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The Hop Total Hub Number of Graphs

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Abstract
Hub number is a graph parameter introduced by modelling a transportation problem for rapid transit in any system. In this paper, we coin a new hub parameter called hup total hub number of graphs and we determine the hop total hub number of some standard graphs. Also upper and lower bounds for the hop hub number are obtained.

Keywords: Hub Number, Total Hub Number, Domination Number, Connected Domination Number, Hop Total Hub Number.

Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. For graph theoretic terminology, we refer to [2]. Let G be such a graph, and let p and q be the number of its vertices and edges, respectively. Then we say that G is an (p, q)-graph. A graph is said to be connected if every pair of its vertices are joined by a path. A graph which is not connected is said to be disconnected. A maximal connected subgraph of G is called a component of G. The maximum degree of a graph G is denoted by $\triangle(G)$ while $\delta(G)$ denotes the minimum degree of G. A vertex of degree one is called a pendant vertex. The vertex which is adjacent to the pendant vertex is called a support vertex. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [3]. A set $S \subseteq V$ of a graph G is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$ [1].

Consider the graphs that represent transportation networks, that is the vertices can be taken to be locations or destinations, and an edge exists between two vertices precisely when there is an "easy passage" between the corresponding locations. For example, a city's network of streets, with vertices representing intersections or other points of intersect, and edges road segments. We are connected with a certain kind of connectivity, specifically we want a set S such that any traffic between disparate points in our network passes solely through vertices in this set.

In 2006, M. Walsh [13] introduced the concept of hub number to provide an optimal solution for rapid transit from one place to another in any system. A hub set S of G is a subset of vertices in G such that any two

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vertices $u, v \in V(G) - S$ are connected by a path with all internal vertices from S. (This includes the degenerate cases where the path consists of the single edge uv or a single vertex u if $u = v$ call such an $S - path$ trivial.) The minimum cardinality of a hub set is the hub number of G, and is denoted by $h(G)$. In 2014, Veena et al [11] introduced the concept of the total hub set in graphs. A total hub set S of G is a subset of $V(G)$ such that every pair of vertices (whether adjacent or nonadjacent) of $V - S$ are connected by a path, whose all intermediate vertices are in S. The total hub number $h_t(G)$ is then defined to be the minimum cardinality of a total hub set of G.

The concept of hub number quantifies the connectivity of vertices in graphs. Due to which it has wide application in the field of networks. For this reason, several hub parameters have been explored and studied extensively [5–7, 9, 10, 12]. Motivated by this, in this article we try to term a new hub parameter called hop hub number of graphs as follows.

Definition 1.1. *A total hub set $S \subseteq V$ is a hop total hub set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop hub set of G is called the hop hub number and is denoted by $h_{ht}(G)$.*

We need the following to prove main result.

Proposition 1.1 [11]. For any connected graph G, $\gamma(G) \leq h_t(G)$.

Main Results

It is clear that $h_{ht}(G)$ is well-defined for any graph G, since $V(G)$ is a hop total hub set. In all situations of interest, we will assume G to be connected, if G is a disconnected graph then any hop total hub set must contain union of the set of vertices from all but one largest component, and the hop total hub set of the largest component.

It is obvious that any hop total hub set in a graph G is also a total hub set, and thus we obtain the obvious bound $h_t(G) \leq h_{ht}(G)$ and the inequality is sharp if $G \cong T$, and $h(G) \geq 2$.

We now proceed to compute $h_{ht}(G)$ for some standard graphs. It can be easily verified that

Proposition 2.2. 1. *For any complete graph K_p, $h_{ht}(K_p) = p$*

2. *For any path P_p with $p \geq 4$, $h_{ht}(P_p) = p - 2$.*

3. *For any cycle C_p,*

$$h_{ht}(C_p) = \begin{cases} 2, & \text{if } p = 4\,; \\ 3, & \text{if } p = 3\,; \\ p - 3, & \text{if } p \geq 5\,. \end{cases}$$

.

4. *For the wheel $W_{1,p-1}$, $p \geq 5$, $h_{ht}(W_{p-1}) = 3$.*

5. *For the double star $S_{n,m}$, $n, m \geq 1$, $h_{ht}(S_{n,m}) = 2$.*

Theorem 2.3. *If $G \cong K_{m_1, m_2, \ldots, m_k}$ be the complete $k-$partite graph, with $m_1 \leq m_2 \leq m_3 \leq \ldots \leq m_k$. Then.*

$$h_{ht}(G) = \begin{cases} 2, & \text{if } k = 2\,; \\ k, & \text{if } k \geq 3\,. \end{cases}$$

.

Proof. Let $m_1, m_2, \ldots, m_k$ be the component of partitions of G. the following two cases are considered:

Case 1: When $k = 2$, we have G is a complete bipartite graph. Suppose that $S = \{v_1, v_2\}$ such that $v_1 \in m_1$ and $v_2 \in m_2$, we have for any two vertices of $V(G) - S$ are connected by a path whose internal vertex in S and for any vertex $u \in V(G) - S$, there exists a vertex v_1 or v_2 in S such that the distance between u and v_1 or u and v_2 is equal to two, clearly S is a minimum hop hub set of G. Therefore $h_{ht}(G) = 2$.

