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A Novel Multivariate Integral Ratio Operator: Theory and Applications Including Inequalities

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Abstract
The interest in creating new multivariate integral operators arises from the need to capture and model complex interactions between multiple variables involved in real-world applications. In this article, we emphasize a new multivariate integral operator mainly based on a one-parameter ratio-type transformation. It has the characteristics of being nonlinear and benefits from many comprehensive properties that make it attractive from a mathematical point of view. In particular, the following are highlighted: (i) it is a solution for some original functional equations; (ii) it is able to generate the integral of the main function raised to a certain integer power; (iii) it can be expanded as a simple functional series; (iv) it fulfills various inequalities, which can be particularly sharp; and (v) it has a simple expression for a wide panel of univariate and multivariate functions. For all these aspects, detailed proofs based on various mathematical tools are given. Finally, applications are provided to illustrate the main findings. These include the revisit of some known inequalities for simple and special functions and new ones. Some graphics illustrate them for a direct visual check.

Keywords: multivariate integral operators; ratio transformation; inequalities; special functions.

Introduction

Integral operators are of interest due to their capacity to provide a robust mathematical framework for the analysis and transformation of functions, playing pivotal roles in fields such as calculus, functional analysis, probability, and signal processing. The creation of novel integral operators holds significance in offering new analytical tools and addressing specific issues or phenomena beyond the reach of existing operators. In the univariate context, notable linear integral operators include the Fourier operator, Laplace operator (see [1]), Sumudu operator (see [2]), Elzaki operator (see [3]), Natural operator (see [4]), Formable operator (see [5]), and Jafari operator (see [6]). On the other hand, important nonlinear integral operators include the nonlinear Fourier operator (see [7]), Kamimura operator (see [8]), Urysohn operator (see [9] and [10]), and Hammerstein operator (see [11]). Furthermore, to effectively capture complex dependencies among multiple variables in mathematical, scientific, and engineering applications, the development of innovative multivariate integral operators has garnered substantial attention. In the family of linear operators of this kind, we may refer to the triple Laplace-Aboodh-Sumudu operator (see [12]), double Laplace operator (see [13]), double Sumudu operator (see again [13]), triple Laplace operator (see [14] and [15]), triple Elzaki operator (see [16]), double fuzzy Natural operator (see [17]), and fractional order multiple integral operator (see [18]). Among the inte-

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gral operators mixing multivariate and nonlinearity, there are the special convolution-type operator (see [19]), special Kantorovich operator (see [20]), C operator (see [21]) and generalized C operator (see [22]).

In fact, innovative multivariate nonlinear integral operators have received relatively limited attention in the literature, although they hold substantial potential for the development of new theories and applications. This article justifies these assertions. We introduce and study a new operator, mainly based on a complete one-parameter ratio transformation, that can be written as follows:

$$\text{"}T(f)(s) = \int_{\Omega} \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x}\text{"}.$$

The details will be given later, in Definition 1. This multivariate nonlinear integral operator has several remarkable features, including:

(i) Being one of the few integral operators satisfying the simple functional equations: "$T(f + b)(s) - [1/(1 + bs)]T(f)[s/(1 + bs)] = 0$" and "$T(1/f)(s) + T(f)(1/s) = 1$";

(ii) Demonstrating the ability to generate the integral of f raised to a specific integer power; we highlight a comprehensive transformation D_m such that "$D_m[T(f)(s)] = \int_{\Omega}[f(\mathbf{x})]^m d\mathbf{x}$", where m is a positive integer;

(iii) Exhibiting the capacity for expansion into a simple infinite series;

(iv) Adhering to various manageable and general inequalities, including convex, concave, and Cauchy-Schwarz type inequalities, which can be of a high level of precision;

(v) Finally, providing a straightforward expression for a broad range of univariate and multivariate functions f.

The findings are illustrated by demonstrating sharp inequalities involving a wide panel of standard and special functions (logarithmic, exponential, trigonometric, exponential integral, etc.). Some well-known inequalities are revisited, and some new ones are established. Thus, we contribute to the field of analysis by introducing an under-explored area of multivariate nonlinear integral operators, demonstrating their potential in analysis.

The remainder of the article is as follows: Section is devoted to the detailed description of the new operator and its basic properties. Some related general inequalities are demonstrated in Section . Section presents a wide panel of univariate and multivariate functions that the new operator transforms into straightforward functions. Applications on diverse inequalities are described in Section . Finally, a conclusion is given in Section .

Definition and basic properties

We begin by elucidating the proposed operator.

Definition 1 *Let $s \in \mathbb{R}$, n be a positive integer and $\Omega = \Omega_n = [0, 1]^n$. Let us set $\mathbf{x} = \mathbf{x}_n = (x_1, \ldots, x_n)$ and $d\mathbf{x} = d\mathbf{x}_n = \prod_{i=1}^{n} dx_i$. Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$, $\mathbf{x} \in \Omega$, be a function with values in $\mathbb{R}$. Under this setting, we define the integral ratio (IR) operator of f at the point s as*

$$T(f)(s) = \int_{\Omega} \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x}.$$

Important notes:

- *Throughout the article, when the IR operator is mentioned (or any other quantity), it is assumed to exist in the mathematical sense; we will voluntary omit the list of possible existence assumptions to ease the reading and highlight the main properties of the IR operator; these assumptions often depend on a case-by-case basis.*

- *Eventually, we can replace* Ω *by* $\Omega = \bigtimes_{i=1}^{n} I_i$*, where* $I_i \in \{[0,1], (0,1], [0,1), (0,1)\}$ *for each* $i = 1, \ldots, n$ *in the definition. This does not affect its integral definition of the IR operator but can relax some coming bounded assumptions on* f*. In light of this, these functional assumptions benefit from a higher level of flexibility.*

Clearly, the nonlinearity of the IR operator comes from the ratio form of the integrated function; immediately, for any $a \in \mathbb{R}$, we have $T(af)(s) = T(f)(as) \neq aT(f)(s)$ in the general case. Its multivariate nature comes from the consideration of f defined on the multivariate domain Ω.

There are numerous simple conditions on f where the IR operator exists. For instance, if $sf(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, the IR operator of f obviously exists, and

$$|T(f)(s)| = \int_{\Omega} \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} \leq \int_{\Omega} d\mathbf{x} = \prod_{i=1}^{n} \left(\int_{[0,1]} dx_i \right) = 1.$$

Furthermore, from an analytical viewpoint, the IR operator has a simple expression. Indeed, its integral-ratio-type construction opens the door to the use of existing mathematical tools (pivotal theorems, integral calculus techniques, series expansions, inequalities, etc.), making it manageable in several aspects, as already listed in the introduction.

Also, in the case of a function f has its support included into Ω, say $\Xi \subseteq \Omega$, we can decompose the corresponding IR operator as

$$T(f)(s) = \int_{\Xi} \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} + \int_{\Omega \setminus \Xi} d\mathbf{x},$$

where the last integral term represents the "volume of $\Omega \setminus \Xi$" (in the dimension n sense). Thus defined, sophisticated multivariate functions f can be transformed.

We now study the IR operator in an in-depth manner, beginning with its basic properties in the proposition below.

Proposition 2.1 *Let* $s \in \mathbb{R}$ *and* $f(\mathbf{x})$, $\mathbf{x} \in \Omega$*, be a function. The IR operator of* f *satisfies the following properties:*

1. *For* $f(\mathbf{x}) = a$ *with* $a \in \mathbb{R}$ *such that* $1 + sa \neq 0$*, we have*

$$T(f)(s) = \frac{1}{1+sa}.$$

   In particular, we have $T(0)(s) = T(f)(0) = 1$*.*

2. *Under the assumption* $sf(\mathbf{x}) \geq -1$ *for any* $\mathbf{x} \in \Omega$*, we have* $T(f)(s) \geq 0$*, and, under the assumption* $sf(\mathbf{x}) \leq -1$ *for any* $\mathbf{x} \in \Omega$*, we have* $T(f)(s) \leq 0$*.*

3. *For* $f_*(\mathbf{x}) = f(\mathbf{x}_*)$*, where* $\mathbf{x}_* = (x_1^*, \ldots, x_n^*)$ *with* $x_i^* \in \{x_i, 1 - x_i\}$ *for each* $i = 1, \ldots, n$*, we have* $T(f_*)(s) = T(f)(s)$*.*

4. *For any* $b \in \mathbb{R}$ *such that* $1 + bs \neq 0$*, the following equality holds:*

$$T(f+b)(s) = \frac{1}{1+bs} T(f) \left(\frac{s}{1+bs} \right).$$

5. *For* $f(\mathbf{x}) = (1/s)[1/g(\mathbf{x}) - 1]$*, where* $g(\mathbf{x})$, $\mathbf{x} \in \Omega$*, denotes a (at least integrable) function, the IR operator of* f *is reduced to the integral of* g*, i.e.,*

$$T(f)(s) = \int_{\Omega} g(\mathbf{x}) d\mathbf{x}.$$

6. *The following functional equation is satisfied:*

$$T\left(\frac{1}{f}\right)(s) + T(f)\left(\frac{1}{s}\right) = 1.$$

7. *For any positive integer m, under the multivariate Leibnitz integral rule assumptions, we have*

$$\frac{(-1)^m}{m!}\frac{\partial^m}{\partial s^m}T(f)(s)\bigg|_{s=0} = \int_\Omega [f(\mathbf{x})]^m d\mathbf{x}.$$

8. *Under the multivariate interchange of integral and sum rule assumptions*

   - *and the assumption $|sf(\mathbf{x})| < 1$ for any $\mathbf{x} \in \Omega$, the following serie expansion holds:*

$$T(f)(s) = \sum_{k=0}^{\infty}(-s)^k \int_\Omega [f(\mathbf{x})]^k d\mathbf{x}.$$

   - *and the assumption $|sf(\mathbf{x})| > 1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) = \sum_{k=0}^{\infty}(-1)^k s^{-(k+1)} \int_\Omega [f(\mathbf{x})]^{-(k+1)} d\mathbf{x}.$$

   Note: it may be allowed to have $|sf(\mathbf{x})| = 1$ for countable values of $\mathbf{x} \in \Omega$; this must be studied on a case-by-case basis.

9. *Under the multivariate interchange of two integrals rule assumptions, for any $t \in \mathbb{R} \setminus \{0\}$, we have*

$$U(f)(t) = \int_{[0,1]} T(f)(st)ds = \int_\Omega \frac{\log[1+tf(\mathbf{x})]}{tf(\mathbf{x})}d\mathbf{x}, \tag{1}$$

   which also represents a new multivariate nonlinear integral operator, to the best of our knowledge.

Proof.

1. For $f(\mathbf{x}) = a$ with $a \in \mathbb{R}$ and $1 + sa \neq 0$, since $\int_\Omega d\mathbf{x} = 1$, we get

$$T(f)(s) = \int_\Omega \frac{1}{1+sa}d\mathbf{x} = \frac{1}{1+sa}\int_\Omega d\mathbf{x} = \frac{1}{1+sa}.$$

   In particular, obviously, we have $T(0)(s) = 1/(1 + s \times 0) = 1$, which also corresponds to $T(f)(0)$.

2. Under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, it is clear that $1/[1 + sf(\mathbf{x})] \geq 0$, implying that $T(f)(s) \geq 0$. Similarly, under the assumption $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, it is immediate that $T(f)(s) \leq 0$.

3. Let us perform the multivariate change of variables $\mathbf{x}_* = (x_1^*, \ldots, x_n^*)$ with $x_i^* \in \{x_i, 1 - x_i\}$ for each $i = 1, \ldots, n$. Then the domain of integration Ω remains unchanged, and the corresponding Jacobian is $(-1)^r$, where r denotes the number of x_i^* such that $x_i^* = 1 - x_i$ for any $i = 1, \ldots, n$. As a result, we have

$$T(f_*)(s) = \int_\Omega \frac{1}{1+sf_*(\mathbf{x})}d\mathbf{x}_* = \int_\Omega \frac{1}{1+sf(\mathbf{x})}|(-1)^r|d\mathbf{x} = T(f)(s).$$

4. For any $b \in \mathbb{R}$ such that $1 + bs \neq 0$, we have

$$\begin{aligned} T(f+b)(s) &= \int_\Omega \frac{1}{1+s[f(\mathbf{x})+b]}d\mathbf{x} = \int_\Omega \frac{1}{1+bs+sf(\mathbf{x})}d\mathbf{x} \\ &= \frac{1}{1+bs}\int_\Omega \frac{1}{1+[s/(1+bs)]f(\mathbf{x})}d\mathbf{x} \\ &= \frac{1}{1+bs}T(f)\left(\frac{s}{1+bs}\right). \end{aligned}$$

5. For $f(\mathbf{x}) = (1/s)[1/g(\mathbf{x}) - 1]$, where $g(\mathbf{x})$, $\mathbf{x} \in \Omega$, we have

$$T(f)(s) = \int_{\Omega} \frac{1}{1 + s(1/s)[1/g(\mathbf{x}) - 1]} d\mathbf{x} = \int_{\Omega} g(\mathbf{x}) d\mathbf{x}.$$

