\frac{|a|^{i+1}}{(i+1)!}\int_{[0,1]^n} |k(x_{(n)})|^{2i} dx_{(n)}}.\end{aligned}$$

Owing to the Cauchy-Schwarz inequality (sum version), we obtain

$$\begin{aligned}|C_n(hk;a)| &\le \sqrt{\sum_{i=0}^{+\infty} \frac{|a|^{i+1}}{(i+1)!}\int_{[0,1]^n} |h(x_{(n)})|^{2i} dx_{(n)}}\sqrt{\sum_{i=0}^{+\infty} \frac{|a|^{i+1}}{(i+1)!}\int_{[0,1]^n} |k(x_{(n)})|^{2i} dx_{(n)}} \\ &= \sqrt{C_n(h^2;|a|)}\sqrt{C_n(k^2;|a|)}.\end{aligned}$$

This ends the proof. □

The bounds obtained reveal the limits of the potential of the C operator and unlock deeper information. The proposition below is about a Lipschitz-type inequality satisfied by the C operator.

Proposition 2.11 *For any functions h and k defined on $[0, 1]^n$ such that there exist two constants m and M satisfying, for any $x_{(n)} \in [0, 1]^n$, $m \le h(x_{(n)}) \le M$ and $m \le k(x_{(n)}) \le M$, we have*

$$|C_n(h;a) - C_n(k;a)| \le Q \int_{[0,1]^n} |h(x_{(n)}) - k(x_{(n)})| dx_{(n)},$$

where

$$Q = \begin{cases} \dfrac{\exp(aM)(aM-1)+1}{M^2} & \text{if } a > 0 \\ \dfrac{\exp(am)(am-1)+1}{m^2} & \text{if } a < 0 \end{cases}.$$

Proof. By considering the function g as described in Equation (1), we have

$$\begin{aligned}|C_n(h;a) - C_n(k;a)| &= \left|\int_{[0,1]^n} \left\{g[h(x_{(n)})] - g[k(x_{(n)})]\right\} dx_{(n)}\right| \\ &\le \int_{[0,1]^n} \left|g[h(x_{(n)})] - g[k(x_{(n)})]\right| dx_{(n)}.\end{aligned}$$

Let us now achieve the proof by demonstrating that, for any $x_{(n)} \in [0, 1]^n$, the following inequality holds: $\left|g[h(x_{(n)})] - g[k(x_{(n)})]\right| \le Q|h(x_{(n)}) - k(x_{(n)})|$. Owing to the mean value inequality, for any $(u, v) \in [m, M]^2$, we obtain

$$|g(u) - g(v)| \le Q|u - v|,$$

where $Q = \sup_{y\in[m,M]} |g'(y)|$. We have

$$g'(y) = \frac{\exp(ay)(ay-1)+1}{y^2}.$$

Note that for $y = 0$, which is the only ambiguous point at a first glance, we have $\lim_{y\to 0} g'(y) = a^2/2$, so Q exists. Moreover, since g is either convex or concave by Lemma 2.3, g' is monotonic, and since $\lim_{|y|\to+\infty} g'(y) \geq 0$, g' is non-negative. Therefore, again based on the convex properties of g as described in Lemma 2.3, we get

$$Q = \sup_{y\in[m,M]} g'(y) = \begin{cases} g'(M) & \text{if } a > 0 \\ g'(m) & \text{if } a < 0 \end{cases} = \begin{cases} \dfrac{\exp(aM)(aM-1)+1}{M^2} & \text{if } a > 0 \\ \dfrac{\exp(am)(am-1)+1}{m^2} & \text{if } a < 0 \end{cases}.$$

Hence, with $u = h(x_{(n)})$ and $v = k(x_{(n)})$, since $(u, v) \in [m, M]^2$ by assumptions, we obtain

$$\left|g[h(x_{(n)})] - g[k(x_{(n)})]\right| \leq Q|h(x_{(n)}) - k(x_{(n)})|,$$

implying the desired Lipschitz-type inequality. The proposition is proved. □

Among the consequences of Proposition 2.11, for any sequence of bounded functions $(f_i)_{i\in\mathbb{N}}$ that converges to a certain function f in the $\mathbb{L}_1$ sense, i.e.,

$$\lim_{i\to+\infty} \int_{[0,1]^n} |f_i(x_{(n)}) - f(x_{(n)})| dx_{(n)} = 0,$$

then $(C_n(f_i; a))_{i\in\mathbb{N}}$ converges in the simple sense to $C_n(f; a)$, i.e.,

$$\lim_{i\to+\infty} |C_n(f_i; a) - C_n(f; a)| = 0.$$

The next section gives more concrete formulas for the C operator by considering standard functions, beginning with the uni-dimensional case, i.e., $n = 1$.