Case 2: When $k \leq 3$, any two vertices u and v are not in the same component of G are adjacent, so the distance between u and v is equal to one but any two vertices in the same component are not adjacent and the distance between them is equal to two. So, we can choose the set S contains one vertex from each component $m_1, m_2, \ldots, m_k$. So, $|S| = k$ is a hop hub set of G and if removed any vertex $v \in m_i$, $1 \leq i \leq k$ from S, we get the distance between any vertex of m_i, $1 \leq i \leq k$ and any vertex of $S - v$ is equal to one. Thus S is a minimum hop hub set of G. Therefore, $h_{ht}(G) = k$. □

Lemma 2.4. *For any graph G. $h_{th}(G) = 2$ if and only if $h(G) \leq 2$.*

Theorem 2.5. *For any tree T of order $p \geq 4$, $h_{ht}(T) = p - n$, where n is the number of pendant vertices.*

Proof. Suppose that H is the set of a pendant vertices of T and $S = V(G) - H$, we have every two vertices of H are connected by a path whose internal vertex in S and for any vertex $v \in H$, there exists a vertex $u \in S$ such that the distance between v and u is equal to two and if removed u from S, there is two vertices outside S have no S-path between them. Thus S a minimum hop hub set of G. Therefore, $h_{ht}(T) = p - n$. □

Lemma 2.6. *For any tree T of order $p \geq 4$, $h(T) = h_t(T) = h_{ht}(T) = \gamma_c(T)$.*

Theorem 2.7. *For any connected graph G, $\gamma(G) \leq h_{ht}(G)$.*

Proof. Since every hop hub set of G is a hub set, we have the proof follows by Proposition 1.1. □

A dominating set of G need not be a hop total hub set of G. For example, in $C_5 = \{v_1, v_2, \ldots, v_5\}$, $\{v_1, v_3\}$ is a dominating set but not a hop total hub set.

Theorem 2.8. *Let G be a connected graph has a pendant vertex v such that . $\triangle(G) = p - 1$. Then $h_{ht}(G) = 2$.*

Proof. Suppose that G is a connected graph with $\triangle(G) = p - 1$ if $S = \{v, u\}$ such that v is a pendant vertex of G and u its support such that $deg(u) = p - 1$, then any two vertices of $V - S$ are connected by a path whose internal vertex is u and the distance between any vertex of $V - S$ and a vertex v is equal to two. So, S is a hop hub set of G. If removed u or v from S, there is two vertices outside S have no S-path between them or there is no vertex in S such that the distance between any vertex of $V - S$ and a vertex of S is equal to two. Therefore, $h_{ht}(G) = 2$. □

Lemma 2.9. *For any graph G, $h_{ht}(G) \leq 2p - (\triangle(G) + 1)$.*

Proof. Since for any graph G, $h_{ht}(G) \leq p$, we get the result.

□

Lemma 2.10. *A graph G is complete graph if and only if, $h_{ht}(G) = p$.*

Theorem 2.11. *Let G be a disconnected graph having $M_1, M_2, \ldots, M_l$ components. Then $h_{ht}(G) = \min_{1 \leq k \leq l}\{X_k\}$, where*

$$X_k = h_{ht}(M_k) + \sum_{i=1, i\neq k}^{l} |V(M_i)|.$$

Proof. From the definition of a hop total hub set, any hop total hub set S of a graph G must contains all the vertices of $k - 1$ components and the vertices of hop total hub set of the remaining component. To show that S is a minimum. The union of all components except one and taking the hop total hub set of the remaining component, we can compute all hop total hub sets of G, and more detailed $S = \bigcup_{i=1, i\neq j}^{k} M_i \cup H_{ht}^k$, where H_{th}^k is a hop total hub set of M_k.

Let $X_k = h_{ht}(M_l) + \sum_{i=1, i\neq k}^{l} |V(M_i)|$, then $\min_{1 \leq k \leq l}\{X_k\} = h_{ht}(G)$. □

Theorem 2.12. *For any connected graph G, $2 \leq h_{ht}(G) \leq p$.*

Proof. By the definition of a hop total hub set S of a graph G we have $|S| \geq 2$ and the upper bound is achieved of $G \cong K_p$. Therefore, $2 \leq h_{ht}(G) \leq p$. □

Theorem 2.13. *Let G be a connected graph of order p, $h_{th}(G) = p$ if and only if $G \cong K_p$.*

Proof. Suppose that $h_{ht}(G) = p$, this means that all vertices of a graph G are adjacent and hence $G \cong K_p$. Conversely, if $G \cong K_p$, the proof follow form Proposition 2.1 part 1. □

Theorem 2.14. *For any connected graph G, if $h_{th}(G) = 2$ then $diam(G) \leq 3$.*

Proof. Suppose that $h_{th}(G) = 2$. We prove that $diam(G) \leq 3$, if $diam(G) > 3$, then by the definition of $diam(G)$, there exists a path between at least five vertices and we get $h_{ht}(G) \geq 3$, but this contradiction that $h_{ht}(G) = 2$, then $diam(G) \leq 3$. □

Remark 2.15. *The converse of Theorem 2.14 is not true. For example for $G \cong K_3$ we have $diam(K_3) = 1$ and $h_{ht}(K_3) = 3$.*

Theorem 2.16. *Let T be a tree, then $h_{ht}(T) = 2$ if and only if $diam(T) \leq 3$.*

Proof. Suppose that $h_{ht}(T) = 2$, then $diam(T) \leq 3$ by Theorem 2.14.
Conversely, suppose that $diam(T) \leq 3$ and we prove that $h_h(T) = 2$. Since $diam(T) \leq 3$ and a tree T has not closed path, then the largest distance in T contains four vertices. Let $T \cong P_4$ since $h_{ht}(P_4) = 2$, by Proposition 2.1 part 2, and without loss of generally, $h_{ht}(T) = 2$. □

Theorem 2.17. *Let G be a graph with p vertices, then*

1. $h_{ht}(G) + h_{ht}(\overline{G}) \leq 2p$.
2. $h_{ht}(G)h_{ht}(\overline{G}) \leq p^2$.

the each inequality is sharp if $G \cong K_P$

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