6. After some mathematical manipulations, we have

$$\begin{aligned} T\left(\frac{1}{f}\right)(s) &= \int_{\Omega} \frac{1}{1 + s(1/f)(\mathbf{x})} d\mathbf{x} = \int_{\Omega} \frac{f(\mathbf{x})/s}{1 + f(\mathbf{x})/s} d\mathbf{x} \\ &= \int_{\Omega} \left(1 - \frac{1}{1 + f(\mathbf{x})/s}\right) d\mathbf{x} = \int_{\Omega} d\mathbf{x} - \int_{\Omega} \frac{1}{1 + f(\mathbf{x})/s} d\mathbf{x} \\ &= 1 - T(f)\left(\frac{1}{s}\right). \end{aligned}$$

As a result, the IR operator satisfies the following nonlinear equation:

$$T\left(\frac{1}{f}\right)(s) + T(f)\left(\frac{1}{s}\right) = 1.$$

7. For any positive integer m, under the multivariate Leibnitz integral rule assumptions, we have

$$\begin{aligned} \frac{\partial^m}{\partial s^m} T(f)(s) &= \frac{\partial^m}{\partial s^m} \left[\int_{\Omega} \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x}\right] = \int_{\Omega} \frac{\partial^m}{\partial s^m} \left[\frac{1}{1 + sf(\mathbf{x})}\right] d\mathbf{x} \\ &= (-1)^m m! \int_{\Omega} [f(\mathbf{x})]^m \frac{1}{[1 + sf(\mathbf{x})]^{m+1}} d\mathbf{x}. \end{aligned}$$

By considering $s = 0$, we get

$$\frac{(-1)^m}{m!} \left.\frac{\partial^m}{\partial s^m} T(f)(s)\right|_{s=0} = \int_{\Omega} [f(\mathbf{x})]^m \frac{1}{[1 + 0 \times f(\mathbf{x})]^{m+1}} d\mathbf{x} = \int_{\Omega} [f(\mathbf{x})]^m d\mathbf{x}.$$

8. Under the multivariate interchange of integral and sum rule assumptions

   - and the assumption $|sf(\mathbf{x})| < 1$ for any $\mathbf{x} \in \Omega$, by applying the following classical geometric series formula: $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $|x| < 1$, we have

$$T(f)(s) = \int_{\Omega} \left\{\sum_{k=0}^{\infty} (-s)^k [f(\mathbf{x})]^k\right\} d\mathbf{x} = \sum_{k=0}^{\infty} (-s)^k \int_{\Omega} [f(\mathbf{x})]^k d\mathbf{x}.$$

   - and the assumption $|sf(\mathbf{x})| > 1$ for any $\mathbf{x} \in \Omega$, by arranging the expression of the IR operator and applying again the classical geometric series formula, we have

$$\begin{aligned} T(f)(s) &= \int_{\Omega} \frac{[sf(\mathbf{x})]^{-1}}{1 + [sf(\mathbf{x})]^{-1}} d\mathbf{x} = \int_{\Omega} \left\{\sum_{k=0}^{\infty} (-1)^k s^{-(k+1)} [f(\mathbf{x})]^{-(k+1)}\right\} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} (-1)^k s^{-(k+1)} \int_{\Omega} [f(\mathbf{x})]^{-(k+1)} d\mathbf{x}. \end{aligned}$$

9. Under the multivariate interchange of two integrals rule assumptions, for any $t \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} U(f)(t) &= \int_{[0,1]} T(f)(st) ds = \int_{[0,1]} \left[\int_{\Omega} \frac{1}{1 + stf(\mathbf{x})} d\mathbf{x}\right] ds \\ &= \int_{\Omega} \left[\int_{[0,1]} \frac{1}{1 + stf(\mathbf{x})} ds\right] d\mathbf{x} = \int_{\Omega} \left.\frac{\log[1 + stf(\mathbf{x})]}{tf(\mathbf{x})}\right|_{s=0}^{s=1} d\mathbf{x} \\ &= \int_{\Omega} \frac{\log[1 + tf(\mathbf{x})]}{tf(\mathbf{x})} d\mathbf{x}. \end{aligned}$$

The proposition is proved. □

Let us now discuss the results in Proposition 2.1. Items 1 and 2 are about simple values and the immediate sign of the IR operator. Item 3 implies that if we have the IR operator of a function f, then we can derive $2^n - 1$ other functions with the same IR operator by changing the components of $\mathbf{x}$ in a specific way. Item 4 shows a relationship between the IR operator of a translated function and the IR operator of this function. Also, it implies that the IR operator satisfies the following nonlinear equation:

$$T(f+b)(s) - \frac{1}{1+bs}T(f)\left(\frac{s}{1+bs}\right) = 0.$$

Item 5 exhibits a specific function depending on a secondary function, denoted by g, for which its IR operator is reduced to the integral of g (independently of s). The construction of g is a bit artificial since it depends on s, but this result shows that all the existing integral quantities can fit with the IR operator. Also, with an appropriate choice for g, we can connect this general integral with some operator defined on Ω. For instance, by taking $n = 1$ and $g(x) = g(x)(r) = [(-1)^r/r!][\log(x)]^r$, where r denotes a positive integer, we obtain the Albazy Altememe transform defined on $[0, 1]$ (see [24]). Item 6 presents an intriguing nonlinear equation satisfied by the IR operator, i.e.,

$$T\left(\frac{1}{f}\right)(s) + T(f)\left(\frac{1}{s}\right) = 1.$$

Item 7 shows that the integral of the integer exponent of a function can be derived from differentiations of its IR operator. This can be interesting when the IR operator of this function is more easily determinable than the integral of its integer exponent version. Item 8 shows some series expansions of the IR operator, implying that it can be used as an intermediary tool to determine some properties of existing series, among others. Finally, item 9 derives a new multivariate nonlinear operator that is defined with logarithmic and ratio functions. This logarithmic-ratio operator will naturally appear in some coming inequalities involving the IR operator.

As an additional comment, based on items 5 and 7 with $m = 1$, we observe that the IR operator satisfies the following differential equation:

$$\left.\frac{\partial}{\partial s}T(f)(s)\right|_{s=0} = -T\left[\frac{1}{s}\left(\frac{1}{f} - 1\right)\right](s),$$

since they are both equal to $\int_\Omega f(\mathbf{x})d\mathbf{x}$.

Another interest of the IR operator is its ability to generate or be involved in multiple kinds of inequalities. The next section is devoted to some of the most attractive of them.

General inequalities

The next result compares $|T(f)(s)|$ and $T(|f|)(-|s|)$.

Proposition 3.2 *Let $s \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function.*

- *Under the assumption $|sf(\mathbf{x})| \leq 1$ for any $\mathbf{x} \in \Omega$, we have*

$$|T(f)(s)| \leq T(|f|)(-|s|).$$

- *Under the assumption $|sf(\mathbf{x})| \geq 1$ for any $\mathbf{x} \in \Omega$, we have*

$$|T(f)(s)| \leq -T(|f|)(-|s|).$$

Proof. By the triangle inequality (for the integral), we have

$$|T(f)(s)| = \left|\int_\Omega \frac{1}{1+sf(\mathbf{x})}d\mathbf{x}\right| \leq \int_\Omega \frac{1}{|1+sf(\mathbf{x})|}d\mathbf{x}.$$

Let us now distinguish the two sets of assumptions.

- Under the assumption $|sf(\mathbf{x})| \leq 1$ for any $\mathbf{x} \in \Omega$, the triangle inequality (for the sum) gives $|1+sf(\mathbf{x})| \geq 1-|sf(\mathbf{x})| \geq 0$, so

$$|T(f)(s)| \leq \int_\Omega \frac{1}{1-|sf(\mathbf{x})|}d\mathbf{x} = T(|f|)(-|s|).$$

- Under the assumption $|sf(\mathbf{x})| \geq 1$ for any $\mathbf{x} \in \Omega$, the triangle inequality gives $|1+sf(\mathbf{x})| \geq -[1-|sf(\mathbf{x})|] \geq 0$, so

$$|T(f)(s)| \leq -\int_\Omega \frac{1}{1-|sf(\mathbf{x})|}d\mathbf{x} = -T(|f|)(-|s|).$$

This ends the proof. □

The proposition below deals with some order properties of the IR operator.

Proposition 3.3 *Let $s \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function.*

- *Under the assumptions $f(\mathbf{x}) \leq g(\mathbf{x})$, $s \geq 0$, $1+sf(\mathbf{x}) \geq 0$ and $1+sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have*

$$T(g)(s) \leq T(f)(s).$$

- *Under the assumptions $f(\mathbf{x}) \leq g(\mathbf{x})$, $s \leq 0$, $1+sf(\mathbf{x}) \geq 0$ and $1+sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \leq T(g)(s).$$

  This inequality also holds if $s \leq 0$, $1+sf(\mathbf{x}) \leq 0$ and $1+sg(\mathbf{x}) \leq 0$ for any $\mathbf{x} \in \Omega$.

Proof.

- Under the assumptions $f(\mathbf{x}) \leq g(\mathbf{x})$, $s \geq 0$, $1+sf(\mathbf{x}) \geq 0$ and $1+sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have $sf(\mathbf{x}) \leq sg(\mathbf{x})$, so $1+sf(\mathbf{x}) \leq 1+sg(\mathbf{x})$, and

$$\frac{1}{1+sg(\mathbf{x})} \leq \frac{1}{1+sf(\mathbf{x})},$$

  which implies that

$$T(g)(s) = \int_\Omega \frac{1}{1+sg(\mathbf{x})}d\mathbf{x} \leq \int_\Omega \frac{1}{1+sf(\mathbf{x})}d\mathbf{x} = T(f)(s).$$

- Under the assumption $f(\mathbf{x}) \leq g(\mathbf{x})$, $s \leq 0$, $1+sf(\mathbf{x}) \geq 0$ and $1+sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have $sf(\mathbf{x}) \geq sg(\mathbf{x})$, so $1+sf(\mathbf{x}) \geq 1+sg(\mathbf{x})$, and

$$\frac{1}{1+sf(\mathbf{x})} \leq \frac{1}{1+sg(\mathbf{x})},$$

  which implies that

$$T(f)(s) = \int_\Omega \frac{1}{1+sf(\mathbf{x})}d\mathbf{x} \leq \int_\Omega \frac{1}{1+sg(\mathbf{x})}d\mathbf{x} = T(g)(s).$$

  We obtain the same result with the same proof for $s \leq 0$, $1+sf(\mathbf{x}) \leq 0$ and $1+sg(\mathbf{x}) \leq 0$ for any $\mathbf{x} \in \Omega$; the order of the inequality involving the ratio terms remains unchanged.

This ends the proof. □

Note that, in the context of Proposition 3.3, the cases $1 + sf(\mathbf{x}) \leq 0$ and $1 + sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, or $1 + sf(\mathbf{x}) \geq 0$ and $1 + sg(\mathbf{x}) \leq 0$ for any $\mathbf{x} \in \Omega$, imply that $T(f)(s) \leq 0$ and $T(g)(s) \geq 0$, or $T(f)(s) \geq 0$ and $T(g)(s) \leq 0$, respectively. Thus, we have obvious order properties thanks to the opposite sign. Their interest is, for this reason, secondary.

Some Lipschitz type inequalities for the IR operator are presented in the next proposition.

Proposition 3.4

1. *Let $s \in \mathbb{R}$, and $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x} \in \Omega$, be two functions. Under the assumptions $sf(\mathbf{x}) \geq 0$ and $sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have*

$$|T(f)(s) - T(g)(s)| \leq |s| \int_{\Omega} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x}.$$

2. *Let $(s, t) \in \mathbb{R}^2$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a (at least integrable) function. Under the assumption $sf(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have*

$$|T(f)(s) - T(f)(t)| \leq \kappa |s - t|,$$

where $\kappa = \int_{\Omega} |f(\mathbf{x})| d\mathbf{x}$.

Proof.

- We have

$$\begin{aligned} T(f)(s) - T(g)(s) &= \int_{\Omega} \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x} - \int_{\Omega} \frac{1}{1 + sg(\mathbf{x})} d\mathbf{x} \\ &= \int_{\Omega} \left[\frac{1}{1 + sf(\mathbf{x})} - \frac{1}{1 + sg(\mathbf{x})} \right] d\mathbf{x} \\ &= s \int_{\Omega} \left\{ \frac{g(\mathbf{x}) - f(\mathbf{x})}{[1 + sf(\mathbf{x})][1 + sg(\mathbf{x})]} \right\} d\mathbf{x}. \end{aligned}$$

  Under the assumptions $sf(\mathbf{x}) \geq 0$ and $sg(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have $[1 + sf(\mathbf{x})][1 + sg(\mathbf{x})] \geq 1$. It follows from the triangle inequality that

$$\begin{aligned} |T(f)(s) - T(g)(s)| &\leq |s| \int_{\Omega} \left\{ \frac{|f(\mathbf{x}) - g(\mathbf{x})|}{[1 + sf(\mathbf{x})][1 + sg(\mathbf{x})]} \right\} d\mathbf{x} \\ &\leq |s| \int_{\Omega} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x}. \end{aligned}$$

- We have

$$\begin{aligned} T(f)(s) - T(f)(t) &= \int_{\Omega} \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x} - \int_{\Omega} \frac{1}{1 + tf(\mathbf{x})} d\mathbf{x} \\ &= \int_{\Omega} \left[\frac{1}{1 + sf(\mathbf{x})} - \frac{1}{1 + tf(\mathbf{x})} \right] d\mathbf{x} \\ &= (t - s) \int_{\Omega} \left\{ \frac{f(\mathbf{x})}{[1 + sf(\mathbf{x})][1 + tf(\mathbf{x})]} \right\} d\mathbf{x}. \end{aligned}$$

  Under the assumption $sf(\mathbf{x}) \geq 0$ and $tf(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have $[1 + sf(\mathbf{x})][1 + tf(\mathbf{x})] \geq 1$. It follows from the triangle inequality that

$$|T(f)(s) - T(g)(s)| \leq |s - t| \int_{\Omega} \left\{ \frac{|f(\mathbf{x})|}{[1 + sf(\mathbf{x})][1 + tf(\mathbf{x})]} \right\} d\mathbf{x} \leq \kappa |s - t|.$$

The desired inequalities are proved. □

In particular, item 1 in Proposition 3.4 implies that, for any positive sequence of functions $(f_n)_{n\in\mathbb{N}}$ and f such that

$$\lim_{n\to\infty}\int_\Omega |f_n(\mathbf{x})-f(\mathbf{x})|d\mathbf{x}=0,$$

we have

$$\lim_{n\to\infty}|T(f_n)(s)-T(f)(s)|=0,$$

for any $s>0$. In other terms, the L_1 convergence of $(f_n)_{n\in\mathbb{N}}$ to f implies the simple convergence of $[T(f_n)(s)]_{n\in\mathbb{N}}$ to $T(f)(s)$.