Some formulas

The C operator of some specific functions can be expressed via standard and special functions. Several results in this vein are presented below.

Focus on the case $n = 1$

In this part, we focus on the C operator defined with $n = 1$, i.e.,

$$C_1(f; a) = \int_{[0,1]} \frac{\exp[af(x)] - 1}{f(x)} dx.$$

Naturally, Propositions 2.2, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10 and 2.11 hold with $n = 1$. For this simple case, more results can be given, including one connecting the C operator and the Laplace transform as described below.

Proposition 3.12 *Assume that f is bijective and differentiable. Then, under the standard Leibniz integral rule assumptions, we have*

$$\frac{\partial}{\partial a} C_1(f; a) = L(\ell)(-a),$$

where

$$\ell(y) = \frac{1}{f'[f^{-1}(y)]} 1_{[f(0),f(1)]}(y),$$

1 denotes the indicator function, and $L(\ell)(b) = \int_{\mathbb{R}} \exp(-by)\ell(y)dy$ is the Laplace transform of the function ℓ at b.

Proof. By the last point of Proposition 2.2 with $n = 1$, we have

$$\frac{\partial}{\partial a} C_1(f; a) = \int_{[0,1]} \exp[af(x)]dx.$$

By applying the change of variables $y = f(x)$, we immediately obtain

$$\frac{\partial}{\partial a} C_1(f; a) = \int_{[f(0),f(1)]} \exp(ay) \frac{1}{f'[f^{-1}(y)]} dy = L(\ell)(-a).$$

This ends the proof. □

Thus, connections exist between the C operator and the Laplace transform, but they are not direct, and some stringent assumptions on f are supposed.

The next result shows some examples of functions such that the associated C operator has a simple analytical expression.

Proposition 3.13

1. *For $f(x)$ equals to any constant, say c, we have*

$$C_1(f; a) = \frac{\exp(ac) - 1}{c}.$$

2. *For $f(x) = \sqrt{x + b}$ with $b \geq 0$, we have*

$$C_1(f; a) = \frac{2}{a} \left\{ \exp\left[a\sqrt{b+1}\right] - \exp\left[a\sqrt{b}\right] \right\} + 2\left[\sqrt{b} - \sqrt{b+1}\right].$$

   In particular, for $f(x) = \sqrt{x}$, i.e., by taking $b = 0$ in the formula above, we have

$$C_1(f; a) = \frac{2}{a}[\exp(a) - 1] - 2.$$

3. *For $f(x) = \log(x)$ and $a > -1$, we have*

$$C_1(f; a) = \log(1 + a).$$

Proof.

1. For $f(x)$ equals to any constant, say c, we have

$$C_1(f; a) = \int_{[0,1]} \frac{\exp(ac) - 1}{c} dx = \frac{\exp(ac) - 1}{c} \int_{[0,1]} dx = \frac{\exp(ac) - 1}{c}.$$

2. For $f(x) = \sqrt{x + b}$ with $b \geq 0$, we have

$$\begin{aligned} C_1(f; a) &= \int_{[0,1]} \frac{\exp\left[a\sqrt{x+b}\right] - 1}{\sqrt{x+b}} dx = \int_{[0,1]} \frac{\exp\left[a\sqrt{x+b}\right]}{\sqrt{x+b}} dx - \int_{[0,1]} \frac{1}{\sqrt{x+b}} dx \\ &= \frac{2}{a} \exp\left[a\sqrt{x+b}\right]\Big|_{x=0}^{x=1} - 2\sqrt{x+b}\Big|_{x=0}^{x=1} \\ &= \frac{2}{a} \left\{ \exp\left[a\sqrt{b+1}\right] - \exp\left[a\sqrt{b}\right] \right\} + 2\left[\sqrt{b} - \sqrt{b+1}\right]. \end{aligned}$$

3. For $f(x) = \log(x)$ and $a > -1$, we have
$$C_1(f;a) = \int_{[0,1]} \frac{\exp[a\log(x)] - 1}{\log(x)} dx = \int_{[0,1]} \frac{x^a - 1}{\log(x)} dx.$$
Therefore, by the last point of Proposition 2.2 with $n = 1$, we have
$$\frac{\partial}{\partial a} C_1(f;a) = \int_{[0,1]} x^a dx = \frac{1}{1+a}.$$
Hence, upon integration with respect to a and since $C_1(f;0) = 0$ (by the first point of Proposition 2.2), we have
$$C_1(f;a) = \int_{[0,a]} \frac{1}{1+y} dy = \log(1+y)|_{y=0}^{y=a} = \log(1+a).$$

The desired formulas are obtained. □

Proposition 3.13 shows that the C operator can have quite manageable expressions for some classical functions. Also, based on the third point of Proposition 2.2, note that Proposition 3.13 applies for $f_*(x) = f(1-x)$, with all the functions f considered, i.e., $f_*(x) = \sqrt{1-x+b}$ and $f_*(x) = \log(1-x)$.

As an example of application, if we consider the function $f(x) = \log(x)$ and $a \geq 0$, since $\xi = \int_{[0,1]} \log(x) dx = -1$, then Propositions 3.13 and 2.8 give
$$\log(1+a) = C_1(f;a) \geq \frac{\exp(a\xi) - 1}{\xi} = 1 - \exp(-a).$$
To the best of our knowledge, this inequality is new in the literature. It is sharper than the following famous logarithmic inequality: $\log(1+a) \geq a/(1+a)$, but it doesn't improve the one established in [13], i.e., $\log(1+a) \geq a/(1+a/2)$ for $a \geq 0$. It also gives the following simple logarithmic-logarithmic inequality by taking $a = -\log(x)$ for $x \in (0,1)$:
$$\log[1 - \log(x)] \geq 1 - x.$$
For any $a \in (-1, 0)$, still thanks to Propositions 3.13 and 2.8, we establish that
$$\log(1+a) = C_1(f;a) \leq \frac{\exp(a\xi) - 1}{\xi} = 1 - \exp(-a),$$
which is also new.

The next proposition shows some specific examples of functions such that the associated C operator involves well-known special functions. It is supposed that $a \in \mathbb{R}$ by default.

Proposition 3.14

1. *For $f(x) = x$, we have*
$$C_1(f;a) = -\operatorname{Ein}(-a),$$
where $\operatorname{Ein}(x) = \int_0^x [1 - \exp(-t)]/t dt$ is the modified exponential integral function (see [18]).

2. *For $f(x) = x + b$ with $b > 0$, we have*
$$C_1(f;a) = \operatorname{Ei}[a(b+1)] - \operatorname{Ei}(ab) + \log\left(\frac{b}{1+b}\right),$$
where $\operatorname{Ei}(x) = -\int_{[-x,+\infty)} [\exp(-t)/t] dt$ is the standard exponential integral function, which must be interpreted as the following Cauchy principal value:
$$\operatorname{Ei}(x) = -\lim_{s \to 0} \left(\int_{[-x,s)} \frac{\exp(-t)}{t} dt + \int_{(s,+\infty)} \frac{\exp(-t)}{t} dt \right).$$

3. *For $f(x) = \log[x(1-x)]$ and $a \geq 0$, we have*

$$C_1(f;a) = \int_{[0,a]} B(y+1, y+1)dy,$$

where $B(x,y) = \int_{[0,1]} t^{x-1}(1-t)^{y-1}dt$ is the standard beta function.

4. *For $f(x) = \log[-b\log(x)]$ with $b > 0$ and $a \geq 0$, we have*

$$C_1(f;a) = \int_{[0,a]} b^y \Gamma(y+1)dy,$$

where $\Gamma(x) = \int_{[0,+\infty)} t^{x-1}\exp(-t)dt$ is the standard gamma function.

Proof.

1. For $f(x) = x$, by applying the change of variables $y = -ax$, we have

$$C_1(f;a) = \int_{[0,1]} \frac{\exp(ax)-1}{x}dx = \int_{[0,-a]} \frac{\exp(-y)-1}{y}dy = -\operatorname{Ein}(-a).$$

2. For $f(x) = x+b$ with $b > 0$, by applying the change of variables $y = -a(x+b)$, we have

$$\begin{aligned} C_1(f;a) &= \int_{[0,1]} \frac{\exp[a(x+b)]-1}{x+b}dx = \int_{[-ab,-a(b+1)]} \frac{\exp(-y)-1}{y}dy \\ &= \int_{[-ab,+\infty)} \frac{\exp(-y)}{y}dy - \int_{[-a(b+1),+\infty)} \frac{\exp(-y)}{y}dy - \int_{[-ab,-a(b+1)]} \frac{1}{y}dy \\ &= \operatorname{Ei}[a(b+1)] - \operatorname{Ei}(ab) - \log(|y|)\Big|_{y=-ab}^{y=-a(b+1)} \\ &= \operatorname{Ei}[a(b+1)] - \operatorname{Ei}(ab) + \log\left(\frac{b}{1+b}\right). \end{aligned}$$