Item 2 implies that the IR operator $T(f)(s)$ can be Lipschitz continuous with respect to s.

The proposition below is about inequalities involving the IR operator of the product of functions.

Proposition 3.5 *Let $s\in\mathbb{R}$, $p\geq 1$ and $q\geq 1$ such that $1/p+1/q=1$, and $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x}\in\Omega$, be two functions. Under the multivariate interchange of integral and sum rule assumptions, and the assumptions $|sf(\mathbf{x})|<1$, $|sg(\mathbf{x})|<1$ and $|sf(\mathbf{x})g(\mathbf{x})|<1$ for any $\mathbf{x}\in\Omega$, we have*

$$|T(fg)(s)|\leq\left[T(|f|^p)(-|s|^p)\right]^{1/p}\left[T(|g|^q)(-|s|^q)\right]^{1/q}.$$

Proof. Under the multivariate interchange of integral and sum rule assumptions, and the assumptions $|sf(\mathbf{x})|<1$, $|sg(\mathbf{x})|<1$ and $|sf(\mathbf{x})g(\mathbf{x})|<1$ for any $\mathbf{x}\in\Omega$, by applying (multiple times) item 8 of Proposition 2.1, (multiple times) the triangle inequality, and the Hölder inequality under its integral and series version in a row, we have

$$\begin{aligned}|T(fg)(s)|&=\left|\sum_{k=0}^{\infty}(-s)^k\int_\Omega[(fg)(\mathbf{x})]^k d\mathbf{x}\right|\\&\leq\sum_{k=0}^{\infty}|s|^k\int_\Omega|f(\mathbf{x})|^k|g(\mathbf{x})|^k d\mathbf{x}\\&\leq\sum_{k=0}^{\infty}|s|^k\left[\int_\Omega|f(\mathbf{x})|^{pk}d\mathbf{x}\right]^{1/p}\left[\int_\Omega|g(\mathbf{x})|^{qk}d\mathbf{x}\right]^{1/q}\\&=\sum_{k=0}^{\infty}\left[(|s|^p)^k\int_\Omega\{|f(\mathbf{x})|^p\}^k d\mathbf{x}\right]^{1/p}\left[(|s|^q)^k\int_\Omega\{|g(\mathbf{x})|^q\}^k d\mathbf{x}\right]^{1/q}\\&\leq\left[\sum_{k=0}^{\infty}(|s|^p)^k\int_\Omega\{|f(\mathbf{x})|^p\}^k d\mathbf{x}\right]^{1/p}\left[\sum_{k=0}^{\infty}(|s|^q)^k\int_\Omega\{|g(\mathbf{x})|^q\}^k d\mathbf{x}\right]^{1/q}\\&=\left[T(|f|^p)(-|s|^p)\right]^{1/p}\left[T(|g|^q)(-|s|^q)\right]^{1/q}.\end{aligned}$$

The stated result is obtained. □

Proposition 3.5 shows that the IR operator satisfies a kind of Hölder inequality.

The proposition below deals with an inequality for the IR operator of the sum of functions.

Proposition 3.6 *Let $s\in\mathbb{R}$, m be a positive integer, and $f_1(\mathbf{x}),\ldots,f_m(\mathbf{x})$, $\mathbf{x}\in\Omega$, be m functions. Under the assumptions $sf_j(\mathbf{x})\in[0,1]$ for any $\mathbf{x}\in\Omega$ and $j=1,\ldots,m$, and $f_1(\mathbf{x}),\ldots,f_m(\mathbf{x})$ are all non-increasing or all non-decreasing in each of its variables (with respect to x_i for any $i=1,\ldots,n$), we have*

$$T\left(\sum_{j=1}^{m}f_j\right)(s)\geq\prod_{j=1}^{m}T(f_j)(s).$$

Proof. We have

$$T\left(\sum_{j=1}^{m} f_j\right)(s) = \int_{\Omega} \frac{1}{1 + s\sum_{j=1}^{m} f_j(\mathbf{x})} d\mathbf{x}.$$

The rest of proof is based on the application of two known inequalities: the Weierstrass product inequality and the multivariate continuous version of the Chebyshev sum inequality, both recalled in Appendix (see also [23]). Under the assumption $sf_j(\mathbf{x}) \in [0, 1]$ for any $\mathbf{x} \in \Omega$ and $j = 1, \ldots, m$, the Weierstrass product inequality implies that

$$1 + s\sum_{j=1}^{m} f_j(\mathbf{x}) \le \prod_{j=1}^{m} [1 + sf_j(\mathbf{x})].$$

Hence, since $1 + sf_j(\mathbf{x}) \ge 0$ for any $i = 1, \ldots, n$, we have

$$\frac{1}{1 + s\sum_{j=1}^{m} f_j(\mathbf{x})} \ge \prod_{j=1}^{m} \frac{1}{1 + sf_j(\mathbf{x})}.$$

Therefore, we get

$$T\left(\sum_{j=1}^{m} f_j\right)(s) \ge \int_{\Omega} \left[\prod_{j=1}^{m} \frac{1}{1 + sf_j(\mathbf{x})}\right] d\mathbf{x}.$$

Now, under the assumption that $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are all non-increasing or all non-decreasing in each of its variables, owing to the multivariate continuous version of the Chebyshev sum inequality (derived from the univariate case by a simple recurrence with the use of $\int_{\Omega} d\mathbf{x} = 1$), we obtain

$$\int_{\Omega} \left[\prod_{j=1}^{m} \frac{1}{1 + sf_j(\mathbf{x})}\right] d\mathbf{x} \ge \prod_{j=1}^{m} \left[\int_{\Omega} \frac{1}{1 + sf_j(\mathbf{x})} d\mathbf{x}\right] = \prod_{j=1}^{m} T(f_j)(s).$$

This concludes the proof of the proposition. □

Proposition 3.6 implies that we can lower bound the IR of a sum of several functions based on the IR operator of each of the involved functions.

For some coming results, the following lemma is needed.

Lemma 3.7 *Let* $\Upsilon \subseteq \mathbb{R}$, $s \in \mathbb{R}$ *and*

$$\psi(x)(s) = \frac{1}{1 + sx}, \quad x \in \Upsilon. \tag{2}$$

Then

- *if* $sx \ge -1$, $\psi(x)(s)$ *is convex with respect to* x *(for a given* s*), and with respect to* s *(for a given* x*),*
- *if* $sx \le -1$, $\psi(x)(s)$ *is concave with respect to* x *(for a given* s*), and with respect to* s *(for a given* x*).*

Proof. The proof is based on the second derivative of $\psi(x)(s)$. Let us first consider it with respect to x. We have

$$\frac{\partial^2}{\partial x^2}\psi(x)(s) = 2\frac{s^2}{(1 + sx)^3}.$$

Let us now distinguish the two sets of assumptions.

- It is clear that, if $sx \geq -1$, we have $s^2 \geq 0$ and $(1+sx)^3 \geq 0$, so $\partial^2\psi(x)(s)/(\partial x^2) \geq 0$, which implies that $\psi(x)(s)$ is convex with respect to x. Since x and s have symmetric roles, i.e., "$\psi(x)(s) = \psi(s)(x)$" (without considering the sets of definition), the same conclusion holds for the variable s.

- Conversely, if $sx \leq -1$, we have $s^2 \geq 0$ and $(1+sx)^3 \leq 0$, so $\partial^2\psi(x)(s)/(\partial x^2) \leq 0$, which implies that $\psi(x)(s)$ is concave with respect to x. The same result holds with respect to s.

The lemma is established. □

The convex and concave properties of the function $\psi(x)(s)$ in Equation (2) are at the center of some important inequalities satisfied by the IR operator. In particular, under some assumptions, it makes the IR operator convex. This property ensures that the IR operator of a convex combination of functions is less than or equal to the convex combination of their individual IR operators. This claim is detailed in the next result.

Proposition 3.8 *Let $s \in \mathbb{R}$, $\alpha \in [0,1]$, and $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x} \in \Omega$, be two functions.*

- *Under the assumptions $sf(\mathbf{x}) \geq -1$ and $sg(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T[\alpha f + (1-\alpha)g](s) \leq \alpha T(f)(s) + (1-\alpha)T(g)(s).$$

- *Under the assumptions $sf(\mathbf{x}) \leq -1$ and $sg(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T[\alpha f + (1-\alpha)g](s) \geq \alpha T(f)(s) + (1-\alpha)T(g)(s).$$

Proof. Based on the function $\psi(x)(s)$ in Equation (2), we can write the IR operator of f as

$$T(f)(s) = \int_{\Omega} \psi[f(\mathbf{x})](s) d\mathbf{x}.$$

Let us now distinguish the two sets of assumptions.

- Under the assumptions $sf(\mathbf{x}) \geq -1$ and $sg(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have

$$1 + s[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})] = \alpha[1 + sf(\mathbf{x})] + (1-\alpha)[1 + sg(\mathbf{x})] \geq 0.$$

  Over the corresponding interval of values of x (for a given s), based on Lemma 3.7, $\psi(x)(s)$ is convex with respect to x, implying that $\psi[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})](s) \leq \alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[g(\mathbf{x})](s)$. As a result, we have

$$\begin{aligned} T[\alpha f + (1-\alpha)g](s) &= \int_{\Omega} \psi[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})](s) d\mathbf{x} \\ &\leq \int_{\Omega} \{\alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[g(\mathbf{x})](s)\} d\mathbf{x} \\ &= \alpha \int_{\Omega} \psi[f(\mathbf{x})](s) d\mathbf{x} + (1-\alpha) \int_{\Omega} \psi[g(\mathbf{x})](s) d\mathbf{x} \\ &= \alpha T(f)(s) + (1-\alpha)T(g)(s). \end{aligned}$$

- Under the assumptions $sf(\mathbf{x}) \leq -1$ and $sg(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have

$$1 + s[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})] = \alpha[1 + sf(\mathbf{x})] + (1-\alpha)[1 + sg(\mathbf{x})] \leq 0.$$

  Over the corresponding interval of values of x (for a given s), based on Lemma 3.7, $\psi(x)(s)$ is concave with respect to x, implying that $\psi[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})](s) \geq \alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[g(\mathbf{x})](s)$. As a result, we have

$$\begin{aligned} T[\alpha f + (1-\alpha)g](s) &= \int_{\Omega} \psi[\alpha f(\mathbf{x}) + (1-\alpha)g(\mathbf{x})](s) d\mathbf{x} \\ &\geq \int_{\Omega} \{\alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[g(\mathbf{x})](s)\} d\mathbf{x} \\ &= \alpha \int_{\Omega} \psi[f(\mathbf{x})](s) d\mathbf{x} + (1-\alpha) \int_{\Omega} \psi[g(\mathbf{x})](s) d\mathbf{x} \\ &= \alpha T(f)(s) + (1-\alpha)T(g)(s). \end{aligned}$$

The desired inequalities are obtained. This ends the proof of the proposition. □

The convex and concave inequalities in Proposition 3.8 may facilitate mathematical analysis and optimization, providing a foundation for exploring relationships between various functions within the context of the IR operator. In particular, the result below is a consequence.

Proposition 3.9 *Let $s \in \mathbb{R}$, m be a positive integer, and $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$, $\mathbf{x} \in \Omega$, be m functions. Under the assumptions $sf_j(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$ and $j = 1, \ldots, m$, we have*

$$T\left(\sum_{j=1}^{m} f_j\right)(s) \leq \frac{1}{m} \sum_{j=1}^{m} T(f_j)(sm).$$

Proof. Under the assumptions $sf_j(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$ and $j = 1, \ldots, m$, it is clear that $(s/m)f_j(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$ and $j = 1, \ldots, m$. Therefore, owing to the convexity of the IR operator proved in Proposition 3.8 and a direct induction, we have

$$T\left(\sum_{j=1}^{m} f_j\right)(s) = T\left[\frac{1}{m}\sum_{j=1}^{m}(mf_j)\right](s) = T\left(\frac{1}{m}\sum_{j=1}^{m} f_j\right)(ms) \leq \frac{1}{m}\sum_{j=1}^{m} T(f_j)(sm).$$

The result is demonstrated. □

In some sense, Proposition 3.9 completes Proposition 3.6 by determining an upper bound for $T\left(\sum_{j=1}^{m} f_j\right)(s)$, but under different assumptions on the involved quantities.