3. For $f(x) = \log[x(1-x)]$ and $a \geq 0$, we have

$$C_1(f;a) = \int_{[0,1]} \frac{\exp\{a\log[x(1-x)]\}-1}{\log[x(1-x)]}dx = \int_{[0,1]} \frac{x^a(1-x)^a-1}{\log[x(1-x)]}dx.$$

Therefore, by the last point of Proposition 2.2 with $n = 1$, we have

$$\frac{\partial}{\partial a}C_1(f;a) = \int_{[0,1]} x^a(1-x)^a dx = B(a+1, a+1).$$

Hence, upon integration with respect to a and since $C_1(f;0) = 0$ (by the first point of Proposition 2.2), we have

$$C_1(f;a) = \int_{[0,a]} B(y+1, y+1)dy.$$

4. For $f(x) = \log[-b\log(x)]$ with $b > 0$ and $a \geq 0$, we have

$$C_1(f;a) = \int_{[0,1]} \frac{\exp\{a\log[-b\log(x)]\}-1}{\log[-b\log(x)]}dx = \int_{[0,1]} \frac{[-b\log(x)]^a-1}{\log[-b\log(x)]}dx.$$

Therefore, by the last point of Proposition 2.2 with $n = 1$ and the change of variables $x = \exp(-y)$, we have

$$\frac{\partial}{\partial a}C_1(f;a) = b^a \int_{[0,1]} [-\log(x)]^a dx = b^a \int_{[0,+\infty)} y^a \exp(-y)dy = b^a\Gamma(a+1).$$

Hence, upon integration with respect to a and since $C_1(f;0) = 0$, we have

$$C_1(f;a) = \int_{[0,a]} b^y\Gamma(y+1)dy.$$

This ends the proof. □

As an example of application, if we consider the function $f(x) = \log[x(1-x)]$ and $a \geq 0$, since $\xi = \int_{[0,1]} \log[x(1-x)]dx = -2$, then Propositions 3.14 and 2.8 give

$$\int_{[0,a]} B(y+1, y+1)dy = C_1(f;a) \geq \frac{\exp(a\xi)-1}{\xi} = 2[1-\exp(-2a)].$$

To the best of our knowledge, this inequality is new in the literature.

It is worth mentioning that the derived integrals of special functions in Proposition 3.14 are computationable with most of the existing software like Matlab, Mathematica, etc. since they are often implemented in associated packages. To illustrate this claim, with the help of the R software, we display the values of the C operator for $f(x) = \log[-b\log(x)]$ with $b > 0$, which is given in the fourth point in Proposition 3.14, i.e.,

$$C_1(f;a) = \int_{[0,a]} b^y \Gamma(y+1)dy.$$

Table 1 gives these values for varying a and b.

Table 1: Numerical values of the C operator of $f(x) = \log[-b\log(x)]$ for varying values of a and b

$a =\longrightarrow$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$b = 0.1$	0	0.153	0.244	0.301	0.338	0.362	0.379	0.391	0.399	0.405	0.41
$b = 0.7$	0	0.184	0.346	0.494	0.636	0.776	0.917	1.064	1.22	1.389	1.575
$b = 1$	0	0.191	0.371	0.548	0.73	0.923	1.132	1.366	1.632	1.942	2.308
$b = 1.5$	0	0.199	0.402	0.619	0.861	1.139	1.466	1.863	2.353	2.97	3.762
$b = 2$	0	0.204	0.426	0.677	0.973	1.333	1.783	2.36	3.115	4.123	5.494
$b = 4$	0	0.219	0.493	0.849	1.331	2.005	2.973	4.399	6.543	9.833	14.97
$b = 10$	0	0.242	0.603	1.17	2.091	3.639	6.311	11.037	19.578	35.323	64.867
$b = 15$	0	0.252	0.662	1.358	2.587	4.825	9.015	17.054	32.814	64.321	128.445
$b = 25$	0	0.267	0.746	1.65	3.416	6.98	14.373	30.085	64.205	139.768	310.121

Complementary study

We now list a certain number of sophisticated formulas involving the C operator. Some of them connect the C operator with referenced special functions, including famous integral operators. The detailed proofs are omitted for the sake of space.