The proposition below also investigates the convex and concave properties with respect to the parameter s of the IR operator.

Proposition 3.10 *Let $(s,t) \in \mathbb{R}^2$, $\alpha \in [0,1]$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function.*

- *Under the assumption $sf(\mathbf{x}) \geq -1$ and $tf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)[\alpha s + (1-\alpha)t] \leq \alpha T(f)(s) + (1-\alpha)T(f)(t).$$

- *Under the assumptions $sf(\mathbf{x}) \leq -1$ and $tf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)[\alpha s + (1-\alpha)t] \geq \alpha T(f)(s) + (1-\alpha)T(f)(t).$$

Proof. As in the proof of Proposition 3.8, based on the function $\psi(x)(s)$ as described in Lemma 3.7, we can write the IR operator of f as

$$T(f)(s) = \int_{\Omega} \psi[f(\mathbf{x})](s) d\mathbf{x}.$$

Let us now distinguish the two sets of assumptions.

- Under the assumptions $sf(\mathbf{x}) \geq -1$ and $tf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have

$$1 + [\alpha s + (1-\alpha)t]f(\mathbf{x}) = \alpha[1 + sf(\mathbf{x})] + (1-\alpha)[1 + tf(\mathbf{x})] \geq 0.$$

Over the corresponding interval of values of s (for a given x), based on Lemma 3.7, $\psi(x)(s)$ is convex with respect to s, implying that $\psi[f(\mathbf{x})][\alpha s + (1-\alpha)t] \leq \alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[f(\mathbf{x})](t)$. As a result, we have

$$\begin{aligned} T(f)[\alpha s + (1-\alpha)t] &= \int_\Omega \psi[f(\mathbf{x})][\alpha s + (1-\alpha)t]d\mathbf{x} \\ &\leq \int_\Omega \{\alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[f(\mathbf{x})](t)\}\, d\mathbf{x} \\ &= \alpha \int_\Omega \psi[f(\mathbf{x})](s)d\mathbf{x} + (1-\alpha)\int_\Omega \psi[f(\mathbf{x})](t)d\mathbf{x} \\ &= \alpha T(f)(s) + (1-\alpha)T(f)(t). \end{aligned}$$

- Under the assumptions $sf(\mathbf{x}) \leq -1$ and $tf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have

$$1 + [\alpha s + (1-\alpha)t]f(\mathbf{x}) = \alpha[1 + sf(\mathbf{x})] + (1-\alpha)[1 + tf(\mathbf{x})] \leq 0.$$

Over the corresponding interval of values of s (for a given x), based on Lemma 3.7, $\psi(x)(s)$ is concave with respect to s, implying that $\psi[f(\mathbf{x})][\alpha s + (1-\alpha)t] \geq \alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[f(\mathbf{x})](t)$. As a result, we have

$$\begin{aligned} T(f)[\alpha s + (1-\alpha)t] &= \int_\Omega \psi[f(\mathbf{x})][\alpha s + (1-\alpha)t]d\mathbf{x} \\ &\geq \int_\Omega \{\alpha\psi[f(\mathbf{x})](s) + (1-\alpha)\psi[f(\mathbf{x})](t)\}\, d\mathbf{x} \\ &= \alpha \int_\Omega \psi[f(\mathbf{x})](s)d\mathbf{x} + (1-\alpha)\int_\Omega \psi[f(\mathbf{x})](t)d\mathbf{x} \\ &= \alpha T(f)(s) + (1-\alpha)T(f)(t). \end{aligned}$$

The desired inequalities are established. This ends the proof of the proposition. □

The result below is about simple bounds of the IR operator involving the integral of the main function.

Proposition 3.11 *Let $s \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function. Then*

- *under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \geq \frac{1}{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

- *under the assumption $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \leq \frac{1}{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

Proof. We propose two proofs for these results: one is based on the Jensen inequality, and the other is based on the Cauchy-Schwarz inequality.

Using the Jensen inequality: As in the proof of Proposition 3.8, based on the function $\psi(x)(s)$ as described in Lemma 3.7, we can write the IR operator of f as

$$T(f)(s) = \int_\Omega \psi[f(\mathbf{x})](s)d\mathbf{x}.$$

Let us now distinguish the two sets of assumptions.

- Under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, based on Lemma 3.7, $\psi(x)(s)$ is convex with respect to x for the associated interval of values. It follows from the Jensen inequality and $\int_\Omega d\mathbf{x} = 1$ that

$$T(f)(s) = \int_\Omega \psi[f(\mathbf{x})](s)d\mathbf{x} \geq \psi\left[\int_\Omega f(\mathbf{x})d\mathbf{x}\right](s) = \frac{1}{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

- Similarly, under the assumptions $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, based on Lemma 3.7, $\psi(x)(s)$ is concave with respect to x for the associated interval of values. It follows from the Jensen inequality and $\int_\Omega d\mathbf{x} = 1$ that

$$T(f)(s) = \int_\Omega \psi[f(\mathbf{x})](s)d\mathbf{x} \leq \psi\left[\int_\Omega f(\mathbf{x})d\mathbf{x}\right](s) = \frac{1}{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

Using the Cauchy-Schwarz inequality: A completely different approach is proposed here, beyond the convexity concept. Let us now distinguish the two sets of assumptions.

- Under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, it follows from $\int_\Omega d\mathbf{x} = 1$ and the Cauchy-Schwarz inequality that

$$\begin{aligned} 1 &= \int_\Omega d\mathbf{x} = \int_\Omega \sqrt{\frac{1 + sf(\mathbf{x})}{1 + sf(\mathbf{x})}} d\mathbf{x} \leq \sqrt{\int_\Omega \frac{1}{1 + sf(\mathbf{x})} d\mathbf{x}} \sqrt{\int_\Omega [1 + sf(\mathbf{x})] d\mathbf{x}} \\ &= \sqrt{T(f)(s)} \sqrt{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}. \end{aligned}$$

Hence, by taking the square of both sides of this inequality and making a division, we get

$$T(f)(s) \geq \frac{1}{1 + s\int_\Omega f(\mathbf{x})} d\mathbf{x}.$$

- Similarly, under the assumptions $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, it follows from $\int_\Omega d\mathbf{x} = 1$ and the Cauchy-Schwarz inequality that

$$\begin{aligned} 1 &= \int_\Omega d\mathbf{x} = \int_\Omega \sqrt{\frac{-[1 + sf(\mathbf{x})]}{-[1 + sf(\mathbf{x})]}} d\mathbf{x} \leq \sqrt{\int_\Omega \frac{1}{-[1 + sf(\mathbf{x})]} d\mathbf{x}} \sqrt{\int_\Omega \{-[1 + sf(\mathbf{x})]\} d\mathbf{x}} \\ &= \sqrt{-T(f)(s)} \sqrt{-\left[1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}\right]}. \end{aligned}$$

Thus, by squaring the two sides of this inequality and making a division, we obtain

$$-T(f)(s) \geq \frac{1}{-\left[1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}\right]},$$

which implies that

$$T(f)(s) \leq \frac{1}{1 + s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

With two different proofs, the desired inequalities are established. This ends the proof of the proposition. □

The result below explores a hierarchy that exists between the IR operator and another type of multivariate integral transform.

Proposition 3.12 *Let $s \in [0, 1]$, and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function. Then, under the assumption $f(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \leq V(f)(s) \leq [T(f)(1)]^{1/s},$$

where

$$V(f)(s) = \int_{\Omega} \frac{1}{[1+f(\mathbf{x})]^s} d\mathbf{x}, \tag{3}$$

which also represents a new multivariate nonlinear integral operator, to the best of our knowledge.

Proof. The reversed Bernoulli inequality can be formulated as follows: $(1+x)^r \leq 1+rx$ for any $r \in [0, 1]$ and $x \geq -1$. Therefore, under the assumption $f(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$ and $s \in [0, 1]$, we have $1+sf(\mathbf{x}) \geq [1+f(\mathbf{x})]^s \geq 0$, which implies that

$$T(f)(s) = \int_{\Omega} \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} \leq \int_{\Omega} \frac{1}{[1+f(\mathbf{x})]^s} d\mathbf{x} = V(f)(s).$$

The left side of the inequality is demonstrated. For the right side, it follows from the Hölder inequality with the parameter $1/s \geq 1$ and $\int_{\Omega} d\mathbf{x} = 1$ that

$$V(f)(s) \leq \left[\int_{\Omega} \frac{1}{1+f(\mathbf{x})} d\mathbf{x}\right]^s \left(\int_{\Omega} d\mathbf{x}\right)^{1-s} = [T(f)(1)]^s.$$

The stated inequality is established. □

A complementary result to Proposition 3.12 is provided below.

Proposition 3.13 *Let $s \in \mathbb{R} \setminus (0, 1)$, and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function. Under the assumptions $f(\mathbf{x}) \geq -1$ and $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \geq V(f)(s),$$

where $V(f)(s)$ is defined in Equation (3).

Proof. The (standard) Bernoulli inequality can be stated as follows: $(1+x)^r \geq 1+rx$ for any $r \in \mathbb{R} \setminus (0, 1)$ and $x \geq -1$. As a result, under the assumptions $f(\mathbf{x}) \geq -1$ and $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we obtain $[1+f(\mathbf{x})]^s \geq 1+sf(\mathbf{x}) \geq 0$, so

$$T(f)(s) = \int_{\Omega} \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} \geq \int_{\Omega} \frac{1}{[1+f(\mathbf{x})]^s} d\mathbf{x} = V(f)(s).$$

This ends the proof. □

We now examine some inequalities involving two already-presented operators: the IR operator and the one described in Equation (1).

Proposition 3.14 *Let $s \in \mathbb{R}$, $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function, and $U(f)(s)$ be the multivariate nonlinear operator defined in Equation (1) (with $t = s$).*

- *Under the assumption $sf(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \leq U(f)(s).$$

- *Under the assumption $sf(\mathbf{x}) \in (-1, 0)$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \geq U(f)(s).$$

Proof. Based on the following well-known logarithmic inequality: $\log(1+x) \geq x/(1+x)$ for any $x > -1$, we get

$$\log[1+sf(\mathbf{x})] \geq \frac{sf(\mathbf{x})}{1+sf(\mathbf{x})}.$$

Let us now distinguish the two sets of assumptions.

- Under the assumption $sf(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \Omega$, we obtain $\log[1+sf(\mathbf{x})]/[sf(\mathbf{x})] \geq 1/[1+sf(\mathbf{x})]$. Therefore, upon integration, we have

$$T(f)(s) = \int_\Omega \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} \leq \int_\Omega \frac{\log[1+sf(\mathbf{x})]}{sf(\mathbf{x})} d\mathbf{x} = U(f)(s).$$

- Under the assumption $sf(\mathbf{x}) \in (-1, 0)$ for any $\mathbf{x} \in \Omega$, we have $\log[1+sf(\mathbf{x})]/[sf(\mathbf{x})] \leq 1/[1+sf(\mathbf{x})]$, which implies that

$$T(f)(s) = \int_\Omega \frac{1}{1+sf(\mathbf{x})} d\mathbf{x} \geq \int_\Omega \frac{\log[1+sf(\mathbf{x})]}{sf(\mathbf{x})} d\mathbf{x} = U(f)(s).$$

This ends the proof. □

In the proposition below, the IR operator is compared to a variant of the C operator (see [21]).

Proposition 3.15 *Let $s \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \Omega$, be a function.*

1. *Under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \geq W(f)(s) \geq 1 - s\int_\Omega f(\mathbf{x}) d\mathbf{x},$$

where

$$W(f)(s) = \int_\Omega e^{-sf(\mathbf{x})} d\mathbf{x}. \tag{4}$$

2. *Under the assumption $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(s) \leq W(f)(s).$$

3. *Under the assumption $sf(\mathbf{x}) \leq 1$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(-s) \geq \frac{1}{W(f)(s)}.$$

4. *Under the assumption $sf(\mathbf{x}) \in (0, 1)$ for any $\mathbf{x} \in \Omega$, we have*

$$T(f)(-s) \geq \frac{[W(f)(-s)]^2}{1 + s\int_\Omega f(\mathbf{x}) d\mathbf{x}}.$$

Proof.

1. Under the assumption $sf(\mathbf{x}) \geq -1$ for any $\mathbf{x} \in \Omega$, by using the following exponential inequality: $e^x \geq 1+x$ for any $x \in \mathbb{R}$, we have $e^{sf(\mathbf{x})} \geq 1+sf(\mathbf{x}) \geq 0$, which implies that

$$\frac{1}{1+sf(\mathbf{x})} \geq e^{-sf(\mathbf{x})}.$$

Therefore, we have

$$T(f)(s) = \int_\Omega \frac{1}{1+sf(\mathbf{x})}d\mathbf{x} \geq \int_\Omega e^{-sf(\mathbf{x})}d\mathbf{x} = W(f)(s).$$

Also, by using again the general exponential inequality, we get $e^{-sf(\mathbf{x})} \geq 1-sf(\mathbf{x})$ (which is not necessarily positive). Therefore, we have

$$W(f)(s) \geq \int_\Omega [1-sf(\mathbf{x})]d\mathbf{x} = 1 - s\int_\Omega f(\mathbf{x})d\mathbf{x}.$$

2. Under the assumption $sf(\mathbf{x}) \leq -1$ for any $\mathbf{x} \in \Omega$, so $1+sf(\mathbf{x}) \leq 0$, we still have $e^{sf(\mathbf{x})} \geq 1+sf(\mathbf{x})$ and, upon division, we get

$$\frac{1}{1+sf(\mathbf{x})} \leq e^{-sf(\mathbf{x})}.$$

This implies that

$$T(f)(s) = \int_\Omega \frac{1}{1+sf(\mathbf{x})}d\mathbf{x} \leq \int_\Omega e^{-sf(\mathbf{x})}d\mathbf{x} = W(f)(s).$$

3. Under the assumption $sf(\mathbf{x}) \leq 1$ for any $\mathbf{x} \in \Omega$, by using the following exponential inequality: $e^x \geq 1+x$ for any $x \in \mathbb{R}$, we have $e^{-sf(\mathbf{x})} \geq 1-sf(\mathbf{x}) \geq 0$, which implies that

$$e^{sf(\mathbf{x})} \leq \frac{1}{1-sf(\mathbf{x})}.$$

Therefore, by applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} 1 = \int_\Omega d\mathbf{x} &= \int_\Omega e^{-sf(\mathbf{x})/2}e^{sf(\mathbf{x})/2}d\mathbf{x} \leq \sqrt{\int_\Omega e^{-sf(\mathbf{x})}d\mathbf{x}}\sqrt{\int_\Omega e^{sf(\mathbf{x})}d\mathbf{x}} \\ &\leq \sqrt{\int_\Omega e^{-sf(\mathbf{x})}d\mathbf{x}}\sqrt{\int_\Omega \frac{1}{1-sf(\mathbf{x})}d\mathbf{x}} = \sqrt{W(f)(s)}\sqrt{T(f)(-s)}. \end{aligned}$$

By isolating $T(f)(-s)$ through standard manipulations, we obtain

$$T(f)(-s) \geq \frac{1}{W(f)(s)}.$$

4. Under the assumption $sf(\mathbf{x}) \in (0,1)$ for any $\mathbf{x} \in \Omega$, by using the following referenced exponential inequality: $e^{2x} \leq (1+x)/(1-x)$ for any $x \in (0,1)$, and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} W(f)(-s) = \int_\Omega e^{sf(\mathbf{x})}d\mathbf{x} &\leq \int_\Omega \sqrt{\frac{1+sf(\mathbf{x})}{1-sf(\mathbf{x})}}d\mathbf{x} \\ &\leq \sqrt{\int_\Omega [1+sf(\mathbf{x})]d\mathbf{x}}\sqrt{\int_\Omega \frac{1}{1-sf(\mathbf{x})}d\mathbf{x}} \\ &= \sqrt{1+s\int_\Omega f(\mathbf{x})d\mathbf{x}}\sqrt{T(f)(-s)}. \end{aligned}$$

Therefore, after some manipulations, we get

$$T(f)(-s) \geq \frac{[W(f)(-s)]^2}{1+s\int_\Omega f(\mathbf{x})d\mathbf{x}}.$$

This ends the proof. □

The multivariate integral operator in Equation (4) corresponds to the derivative of the C operator with respect to the parameter β (with reference to the notations in [21]), and then taken at $\beta = -s$. Thus, Proposition 3.15 discusses how this simple variant can be compared with the IR operator. In addition, upper and lower bounds for the IR operator can thus be easily derived.

The rest of the article is devoted to the expressions and applications of the IR operator by considering specific functions f of diverse dimensions.

IR operator calculations

The IR operator can be calculated for a wide panel of functions. The most immediate of them are considered in this section.

Univariate case

The IR operator of some univariate functions f is investigated, corresponding to the case $n = 1$, i.e., $\Omega = [0, 1]$ and $\mathbf{x} = x$.

For $f(x) = x$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx} dx = \frac{1}{s} \log(1+s),$$

with $s > -1$. The same goes for $f(x) = 1-x$, $x \in [0, 1]$.

For $f(x) = 1/x$, $x \in (0, 1]$, we obtain

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s(1/x)} dx = 1 + s\log\left(\frac{s}{s+1}\right),$$

with $s > 0$. The same holds for $f(x) = 1/(1-x)$, $x \in [0, 1)$.

For $f(x) = x^2$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx^2} dx = \frac{1}{\sqrt{s}} \arctan[\sqrt{s}],$$

with $s > 0$. The same goes for $f(x) = (1-x)^2$, $x \in [0, 1]$.

For $f(x) = 1/x^2$, $x \in (0, 1]$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s(1/x^2)} dx = 1 - \sqrt{s}\arctan\left[\frac{1}{\sqrt{s}}\right],$$

with $s > 0$. The same is true for $f(x) = 1/(1-x)^2$, $x \in [0, 1)$.

For $f(x) = \sqrt{x}$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s\sqrt{x}} dx = \frac{2}{s^2}[s - \log(1+s)],$$

with $s > -1$. The same goes for $f(x) = \sqrt{1-x}$, $x \in [0, 1]$.

For $f(x) = 1/\sqrt{x}$, $x \in (0, 1]$, we establish that

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s[1/\sqrt{x}]} dx = 1 + 2s\left[s\log\left(1+\frac{1}{s}\right) - 1\right],$$

with $s > 0$. The same holds for $f(x) = 1/\sqrt{1-x}$, $x \in [0, 1)$.

For $f(x) = \sqrt{x+b}$, $x \in [0,1]$, with $b > 0$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]} \frac{1}{1+s\sqrt{x+b}} dx \\ &= \frac{2}{s^2}\left\{ s\left[\sqrt{b+1}-\sqrt{b}\right] + \log\left[1+s\sqrt{b}\right] - \log\left[1+s\sqrt{b+1}\right]\right\}, \end{aligned}$$

with $s > -1/\sqrt{b}$. The same goes for $f(x) = \sqrt{1-x+b}$, $x \in [0,1]$.

For $f(x) = 1/\sqrt{x+b}$, $x \in [0,1]$, with $b > 0$, we obtain

$$\begin{aligned} T(f)(s) &= \int_{[0,1]} \frac{1}{1+s/\sqrt{x+b}} dx \\ &= 1 - 2s\left[\sqrt{b+1}-\sqrt{b}\right] - 2s^2\left\{\log\left[\sqrt{b}+s\right] - \log\left[\sqrt{b+1}+s\right]\right\}, \end{aligned}$$

with $s > -1/\sqrt{b}$. The same is true for $f(x) = 1/\sqrt{1-x+b}$, $x \in [0,1]$.

For $f(x) = e^{bx}$, $x \in [0,1]$, with $b \in \mathbb{R}$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+se^{bx}} dx = 1 + \frac{1}{b}\log\left(\frac{s+1}{se^b+1}\right),$$

with $s > -\min(1, e^{-b})$. The same goes for $f(x) = e^{b(1-x)}$, $x \in [0,1]$.

For $f(x) = -\log(x)$, $x \in (0,1]$, we find that

$$T(f)(s) = \int_{[0,1]} \frac{1}{1-s\log(x)} dx = -\frac{1}{s}e^{1/s}\operatorname{Ei}\left(-\frac{1}{s}\right),$$

with $s > 0$, and $\operatorname{Ei}(a) = -\int_{[-a,\infty)}(e^{-x}/x)dx$ is the exponential integral function. The same holds for $f(x) = -\log(1-x)$, $x \in [0,1)$.

For $f(x) = 1/[-\log(x)]$, $x \in (0,1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1-s/\log(x)} dx = 1 + se^s\operatorname{Ei}(-s),$$

with $s > 0$. The same goes for $f(x) = 1/[-\log(1-x)]$, $x \in [0,1)$.

For $f(x) = \sin(x)$, $x \in [0,1]$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s\sin(x)} dx = \frac{2}{\sqrt{1-s^2}}\operatorname{arccot}\left[\frac{s+\cot(1/2)}{\sqrt{1-s^2}}\right],$$

with $|s| < 1$, $\operatorname{arccot}(x)$ is the inverse cotangent function, i.e., such that $\cot[\operatorname{arccot}(x)] = \operatorname{arccot}[\cot(x)] = x$ with $\cot(x) = \cos(x)/\sin(x)$. The same holds for $f(x) = \sin(1-x)$, $x \in [0,1]$.

For $f(x) = 1/\sin(x)$, $x \in (0,1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s/\sin(x)} dx = 1 - \frac{2s}{\sqrt{1-s^2}}\operatorname{arccoth}\left[\frac{1+s\cot(1/2)}{\sqrt{1-s^2}}\right],$$

with $|s| < 1$, $\operatorname{arccoth}(x)$ is the inverse cotangent hyperbolic function, i.e., such that $\coth[\operatorname{arccoth}(x)] = \operatorname{arccoth}[\coth(x)] = x$ with $\coth(x) = \cosh(x)/\sinh(x)$. The same goes for $f(x) = 1/\sin(1-x)$, $x \in [0,1)$.

For $f(x) = \cos(x)$, $x \in [0,1]$, we obtain

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s\cos(x)} dx = -\frac{2}{\sqrt{1-s^2}}\arctan\left[\frac{(s-1)\tan(1/2)}{\sqrt{1-s^2}}\right],$$

with $|s| < 1$, $\arctan(x)$ is the inverse tangent function, i.e., such that $\tan[\arctan(x)] = \arctan[\tan(x)] = x$ with $\tan(x) = \sin(x)/\cos(x)$. The same holds for $f(x) = \cos(1-x)$, $x \in [0,1]$.

For $f(x) = 1/\cos(x)$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s/\cos(x)} dx = 1 + \frac{2s}{\sqrt{1 - s^2}} \operatorname{arctanh}\left[\frac{(s-1)\tan(1/2)}{\sqrt{1 - s^2}}\right],$$

with $|s| < 1$, $\operatorname{arctanh}(x)$ is the inverse tangent hyperbolic function, i.e., such that $\tanh[\operatorname{arctanh}(x)] = \operatorname{arctanh}[\tanh(x)] = x$ with $\tanh(x) = \sinh(x)/\cosh(x)$. The same is true for $f(x) = 1/\cos(1 - x)$, $x \in [0, 1]$.

For $f(x) = [\sin(x)]^2$, $x \in [0, 1]$, we obtain

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\sin(x)]^2} dx = \frac{1}{\sqrt{1 + s}} \arctan\left[\tan(1)\sqrt{1 + s}\right],$$

with $s > -1$. The same is true for $f(x) = [\sin(1 - x)]^2$, $x \in [0, 1]$.

For $f(x) = 1/[\sin(x)]^2$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s/[\sin(x)]^2} dx = 1 - \sqrt{\frac{s}{1 + s}} \arctan\left[\tan(1)\sqrt{\frac{1 + s}{s}}\right],$$

with $s > -1$. The same holds for $f(x) = 1/[\sin(1 - x)]^2$, $x \in [0, 1)$.

For $f(x) = [\cos(x)]^2$, $x \in [0, 1]$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\cos(x)]^2} dx = \frac{1}{\sqrt{1 + s}} \arctan\left[\frac{\tan(1)}{\sqrt{1 + s}}\right],$$

with $s > -1$. The same holds for $f(x) = [\cos(1 - x)]^2$, $x \in [0, 1]$.

For $f(x) = 1/[\cos(x)]^2$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s/[\cos(x)]^2} dx = 1 - \sqrt{\frac{s}{1 + s}} \arctan\left[\tan(1)\sqrt{\frac{s}{1 + s}}\right],$$

with $s > 0$. The same is true for $f(x) = 1/[\cos(1 - x)]^2$, $x \in [0, 1]$.

For $f(x) = \tan(x)$, $x \in [0, 1]$, we establish that

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s\tan(x)} dx = \frac{1}{1 + s^2} \left\{1 + \log[\cos(1) + s\sin(1)]\right\},$$

with $s > -\cot(1)$. The same holds for $f(x) = \tan(1 - x)$, $x \in [0, 1]$.

For $f(x) = [\tan(x)]^2$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\tan(x)]^2} dx = \frac{1}{s - 1} \left\{\sqrt{s}\arctan[\tan(1)\sqrt{s}] - 1\right\},$$

with $s > 0$. The same goes for $f(x) = [\tan(1 - x)]^2$, $x \in [0, 1]$.

For $f(x) = \cot(x)$, $x \in [0, 1]$, we obtain

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s\cot(x)} dx = \frac{1}{1 + s^2} \left\{1 + s\log(s) - s\log[\sin(1) + s\cos(1)]\right\},$$

with $s > 0$. The same holds for $f(x) = \cot(1 - x)$, $x \in [0, 1)$.

For $f(x) = \sinh(x)$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s\sinh(x)} dx = \frac{2}{\sqrt{1 + s^2}} \operatorname{arccoth}\left[\frac{s + \coth(1/2)}{\sqrt{1 + s^2}}\right],$$

with $s \in \mathbb{R}$. The same goes for $f(x) = \sinh(1 - x)$, $x \in [0, 1]$.

For $f(x) = [\sinh(x)]^2$, $x \in [0, 1]$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\sinh(x)]^2} dx = \frac{1}{\sqrt{1-s}} \operatorname{arctanh}\left[\tanh(1)\sqrt{1-s}\right],$$

with $|s| < 1$. The same is true for $f(x) = [\sinh(1-x)]^2$, $x \in [0, 1]$.

For $f(x) = \cosh(x)$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s\cosh(x)} dx = -\frac{2}{\sqrt{1-s^2}} \operatorname{arctanh}\left[\frac{(s-1)\tanh(1/2)}{\sqrt{1-s^2}}\right],$$

with $|s| < 1$. The same holds for $f(x) = \cosh(1-x)$, $x \in [0, 1]$.