- For $f(x) = x^2$,
  - and $a \geq 0$, we have

    $$C_1(f;a) = 1 + \exp(a)\left\{2\sqrt{a}F[\sqrt{a}] - 1\right\},$$

    where $F(x) = \exp(-x^2)\int_0^x \exp(t^2)dt$ is the Dawson (integral) function (see [6]).

– and $a < 0$, we have

$$C_1(f;a) = -\sqrt{-a\pi}\,\mathrm{erf}[\sqrt{-a}] + 1 - \exp(a),$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)dt$ is the error function.

- For $f(x) = 1/x$ and $a \leq 0$, we have

$$C_1(f;a) = -\frac{1}{2}[a^2\,\mathrm{Ei}(a) + 1] + \frac{1}{2}(1+a)\exp(a).$$

- For $f(x) = 1/(1+x)$, we have

$$C_1(f;a) = \frac{1}{2}\left\{a^2\left[\mathrm{Ei}(a) - \mathrm{Ei}\left(\frac{a}{2}\right)\right] + 2(a+2)\exp\left(\frac{a}{2}\right) - (a+1)\exp(a) - 3\right\}.$$

- For $f(x) = 1/[1+\sqrt{x}]$, we have

$$C_1(f;a) =$$
$$\frac{1}{3}\left\{(a-3)a^2\left[\mathrm{Ei}(a) - \mathrm{Ei}\left(\frac{a}{2}\right)\right] - 5 + \exp(a)[1-(a-2)a] + 2\exp\left(\frac{a}{2}\right)[2+(a-1)a]\right\}.$$

- For $f(x) = x/(1+x)$ and $a \in \mathbb{R}/\{0\}$, we have

$$C_1(f;a) =$$
$$-\exp(a)(a-1)\left[\mathrm{Ei}(-a) - \mathrm{Ei}\left(-\frac{a}{2}\right)\right] + \mathrm{Ei}\left(\frac{a}{2}\right) - 2 + 2\exp\left(\frac{a}{2}\right) - \gamma - \log(|a|),$$

where γ is the Euler-Mascheroni constant, i.e., $\gamma \approx 0.57721$.

- For $f(x) = 1/x^2$ and $a \leq 0$, we have

$$C_1(f;a) = \frac{1}{3}\left\{2\sqrt{\pi}(-a)^{3/2}\,\mathrm{erfc}[\sqrt{-a}] - \exp(a)[\exp(-a) - 2a - 1]\right\},$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complementary error function.

- For $f(x) = \log(1+x)$ and $a > -1$, we have

$$C_1(f;a) = \mathrm{Ei}\left[(1+a)\log(2)\right] - \mathrm{li}(2) - \log(1+a),$$

where $\mathrm{li}(x) = \int_0^x [1/\log(t)]dt$ is the logarithmic integral function.

- For $f(x) = b + \log(1+x)$ with $b > 0$, we have

$$C_1(f;a) = \exp(-b)\left\{\mathrm{Ei}\{(a+1)[b+\log(2)]\} - \mathrm{Ei}[(a+1)b] + \mathrm{Ei}(b) - \mathrm{Ei}[b+\log(2)]\right\}.$$

- For $f(x) = \exp(bx)$ with $b \in \mathbb{R}$, we have

$$C_1(f;a) = -\frac{1}{b}\left\{a\,\mathrm{Ei}(a) - a\,\mathrm{Ei}[a\exp(b)] - \exp(-b) + \exp[a\exp(b) - b] + 1 - \exp(a)\right\}.$$

- For $f(x) = \sqrt{-\log(x)}$ and $a \leq 0$, we have

$$C_1(f;a) = \sqrt{\pi}\left\{\exp\left(\frac{a^2}{4}\right)\left[\mathrm{erf}\left(\frac{a}{2}\right) + 1\right] - 1\right\}.$$

- For $f(x) = 1/\log(x)$ and $a \geq 0$, we have

$$C_1(f; a) = 1 - 2aK_2[2\sqrt{a}],$$

where $K_2(x)$ is the modified Bessel function of the second kind, i.e., satisfying the following differential equation: $x^2y'' + xy' - (x^2 + 4)y = 0$ (see [2]).

- For $f(x) = x^b$ with $b > 0$ and $a \leq 0$, we have

$$C_1(f; a) = \frac{1}{b-1}\left\{1 - \exp(a) + a(-a)^{-1/b}\left[\Gamma\left(\frac{1}{b}\right) - \Gamma\left(\frac{1}{b}, -a\right)\right]\right\},$$

where $\Gamma(x) = \int_{[0,+\infty)} t^{x-1}\exp(-t)dt$ is the standard gamma function, and $\Gamma(x, y) = \int_{[y,+\infty)} t^{x-1}\exp(-t)dt$ is the standard incomplete gamma function.