For $f(x) = [\cosh(x)]^2$, $x \in [0, 1]$, we find that

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\cosh(x)]^2} dx = \frac{1}{\sqrt{1+s}} \operatorname{arctanh}\left[\frac{\tanh(1)}{\sqrt{1+s}}\right],$$

with $|s| < 1$. The same goes for $f(x) = [\cosh(1-x)]^2$, $x \in [0, 1]$.

For $f(x) = \tanh(x)$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s\tanh(x)} dx = \frac{1}{s^2-1} \left\{s\log[\cosh(1) + s\sinh(1)] - 1\right\},$$

with $s > -\coth(1)$. The same is true for $f(x) = \tanh(1-x)$, $x \in [0, 1]$.

For $f(x) = [\tanh(x)]^2$, $x \in [0, 1]$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s[\tanh(x)]^2} dx = \frac{1}{1+s} \left\{1 + \sqrt{s}\arctan[\tanh(1)\sqrt{s}]\right\},$$

with $s > 0$. The same goes for $f(x) = [\tanh(1-x)]^2$, $x \in [0, 1]$.

For $f(x) = \operatorname{cotanh}(x)$, $x \in [0, 1]$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]} \frac{1}{1 + s\operatorname{cotanh}(x)} dx \\ &= \frac{1}{s^2-1} \left\{s\log[\sinh(1) + s\cosh(1)] - s\log(s) - 1\right\}, \end{aligned}$$

with $s > 0$. The same holds for $f(x) = \operatorname{cotanh}(1-x)$, $x \in [0, 1]$.

For $f(x) = x(1+x)$, $x \in [0, 1]$, we calculate

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + sx(1+x)} dx = \frac{2}{\sqrt{s(4-s)}} \arccos\left[\frac{2+s}{2\sqrt{1+2s}}\right],$$

with $s \in (0, 4)$. The same goes for $f(x) = x(2-x)$, $x \in [0, 1]$.

For $f(x) = x(1-x)$, $x \in [0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + sx(1-x)} dx = \frac{4}{\sqrt{s(s+4)}} \operatorname{arctanh}\left[\sqrt{\frac{s}{s+4}}\right],$$

with $s > 0$.

For $f(x) = 1/[x(1-x)]$, $x \in (0, 1)$, we get

$$T(f)(s) = \int_{[0,1]} \frac{1}{1 + s/[x(1-x)]} dx = 1 - \frac{4s}{\sqrt{1+4s}} \operatorname{arctanh}\left[\frac{1}{\sqrt{1+4s}}\right],$$

with $s > 0$.

For $f(x) = x/(1-x)$, $x \in [0, 1)$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx/(1-x)}dx = \frac{1}{(s-1)^2}\{1+s[\log(s)-1]\},$$

with $s > 0$. The same goes for $f(x) = (1-x)/x$, $x \in (0, 1)$.

For $f(x) = x^2/(1-x)^2$, $x \in [0, 1)$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]} \frac{1}{1+sx^2/(1-x)^2}dx \\ &= \frac{1}{2(s+1)^2}\{2-\pi\sqrt{s}+\pi s^{3/2}-2s[\log(s)-1]\}, \end{aligned}$$

with $s > 0$. The same is true for $f(x) = (1-x)^2/x^2$, $x \in (0, 1]$.

For $f(x) = x/(1+x)$, $x \in [0, 1]$, we obtain

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx/(1+x)}dx = \frac{1}{(s+1)^2}[1+s+s\log(2+s)],$$

with $s > -2$. The same goes for $f(x) = (1-x)/(2-x)$, $x \in [0, 1]$.

For $f(x) = (1+x)/x$, $x \in (0, 1]$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx/(1+x)}dx = \frac{1}{(s+1)^2}\left\{1+s\left[1+\log\left(\frac{s}{1+2s}\right)\right]\right\},$$

with $s > 0$. The same goes for $f(x) = (2-x)/(1-x)$, $x \in [0, 1)$.

Multivariate functions

The IR operator of some multivariate functions f is now examined, especially the bivariate case corresponding to $n = 2$, i.e., $\Omega = [0, 1]^2$ and $\mathbf{x} = (x_1, x_2)$ and $d\mathbf{x} = dx_1dx_2$.

For $f(x_1, x_2) = x_1$, $(x_1, x_2) \in [0, 1]^2$, we get

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1+sx_1}dx_1dx_2 = \frac{1}{s}\log(1+s),$$

with $s > -1$. The same is true for $f(x_1, x_2) = 1 - x_1$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = x_1x_2$, $(x_1, x_2) \in [0, 1]^2$, we have

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1+sx_1x_2}dx_1dx_2 = -\frac{1}{s}\operatorname{Li}_2(-s),$$

with $|s| \leq 1$ and $s \neq 0$, where $\operatorname{Li}_n(x) = \sum_{k=1}^{\infty} x^k/k^n$ is the polylogarithm function and n is a positive integer (the case $s = 1$ is obviously excluded for $n = 1$). The same holds for $f(x_1, x_2) = (1-x_1)x_2$, $f(x_1, x_2) = x_1(1-x_2)$ and $f(x_1, x_2) = (1-x_1)(1-x_2)$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = x_1 + x_2$, $(x_1, x_2) \in [0, 1]^2$, we obtain

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1+s(x_1+x_2)}dx_1dx_2 = \frac{1}{s^2}\left[(1+2s)\log(1+2s)-2(1+s)\log(1+s)\right],$$

with $s > -1/2$. The same goes for $f(x_1, x_2) = 1 - x_1 + x_2$, $f(x_1, x_2) = x_1 + 1 - x_2$ and $f(x_1, x_2) = 2 - x_1 - x_2$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = x_1 - x_2$, $(x_1, x_2) \in [0, 1]^2$, we have

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1+s(x_1-x_2)}dx_1dx_2 = \frac{1}{s^2}\left[\log(1-s^2)+2s\operatorname{arctanh}(s)\right],$$

with $|s| < 1$. The same holds for $f(x_1, x_2) = 1 - x_1 - x_2$, $f(x_1, x_2) = x_1 - 1 + x_2$ and $f(x_1, x_2) = x_2 - x_1$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = x_1/x_2$, $(x_1, x_2) \in [0, 1]^2$, we establish that

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1 + sx_1/x_2} dx_1 dx_2 = \frac{1}{2s} \left[s - s^2 \log\left(1 + \frac{1}{s}\right) + \log(1 + s) \right],$$

with $s > 0$. The same is true for $f(x_1, x_2) = (1-x_1)/x_2$, $f(x_1, x_2) = x_1/(1-x_2)$ and $f(x_1, x_2) = (1-x_1)/(1-x_2)$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = x_1/(x_1 + x_2)$, $(x_1, x_2) \in [0, 1]^2$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1 + sx_1/(x_1 + x_2)} dx_1 dx_2 \\ &= 1 - \frac{s}{2(s+1)^2} \left[1 + s - \log(1 + s) + 2s(2 + s) \operatorname{arccoth}(3 + 2s)\right], \end{aligned}$$

with $s > -1$. The same goes for $f(x_1, x_2) = (1 - x_1)/(1 - x_1 + x_2)$, $f(x_1, x_2) = x_1/(x_1 + 1 - x_2)$ and $f(x_1, x_2) = (1 - x_1)/(2 - x_1 - x_2)$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = \sqrt{x_1 + x_2}$, $(x_1, x_2) \in [0, 1]^2$, we obtain

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1 + s\sqrt{x_1 + x_2}} dx_1 dx_2 \\ &= \frac{1}{3s^4} \left\{-6s\left[\sqrt{2} - 2\right] + 8s^3\left[\sqrt{2} - 1\right] + 12(s^2 - 1)\log(1 + s) + 6(1 - 2s^2)\log\left[1 + \sqrt{2}s\right]\right\}, \end{aligned}$$

with $s > -1/\sqrt{2}$. The same goes for $f(x_1, x_2) = \sqrt{1 - x_1 + x_2}$, $f(x_1, x_2) = \sqrt{x_1 + 1 - x_2}$ and $f(x_1, x_2) = \sqrt{2 - x_1 - x_2}$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = \sqrt{x_1 x_2}$, $(x_1, x_2) \in [0, 1]^2$, we have

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1 + s\sqrt{x_1 x_2}} dx_1 dx_2 = \frac{4}{s^2}\left[\mathrm{Li}_2(-s) + s\right],$$

with $|s| \leq 1$. The same holds for $f(x_1, x_2) = \sqrt{(1 - x_1)x_2}$, $f(x_1, x_2) = \sqrt{x_1(1 - x_2)}$ and $f(x_1, x_2) = \sqrt{(1 - x_1)(1 - x_2)}$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = \sqrt{x_1/x_2}$, $(x_1, x_2) \in [0, 1]^2$, we get

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1 + s\sqrt{x_1/x_2}} dx_1 dx_2 \\ &= \frac{1}{2} + \frac{1}{s} - s - s^2 \log(s) + \left(s^2 - \frac{1}{s^2}\right) \log(1 + s), \end{aligned}$$

with $s > 0$. The same goes for $f(x_1, x_2) = \sqrt{(1 - x_1)/x_2}$, $f(x_1, x_2) = \sqrt{x_1/(1 - x_2)}$ and $f(x_1, x_2) = \sqrt{(1 - x_1)/(1 - x_2)}$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = e^{x_1+x_2}$, $(x_1, x_2) \in [0, 1]^2$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1 + se^{x_1+x_2}} dx_1 dx_2 \\ &= 1 + \mathrm{Li}_2(-s) - 2\,\mathrm{Li}_2(-se) + \mathrm{Li}_2(-se^2), \end{aligned}$$

with $|s| \leq e^{-2}$. The same holds for $f(x_1, x_2) = e^{1-x_1+x_2}$, $f(x_1, x_2) = e^{x_1+1-x_2}$ and $f(x_1, x_2) = e^{2-x_1-x_2}$, $(x_1, x_2) \in [0, 1]^2$.

For $f(x_1, x_2) = e^{x_1-x_2}$, $(x_1, x_2) \in [0, 1]^2$, we find that

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1 + se^{x_1-x_2}} dx_1 dx_2 \\ &= 1 - 2\,\mathrm{Li}_2(-s) + \mathrm{Li}_2(-se^{-1}) + \mathrm{Li}_2(-se), \end{aligned}$$

with $|s| \leq e^{-1}$. The same is true for $f(x_1,x_2)=e^{1-x_1-x_2}$, $f(x_1,x_2)=e^{x_1-1+x_2}$ and $f(x_1,x_2)=e^{x_2-x_1}$, $(x_1,x_2)\in[0,1]^2$.

For $f(x_1,x_2)=(x_1+x_2)^2$, $(x_1,x_2)\in[0,1]^2$, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1+s(x_1+x_2)^2} dx_1 dx_2 \\ &= \frac{1}{2s}\left[2\log(1+s)-\log(1+4s)+4\sqrt{s}\,\mathrm{arccot}\left(\frac{1+2s}{\sqrt{s}}\right)\right], \end{aligned}$$

with $s>-1/4$. The same holds for $f(x_1,x_2)=(1-x_1+x_2)^2$, $f(x_1,x_2)=(x_1+1-x_2)^2$ and $f(x_1,x_2)=(2-x_1-x_2)^2$, $(x_1,x_2)\in[0,1]^2$.

For $f(x_1,x_2)=(x_1-x_2)^2$, $(x_1,x_2)\in[0,1]^2$, we calculate

$$\begin{aligned} T(f)(s) &= \int_{[0,1]^2} \frac{1}{1+s(x_1-x_2)^2} dx_1 dx_2 \\ &= \frac{1}{s}\left\{2\sqrt{s}\arctan[\sqrt{s}]-\log(1+s)\right\}, \end{aligned}$$

with $s\geq 0$. The same goes for $f(x_1,x_2)=(1-x_1-x_2)^2$ and $f(x_1,x_2)=(x_1-1+x_2)^2$, $(x_1,x_2)\in[0,1]^2$.

We now consider some bivariate functions having a support different from the whole Ω.

For $f(x_1,x_2)=x_1+x_2$ with $x_2\leq x_1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]}\int_{[0,x_1]} \frac{1}{1+s(x_1+x_2)} dx_2 dx_1 + \int_{[0,1]}\int_{[x_1,1]} dx_2 dx_1 \\ &= \frac{1}{2s^2}\left[(1+2s)\log(1+2s)-2(1+s)\log(1+s)\right]+\frac{1}{2}, \end{aligned}$$

with $s>-1/2$. The same goes for $f(x_1,x_2)=1-x_1+x_2$ with $x_2\leq 1-x_1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, $f(x_1,x_2)=x_1+1-x_2$ with $1-x_2\leq x_1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, and $f(x_1,x_2)=2-x_1-x_2$ with $x_1\leq x_2$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise.

For $f(x_1,x_2)=x_1+x_2$ with $x_1+x_2\leq 1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, we obtain

$$\begin{aligned} T(f)(s) &= \int_{[0,1]}\int_{[0,1-x_1]} \frac{1}{1+s(x_1+x_2)} dx_2 dx_1 + \int_{[0,1]}\int_{[1-x_1,1]} dx_2 dx_1 \\ &= \frac{1}{s^2}\left[s-\log(1+s)\right]+\frac{1}{2}, \end{aligned}$$

with $s>-1$. The same holds for $f(x_1,x_2)=1-x_1+x_2$ with $x_2\leq x_1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, $f(x_1,x_2)=x_1+1-x_2$ with $x_1\leq x_2$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, and $f(x_1,x_2)=2-x_1-x_2$ with $x_1+x_2\geq 1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise.

For $f(x_1,x_2)=x_1/x_2$ with $x_1+x_2\leq 1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]}\int_{[0,1-x_1]} \frac{1}{1+sx_1/x_2} dx_2 dx_1 + \int_{[0,1]}\int_{[1-x_1,1]} dx_2 dx_1 \\ &= \frac{1}{2(s-1)^2}\left\{1+s[\log(s)-1]\right\}+\frac{1}{2}, \end{aligned}$$

with $s>0$. The same is true for $f(x_1,x_2)=(1-x_1)/x_2$ with $x_2\leq x_1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, $f(x_1,x_2)=x_1/(1-x_2)$ with $x_1\leq x_2$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, and $f(x_1,x_2)=(1-x_1)/(1-x_2)$ with $x_1+x_2\geq 1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise.

For $f(x_1,x_2)=x_1-x_2$ with $x_1+x_2\leq 1$, $(x_1,x_2)\in[0,1]^2$, and $f(x_1,x_2)=0$ otherwise, we get

$$\begin{aligned} T(f)(s) &= \int_{[0,1]}\int_{[0,1-x_1]} \frac{1}{1+s(x_1-x_2)} dx_2 dx_1 + \int_{[0,1]}\int_{[1-x_1,1]} dx_2 dx_1 \\ &= \frac{1}{2s^2}\left[\log(1-s^2)+2s\,\mathrm{cotanh}(s)\right]+\frac{1}{2}, \end{aligned}$$

with $|s| < 1$. The same goes for $f(x_1, x_2) = 1 - x_1 - x_2$ with $x_2 \leq x_1$, $(x_1, x_2) \in [0, 1]^2$, and $f(x_1, x_2) = 0$ otherwise, $f(x_1, x_2) = x_1 - 1 + x_2$ with $x_1 \leq x_2$, $(x_1, x_2) \in [0, 1]^2$, and $f(x_1, x_2) = 0$ otherwise, and $f(x_1, x_2) = x_2 - x_1$ with $x_1 + x_2 \geq 1$, $(x_1, x_2) \in [0, 1]^2$, and $f(x_1, x_2) = 0$ otherwise.

For $f(x_1, x_2) = (x_1 - x_2)^2$ with $x_1 + x_2 \leq 1$, $(x_1, x_2) \in [0, 1]^2$, and $f(x_1, x_2) = 0$ otherwise, we have

$$\begin{aligned} T(f)(s) &= \int_{[0,1]} \int_{[0,1-x_1]} \frac{1}{1 + s(x_1 - x_2)^2} dx_2 dx_1 + \int_{[0,1]} \int_{[1-x_1,1]} dx_2 dx_1 \\ &= \frac{1}{\sqrt{s}} \arctan\left[\sqrt{s}\right] - \frac{1}{2s} \log(1+s) + \frac{1}{2}, \end{aligned}$$

with $s > 0$. The same is true for $f(x_1, x_2) = (1 - x_1 - x_2)^2$ with $x_2 \leq x_1$, $(x_1, x_2) \in [0, 1]^2$, and $f(x_1, x_2) = 0$ otherwise, and $f(x_1, x_2) = (x_1 - 1 + x_2)^2$ with $x_1 \leq x_2$, $(x_1, x_2) \in [0, 1]^2$.

For $f(\mathbf{x}) = \prod_{i=1}^n x_i$, $\mathbf{x} \in \Omega$, we obtain

$$T(f)(s) = \int_{[0,1]^n} \frac{1}{1 + s \prod_{i=1}^n x_i} d\mathbf{x} = -\frac{1}{s} \operatorname{Li}_n(-s),$$

with $|s| \leq 1$ and $s \neq 0$ (the case $s = 1$ is excluded for $n = 1$). The same holds for $f_*(\mathbf{x}) = f(\mathbf{x}_*)$, where $\mathbf{x}_* = (x_1^*, \ldots, x_n^*)$ with $x_i^* \in \{x_i, 1 - x_i\}$ for each $i = 1, \ldots, n$.

Applications

This section is devoted to some applications of our findings, mainly on some inequalities that offer alternative approaches or new perspectives on existing results.

Simple logarithmic inequalities

The proposition below examines some known and new lower bounds for the logarithmic functions $\log(1 + s)$ and $-\log(1 - s)$ that can be derived with little effort from some established properties of the IR operator.

Proposition 5.16 *Let $s \in \mathbb{R}$. The following logarithmic inequalities hold:*

1. *For any $s > -1$ and $s \neq 0$, we have*
$$\frac{1}{s} \log(1+s) \geq \frac{2}{2+s}.$$
2. *For any $s \in (-1, 0) \cup (1, \infty)$, we have*
$$\frac{\log(1+s)}{s} \geq \frac{1 - 2^{-(s-1)}}{s-1}.$$
3. *For any $s > -1$ and $s \neq 0$, we have*
$$\frac{1}{s} \log(1+s) \geq \frac{1 - e^{-s}}{s}.$$
4. *For any $s > -1$ and $s \neq 0$, we have*
$$\frac{1}{s} \log(1+s) \geq \frac{s}{e^s - 1}.$$
5. *For any $s \in (0, 1)$, we have*
$$-\frac{1}{s} \log(1-s) \geq \frac{2(e^s - 1)^2}{s^2(2+s)}.$$

Proof. To begin, by taking $f(x) = x$, $x \in [0, 1]$, the IR operator of f is given by

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx} dx = \frac{1}{s}\log(1+s),$$

which is well defined for $s > -1$ and $s \neq 0$. This result is central in the developments below.

1. For any $s > -1$, based on item 1 of Proposition 3.11, we obtain

$$T(f)(s) \geq \frac{1}{1 + s\int_{[0,1]} f(x)dx}.$$

Since $\int_{[0,1]} f(x)dx = \int_{[0,1]} xdx = 1/2$, we immediately get

$$\frac{1}{s}\log(1+s) \geq \frac{1}{1+s/2} = \frac{2}{2+s}.$$

2. For any $s \in (-1, 0) \cup (1, \infty)$, based on Proposition 3.13, we have

$$T(f)(s) \geq V(f)(s),$$

with $V(f)(s) = \int_{[0,1]} 1/[1 + f(x)]^s dx$.
Since $V(f)(s) = \int_{[0,1]} 1/(1+x)^s dx = (1 - 2^{-(s-1)})/(s-1)$, we immediately get

$$\frac{1}{s}\log(1+s) \geq \frac{1 - 2^{-(s-1)}}{s-1}.$$

3. For any $s > -1$, based on item 1 in Proposition 3.15, we obtain

$$T(f)(s) \geq W(f)(s),$$

with $W(f)(s) = \int_{[0,1]} e^{-sf(x)} dx$.
Since $W(f)(s) = \int_{[0,1]} e^{-sx} dx = (1 - e^{-s})/s$, we find that

$$\frac{1}{s}\log(1+s) \geq \frac{1-e^{-s}}{s}.$$

4. For any $t < 1$, owing to item 3 in Proposition 3.15, we get

$$T(f)(-t) \geq \frac{1}{W(f)(t)},$$

with $W(f)(t) = (1 - e^{-t})/t$. By putting $t = -s$, we have $s > -1$ and

$$\frac{1}{s}\log(1+s) \geq \frac{1}{(1 - e^{-(-s)})/(-s)} = \frac{s}{e^s - 1}.$$

5. For any $s \in (0, 1)$, based on item 4 in Proposition 3.15, we obtain

$$T(f)(-s) \geq \frac{[W(f)(-s)]^2}{1 + s\int_{[0,1]} f(x)dx}.$$

Thus, since $W(f)(s) = (1 - e^{-s})/s$ and $\int_{[0,1]} f(x)dx = 1/2$, we find that

$$-\frac{1}{s}\log(1-s) \geq \frac{[(e^s - 1)/s]^2}{1 + s/2} = \frac{2(e^s-1)^2}{s^2(2+s)}.$$

The desired inequalities are obtained, ending the proof. □

Owing to item 1 of Proposition 5.16, for any $s > 0$, we obtain

$$\log(1+s) \geq \frac{2s}{2+s}.$$

Also, for any $s \in (-1, 0)$, we get

$$\log(1+s) \leq \frac{2s}{2+s}.$$

These are well-known sharp logarithmic inequalities (see [25]). The interesting fact is that their derivations are obtained through the use of integral tools. The other logarithmic inequalities in Proposition 5.16 are original, to the best of our knowledge. After a numerical investigation, the following inequalities are conjectured: $2/(2+s) \geq s/(e^s - 1) \geq (1 - e^{-s})/s$ for any $s > -1$, meaning that item 1 of Proposition 5.16 possibly implies items 3 and 4. Item 5 provides an original logarithmic lower bound, to the best of our knowledge. Its sharpness around $s = 0$ is illustrated in Figure 1, with consideration of the following function:

$$g(s) = -\frac{1}{s}\log(1-s) - \frac{2(e^s-1)^2}{s^2(2+s)}$$

for $s \in [0, 0.3]$ to zoom around the region of $s = 0$. As expected, this intermediate function is positive.

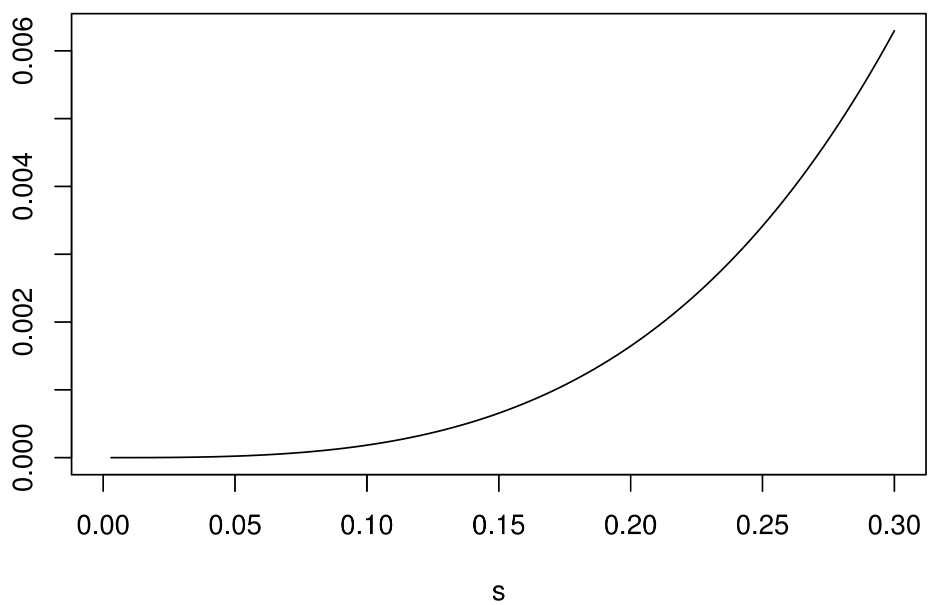

Figure 1: Plots of the function $g(s)$ for $s \in [0, 0.3]$.

For the case $s \in (0, 1)$, alternative original lower bounds for $\log(1+s)$ and $-\log(1-s)$ of the natural form "s plus or minus a function (with respect to s)" are examined below.

Proposition 5.17

1. *For any* $s \in (0, 1]$, *we have*

$$\log(1+s) \geq s - \frac{s^2(1-2^{1-s}s)}{s^2-3s+2}.$$

2. *For any* $s \in (0, 1]$, *we have*

$$-\log(1-s) \geq s + \frac{6[1-(1-s)e^s]^2}{s^2(3+2s)}.$$

Proof. Let us consider $f(x) = \sqrt{x}$, $x \in [0, 1]$. Then the IR operator of f is given by

$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s\sqrt{x}}dx = \frac{2}{s^2}[s - \log(1+s)],$$

which is well defined for $s \in (-1, 1] \setminus \{0\}$. This result is crucial in the developments that follow.

1. Based on Proposition 3.12, we have

$$T(f)(s) \leq V(f)(s),$$

with $V(f)(s) = \int_{[0,1]} 1/[1+f(x)]^s dx$.

After some integration steps, we obtain $V(f)(s) = \int_{[0,1]} 1/[1+\sqrt{x}]^s dx = (2-2^{2-s}s)/(s^2-3s+2)$. Therefore, we have

$$\frac{2}{s^2}[s-\log(1+s)] \leq \frac{2-2^{2-s}s}{s^2-3s+2},$$

which is equivalent to

$$\log(1+s) \geq s - \frac{s^2(1-2^{1-s}s)}{s^2-3s+2}.$$

2. For any $s \in (0, 1)$, based on item 4 in Proposition 3.15, we obtain

$$T(f)(-s) \geq \frac{[W(f)(-s)]^2}{1+s\int_{[0,1]} f(x)dx},$$

where $W(f)(s) = \int_{[0,1]} e^{-sf(x)}dx$.

An integration by parts gives $W(f)(s) = \int_{[0,1]} e^{-s\sqrt{x}}dx = 2[1-(1+s)e^{-s}]/s^2$. Furthermore, since $\int_{[0,1]} f(x)dx = 2/3$, we find that

$$\frac{2}{s^2}[-s-\log(1-s)] \geq \frac{\{2[1-(1-s)e^s]/s^2\}^2}{1+2s/3} = \frac{12[1-(1-s)e^s]^2}{s^4(3+2s)}.$$

This is equivalent to

$$-\log(1-s) \geq s + \frac{6[1-(1-s)e^s]^2}{s^2(3+2s)}.$$

The stated inequalities are established. □

The sharpness of item 1 in Proposition 5.17 is illustrated in Figure 2 through the plot of the following function:

$$h(s) = \log(1+s) - s + \frac{s^2(1-2^{1-s}s)}{s^2-3s+2}$$

for $s \in [0, 1]$. It is positive and has a small maximum value (smaller than 0.005), thus supporting the theory.