- For $f(x) = \operatorname{arcsinh}(x)$ and $a \in (-1, 1)$, we have

$$C_1(f; a) = \frac{1}{2}\left\{-2\,\mathrm{Chi}[\operatorname{arcsinh}(1)] + \mathrm{Ei}[(-1+a)\operatorname{arcsinh}(1)] + \mathrm{Ei}[(1+a)\operatorname{arcsinh}(1)] - \log(1-a^2)\right\},$$

where $\mathrm{Chi}(x) = \gamma + \log(x) + \int_0^x \{[\cosh(t) - 1]/t\}dt$ is the hyperbolic cosine integral.

- For $f(x) = \log[1 + \sqrt{x}]$, we have

$$C_1(f; a) = -2\,\mathrm{Ei}[(1+a)\log(2)] + 2\,\mathrm{Ei}[(2+a)\log(2)] + 2\,\mathrm{li}(2) - 2\,\mathrm{li}(4) + \log(4) - 4\operatorname{arccoth}(3+2a).$$

- For $f(x) = \sqrt{1 + \sqrt{x}}$, we have

$$C_1(f; a) = \frac{1}{a^3}\left\{\left[4a^2 - 8\sqrt{2}a + 8\right]\exp[\sqrt{2}a] + 8\exp(a)(a-1)\right\} + \frac{4}{3}[\sqrt{2} - 2].$$

Again, for numerical purposes, most of the involved integral operators are already implemented in most of the mathematical software, making the C operator quite computable.

The rest of the article investigates some formulas in the general multi-dimensional case.

The higher-dimension case

In this part, we consider the C operator with n supposed to be greater than 1 or 2. The next result shows some examples of functions for which the associated C operator has a simple analytical expression.

Proposition 3.15

1. *For $f(x_{(n)})$ equals to any constant, say c, we have*

$$C_n(f; a) = \frac{\exp(ac) - 1}{c}.$$

2. *For $f(x_{(n)}) = \sum_{i=1}^{n}\log(x_i)$ with n greater or equal to 2 and $a > -1$, we have*

$$C_n(f; a) = \frac{1}{n-1}\left[1 - \frac{1}{(1+a)^{n-1}}\right].$$

For the case $n = 1$, we recall that $C_n(f; a) = \log(1 + a)$ (see Proposition 3.13).

3. *For* $f(x_{(n)}) = \sum_{i=1}^{n} \log[x_i(1-x_i)]$ *and* $a \geq 0$*, we have*

$$C_n(f;a) = \int_{[0,a]} [B(y+1, y+1)]^n \, dy.$$

4. *For* $f(x_{(n)}) = \sum_{i=1}^{n} \log[-b\log(x_i)]$ *with* $b > 0$ *and* $a \geq 0$*, we have*

$$C_n(f;a) = \int_{[0,a]} b^{ny} \, [\Gamma(y+1)]^n \, dy.$$

Proof.

1. For $f(x_{(n)})$ equals to any constant, say c, we have

$$C_n(f;a) = \int_{[0,1]^n} \frac{\exp(ac)-1}{c} dx_{(n)} = \frac{\exp(ac)-1}{c} \int_{[0,1]^n} dx_{(n)} = \frac{\exp(ac)-1}{c}.$$

2. For $f(x_{(n)}) = \sum_{i=1}^{n} \log(x_i) = \log(\prod_{i=1}^{n} x_i)$ with n greater or equal to 2 and $a > -1$, we obtain

$$C_n(f;a) = \int_{[0,1]^n} \frac{\exp[a\log(\prod_{i=1}^{n} x_i)] - 1}{\log(\prod_{i=1}^{n} x_i)} dx_{(n)} = \int_{[0,1]^n} \frac{(\prod_{i=1}^{n} x_i)^a - 1}{\log(\prod_{i=1}^{n} x_i)} dx_{(n)}.$$

It follows from the last point of Proposition 2.2 that

$$\frac{\partial}{\partial a} C_n(f;a) = \int_{[0,1]^n} \left(\prod_{i=1}^{n} x_i^a \right) dx_{(n)} = \prod_{i=1}^{n} \left(\int_{[0,1]} x_i^a dx_i \right) = \frac{1}{(1+a)^n}.$$

Hence, upon integration with respect to a and since $C_n(f;0) = 0$ (by the first point of Proposition 2.2), we have

$$C_n(f;a) = \int_{[0,a]} \frac{1}{(1+y)^n} dy = \frac{1}{(1-n)(1+y)^{n-1}} \Bigg|_{y=0}^{y=a} = \frac{1}{n-1} \left[1 - \frac{1}{(1+a)^{n-1}} \right].$$

3. For $f(x_{(n)}) = \sum_{i=1}^{n} \log[x_i(1-x_i)] = \log\left[\prod_{i=1}^{n} x_i(1-x_i)\right]$ and $a \geq 0$, we have

$$\begin{aligned} C_n(f;a) &= \int_{[0,1]^n} \frac{\exp\left\{a\log\left[\prod_{i=1}^{n} x_i(1-x_i)\right]\right\} - 1}{\log\left[\prod_{i=1}^{n} x_i(1-x_i)\right]} dx_{(n)} \\ &= \int_{[0,1]^n} \frac{\left[\prod_{i=1}^{n} x_i(1-x_i)\right]^a - 1}{\log\left[\prod_{i=1}^{n} x_i(1-x_i)\right]} dx_{(n)}. \end{aligned}$$

Therefore, by the last point of Proposition 2.2, we get

$$\begin{aligned} \frac{\partial}{\partial a} C_n(f;a) &= \int_{[0,1]^n} \left[\prod_{i=1}^{n} x_i^a (1-x_i)^a \right] dx_{(n)} \\ &= \prod_{i=1}^{n} \left(\int_{[0,1]} x_i^a (1-x_i)^a dx_i \right) = [B(a+1, a+1)]^n. \end{aligned}$$

Hence, upon integration with respect to a and since $C_n(f;0) = 0$, we have

$$C_n(f;a) = \int_{[0,a]} [B(y+1, y+1)]^n \, dy.$$

4. For $f(x_{(n)}) = \sum_{i=1}^{n} \log[-b\log(x_i)] = \log\left\{b^n \prod_{i=1}^{n}[-\log(x_i)]\right\}$ with $b > 0$ and $a \geq 0$, we have

$$C_n(f;a) = \int_{[0,1]^n} \frac{\exp\left[a\log\left\{b^n \prod_{i=1}^{n}[-\log(x_i)]\right\}\right]-1}{\log\left\{b^n \prod_{i=1}^{n}[-\log(x_i)]\right\}} dx_{(n)}$$
$$= \int_{[0,1]^n} \frac{\left\{b^n \prod_{i=1}^{n}[-\log(x_i)]\right\}^a - 1}{\log\left\{b^n \prod_{i=1}^{n}[-\log(x_i)]\right\}} dx_{(n)}.$$

Therefore, by the last point of Proposition 2.2 and the change of variables $x_i = \exp(-y)$ for $i = 1, \ldots, n$, we have

$$\frac{\partial}{\partial a}C_n(f;a) = b^{na}\int_{[0,1]^n}\left\{\prod_{i=1}^{n}[-\log(x_i)]^a\right\}dx_{(n)} = b^{na}\prod_{i=1}^{n}\left(\int_{[0,+\infty)} y^a\exp(-y)dy\right)$$
$$= b^{na}\left[\Gamma(a+1)\right]^n.$$

Hence, upon integration with respect to a and since $C_n(f;0) = 0$, we have

$$C_n(f;a) = \int_{[0,a]} b^{ny}\left[\Gamma(y+1)\right]^n dy.$$

This ends the proof. □

We end this part with some others formulas involving the C operator of special functions with $n = 2$ exclusively.

- For $f(x_1, x_2) = \sqrt{x_1 + x_2}$ and $a \neq 0$, we have

$$C_2(f;a) = \frac{4}{3a^3}\left\{6(1-a)\exp(a) - 2[\sqrt{2}-1]a^3 + 3[a\sqrt{2}-1]\exp[a\sqrt{2}] - 3\right\}.$$

- For $f(x_1, x_2) = \sqrt{1 + x_1 + x_2}$ and $a \neq 0$, we have

$$C_2(f;a) =$$
$$\frac{4}{3}\left\{\frac{3}{a^3}\left[\exp(a)(a-1) + (2-2\sqrt{2}a)\exp[\sqrt{2}a] + [\sqrt{3}a-1]\exp[\sqrt{3}a]\right] - 3\sqrt{3} + 4\sqrt{2} - 1\right\}.$$

- For $f(x_1, x_2) = \sqrt{x_1 + x_2}$ if $0 \leq x_2 \leq x_1 \leq 1$, and $f(x_1, x_2) = 0$ otherwise, and $a \neq 0$, we have

$$C_2(f;a) = -\frac{2}{a^3}\{2\exp(a)(a-1) + 2\} + \frac{2}{a^3}\{\exp[\sqrt{2}a][\sqrt{2}a - 1] + 1\} - \frac{4}{3}\sqrt{2} + \frac{4}{3}.$$

- For $f(x_1, x_2) = \sqrt{x_1 + x_2}$ if $0 \leq x_2 \leq (1 + x_1^2)/2 \leq 1$, and $f(x_1, x_2) = 0$ otherwise, and $a \neq 0$, we have

$$C_2(f;a) = -\frac{2}{a^3}\{2\exp(a)(a-1) + 2\} + \frac{2\sqrt{2}}{a^2}\exp\left[\frac{a}{\sqrt{2}}\right]\left\{\exp\left[\frac{a}{\sqrt{2}}\right] - 1\right\} - \frac{3}{\sqrt{2}} + \frac{4}{3}.$$

- For $f(x_1, x_2) = \sqrt{x_1 + x_2}$ if $0 \leq x_2 \leq 1 - x_1 \leq 1$, and $f(x_1, x_2) = 0$ otherwise, and $a \neq 0$, we have

$$C_2(f;a) = -\frac{2}{a^3}\{2\exp(a)(a-1) + 2\} + \frac{2}{a}\exp(a) - \frac{2}{3}.$$

- For $f(x_1, x_2) = \sqrt{x_1 - \log(x_2)}$ and $a \neq 0$, we have

$$C_2(f;a) = -\sqrt{\pi}\left\{\exp\left(1 + \frac{a^2}{4}\right)\left[\operatorname{erf}\left(1 - \frac{a}{2}\right) - 1\right] + \exp\left(\frac{a^2}{4}\right)\left[\operatorname{erf}\left(\frac{a}{2}\right) + 1\right]\right.$$
$$\left. + \exp(1)\operatorname{erfc}(1) - 1\right\} - 2 + 2\frac{\exp(a) - 1}{a}.$$

- For $f(x_1, x_2) = x_1/x_2$ and $a < 0$, we have

$$C_2(f; a) = \\ -\frac{1}{4}\exp(a)\left\{\exp(-a)\left[-2\operatorname{Ei}(a) + a^2\operatorname{Ei}(a) + 1 + 2\gamma + 2\log(-a)\right] - 1 - a\right\}.$$

- For $f(x_1, x_2) = x_1 + \log(x_2)$ and $a > 0$, we have

$$C_2(f; a) = \frac{1}{\exp(1)}\left[\operatorname{Ei}(1) - \operatorname{Ei}(1 + a)\right] + \operatorname{Ei}(a) - \gamma + \log\left(1 + \frac{1}{a}\right).$$

- For $f(x_1, x_2) = \sqrt{x_1 x_2}$ and $a < 0$, we have

$$C_2(f; a) = -\frac{4}{a}\left[a + \log(-a) + \Gamma(0, -a) + \gamma\right].$$

- For $f(x_1, x_2) = \sqrt{x_1 - x_2}$ if $0 \le x_2 \le x_1 \le 1$, and $f(x_1, x_2) = 0$ otherwise, and $a \neq 0$, we have

$$C_2(f; a) = \frac{4}{a^3}\left[\exp(a)(a - 1) + 1\right] - \frac{2}{a} - \frac{4}{3}.$$

- For $f(x_1, x_2) = x_1^2/x_2^2$ if $0 \le x_2 \le x_1 \le 1$, and $f(x_1, x_2) = 0$ otherwise, and $a < 0$, we have

$$C_2(f; a) = \frac{1}{6}\left\{-4a^{3/2}\exp(a)F[\sqrt{a}] + 2\sqrt{\pi}(-a)^{3/2} + (2a + 1)\exp(a) - 1\right\}.$$

To the best of our knowledge, some of the above multi-dimensional integral formulas are new in the literature, which reveal the unexplored potential of the C operator for purposes beyond the theory.

Conclusion

In this article, we have presented a new and original multi-dimensional nonlinear integral operator called the C operator. Its construction is distantly inspired by the Box-Cox transformation. We emphasized the fact that the C operator enjoys attractive mathematical properties, including various scale properties, manageable series expansions and partial derivatives, original lower and upper bounds, and convex properties. A wide panel of functions has expressions involving simple or special functions, making the C operator able to capture complex relationships and dynamics in various fields, such as physics, mathematics, and machine learning, enabling more accurate modeling and analysis of nonlinear phenomena. However, the applied aspect needs further developments that we leave for future work.

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