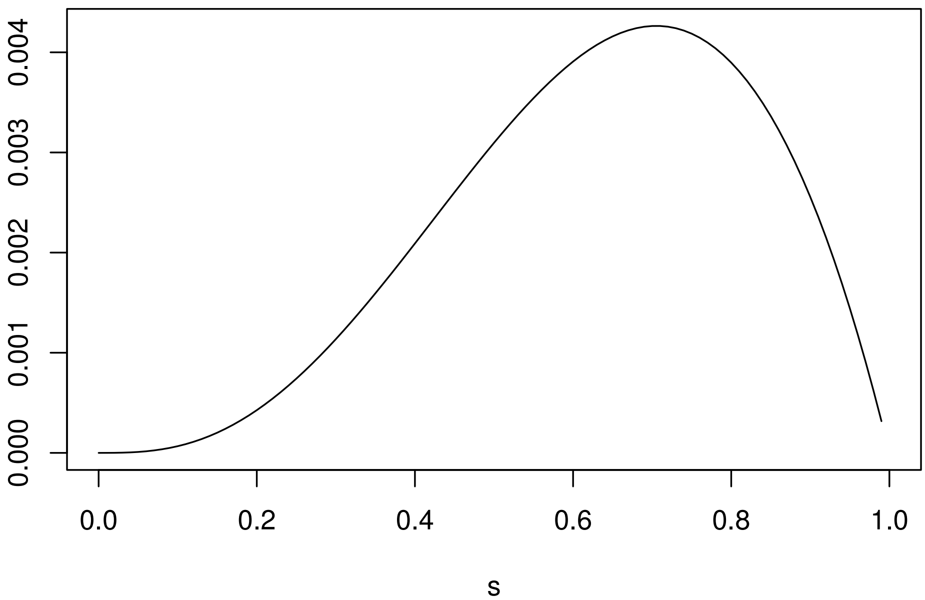

Figure 2: Plots of the function $h(s)$ for $s \in [0, 1]$.

The following proposition investigates a few upper bounds of the logarithmic function $\log(1 + s)$ that can be obtained from our results.

Proposition 5.18 *Let $s \in \mathbb{R}$. The following logarithmic inequalities hold:*

1. *For any $s > -1$ and $s \neq 0$, we have*
$$\log(1+s) \leq s - \frac{3s^2}{2(3+2s)}.$$
2. *For any $s \in (-1, 0) \cup (1, \infty)$, we have*
$$\log(1+s) \leq s - \frac{s^2(1-2^{1-s}s)}{s^2-3s+2}.$$
3. *For any $s > -1$ and $s \neq 0$, we have*
$$\log(1+s) \leq s + (1+s)e^{-s} - 1.$$
4. *For any $s > -1$ and $s \neq 0$, we have*
$$\log(1+s) \leq s - \frac{s^4}{4[1-(1-s)e^s]}.$$

Proof. To begin, by taking $f(x) = \sqrt{x}$, $x \in [0, 1]$, the IR operator of f is given by
$$T(f)(s) = \int_{[0,1]} \frac{1}{1+s\sqrt{x}} dx = \frac{2}{s^2}[s - \log(1+s)],$$
which is well defined for $s > -1$ and $s \neq 0$. This result is central in the developments below.

1. For any $s > -1$, based on item 1 of Proposition 3.11, we obtain

$$T(f)(s) \geq \frac{1}{1 + s\int_{[0,1]} f(x)dx}.$$

Since $\int_{[0,1]} f(x)dx = \int_{[0,1]} \sqrt{x}dx = 2/3$, we immediately get

$$\frac{2}{s^2}[s - \log(1+s)] \geq \frac{1}{1 + 2s/3} = \frac{3}{3+2s},$$

which is equivalent to

$$\log(1+s) \leq s - \frac{3s^2}{2(3+2s)}.$$

2. For any $s \in (-1, 0) \cup (1, \infty)$, based on Proposition 3.13, we have

$$T(f)(s) \geq V(f)(s),$$

with $V(f)(s) = \int_{[0,1]} 1/[1 + f(x)]^s dx$.

Since $V(f)(s) = \int_{[0,1]} 1/[1 + \sqrt{x}]^s dx = (2 - 2^{2-s}s)/(s^2 - 3s + 2)$, we immediately get

$$\frac{2}{s^2}[s - \log(1+s)] \geq \frac{2 - 2^{2-s}s}{s^2 - 3s + 2},$$

which is equivalent to

$$\log(1+s) \leq s - \frac{s^2(1 - 2^{1-s}s)}{s^2 - 3s + 2}.$$

3. For any $s > -1$, based on item 1 in Proposition 3.15, we obtain

$$T(f)(s) \geq W(f)(s),$$

where $W(f)(s) = \int_{[0,1]} e^{-sf(x)}dx$.

Since $W(f)(s) = \int_{[0,1]} e^{-s\sqrt{x}}dx = 2[1 - (1+s)e^{-s}]/s^2$, we obtain

$$\frac{2}{s^2}[s - \log(1+s)] \geq 2\frac{1 - (1+s)e^{-s}}{s^2},$$

which is equivalent to

$$\log(1+s) \leq s + (1+s)e^{-s} - 1.$$

4. For any $t < 1$, owing to item 3 in Proposition 3.15, we get

$$T(f)(-t) \geq \frac{1}{W(f)(t)},$$

with $W(f)(t) = 2[1 - (1+t)e^{-t}]/t^2$. By putting $t = -s$, we have $s > -1$ and

$$\frac{2}{s^2}[s - \log(1+s)] \geq \frac{1}{2[1 - (1-s)e^s]/s^2} = \frac{s^2}{2[1 - (1-s)e^s]},$$

which is equivalent to

$$\log(1+s) \leq s - \frac{s^4}{4[1 - (1-s)e^s]}.$$

The stated inequalities are established. This ends the proof. □

The inequalities in items 1, 3 and 4 in Proposition 5.18 are improvements of the following well-known logarithmic inequality: $\log(1+s) \leq s$ for any $s > -1$. The inequality in item 2 improves it for $s > 1$.

Some other inequalities

In order to highlight the inequalities generating power for the IR operator, additional examples beyond the logarithmic function are given in this section.

The result below is about some arctangent inequalities of potential interest.

Proposition 5.19

1. *For any* $s > 0$, *we have*
$$\arctan(s) \geq \frac{3s}{3+s^2}.$$
2. *For any* $s > 0$, *we have*
$$\arctan(s) \geq \frac{\sqrt{\pi}}{2}\operatorname{erf}(s),$$
where $\operatorname{erf}(a) = [2/\sqrt{\pi}]\int_{[0,a]} e^{-x^2}dx$, $a \geq 0$.

Proof. To begin, by taking $f(x) = x^2$, $x \in [0,1]$, the IR operator of f is given by
$$T(f)(s) = \int_{[0,1]} \frac{1}{1+sx^2}dx = \frac{1}{\sqrt{s}}\arctan[\sqrt{s}],$$
with $s > 0$. This result is at the heart of the coming developments.

1. For any $s > 0$, based on item 1 of Proposition 3.11, we obtain
$$T(f)(s) \geq \frac{1}{1+s\int_{[0,1]} f(x)dx}.$$
Since $\int_{[0,1]} f(x)dx = \int_{[0,1]} x^2 dx = 1/3$, we have
$$\frac{1}{\sqrt{s}}\arctan[\sqrt{s}] \geq \frac{1}{1+s/3} = \frac{3}{3+s}.$$
By replacing s by s^2 in terms of notations, we get
$$\arctan(s) \geq \frac{3s}{3+s^2}.$$
2. For any $s > 0$, based on item 1 in Proposition 3.15, we obtain
$$T(f)(s) \geq W(f)(s),$$
where $W(f)(s) = \int_{[0,1]} e^{-sf(x)}dx$.
By a simple change of variables, we get $W(f)(s) = \int_{[0,1]} e^{-sx^2}dx = \{\sqrt{\pi}\operatorname{erf}[\sqrt{s}]\}/[2\sqrt{s}]$. As a result, we have
$$\frac{1}{\sqrt{s}}\arctan[\sqrt{s}] \geq \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{s}}\operatorname{erf}[\sqrt{s}].$$
By simplifying the term $\sqrt{s}$ in both sides and replacing s by s^2 in terms of notations, we establish that
$$\arctan(s) \geq \frac{\sqrt{\pi}}{2}\operatorname{erf}(s).$$

This ends the proof. □

The inequality of item 1 in Proposition 5.19 improves the following famous arctangent inequality: $\arctan(s) \geq s/(1+s^2)$ for $s \geq 0$, but it does not improve the Shafer inequality, which states that $\arctan(s) \geq 3s/[1+2\sqrt{1+s^2}]$ for $s \geq 0$ (see [26] and [27]). It thus can be viewed as an intermediate inequality, which has the merit of being sharp while keeping a certain simplicity that can be useful in an analysis context; this simplicity is justified by the absence of the square root term in the denominator in comparison to the Shafer inequality.

The result below is about two mixed logarithmic-arctangent inequalities.

Proposition 5.20

1. *For any $s > 0$, we have*

$$2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s) \geq \frac{6s}{6+s}.$$

2. *For any $s > 0$, we have*

$$2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s) \geq \sqrt{\pi}\sqrt{s}\,\mathrm{erf}[\sqrt{s}] + e^{-s} - 1.$$

Proof. By taking $f(x_1, x_2) = (x_1 - x_2)^2$, $(x_1, x_2) \in [0, 1]^2$, after some integral developments, the IR operator of f is given by

$$T(f)(s) = \int_{[0,1]^2} \frac{1}{1 + s(x_1 - x_2)^2} dx_1 dx_2 = \frac{1}{s}\left\{2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s)\right\},$$

with $s > 0$. The proofs below are centered on it.

1. For any $s > 0$, based on item 1 of Proposition 3.11, we obtain

$$T(f)(s) \geq \frac{1}{1 + s\int_{[0,1]^2} f(x_1, x_2)dx_1 dx_2}.$$

   Since $\int_{[0,1]^2} f(x_1, x_2)dx_1 dx_2 = \int_{[0,1]^2}(x_1 - x_2)^2 dx_1 dx_2 = 1/6$, we have

$$\frac{1}{s}\left\{2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s)\right\} \geq \frac{1}{1+s/6} = \frac{6}{6+s},$$

   which implies that

$$2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s) \geq \frac{6s}{6+s}.$$

2. For any $s > 0$, based on item 1 in Proposition 3.15, we obtain

$$T(f)(s) \geq W(f)(s),$$

   where $W(f)(s) = \int_{[0,1]^2} e^{-sf(x_1,x_2)}dx_1 dx_2$.

   By a natural bivariate change of variables and some integral manipulations, we obtain $W(f)(s) = \int_{[0,1]^2} e^{-s(x_1-x_2)^2}dx_1 dx_2 = \left\{\sqrt{\pi}\sqrt{s}\,\mathrm{erf}[\sqrt{s}] + e^{-s} - 1\right\}/s$. Therefore, we have

$$\frac{1}{s}\left\{2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s)\right\} \geq \frac{1}{s}\left\{\sqrt{\pi}\sqrt{s}\,\mathrm{erf}[\sqrt{s}] + e^{-s} - 1\right\},$$

   so

$$2\sqrt{s}\arctan[\sqrt{s}] - \log(1+s) \geq \sqrt{\pi}\sqrt{s}\,\mathrm{erf}[\sqrt{s}] + e^{-s} - 1.$$

This ends the proof. □

To the best of our knowledge, the inequalities demonstrated in Proposition 5.20 are new.

The result below is about an inequality involving the exponential integral function.

Proposition 5.21 *For any $s > 0$, we have*

$$\mathrm{Ei}\left(-\frac{1}{s}\right) \leq -\frac{s}{1+s}e^{-1/s}.$$

Proof. For $f(x) = -\log(x)$, we have

$$T(f)(s) = \int_{[0,1]} \frac{1}{1-s\log(x)}dx = -\frac{1}{s}e^{1/s}\,\mathrm{Ei}\left(-\frac{1}{s}\right),$$

with $s > 0$. Let us now provide two different proofs, based on our findings.

Proof 1: For any $s > 0$, based on item 1 of Proposition 3.11, we obtain

$$T(f)(s) \geq \frac{1}{1+s\int_{[0,1]} f(x)dx}.$$

Since $\int_{[0,1]} f(x)dx = \int_{[0,1]}[-\log(x)]dx = 1$, we have

$$-\frac{1}{s}e^{1/s}\,\mathrm{Ei}\left(-\frac{1}{s}\right) \geq \frac{1}{1+s},$$

which is equivalent to

$$\mathrm{Ei}\left(-\frac{1}{s}\right) \leq -\frac{s}{1+s}e^{-1/s}.$$

Proof 2: For any $s > 0$, based on item 1 in Proposition 3.15, we obtain

$$T(f)(s) \geq W(f)(s),$$

where $W(f)(s) = \int_{[0,1]} e^{-sf(x)}dx$.
Since $W(f)(s) = \int_{[0,1]} e^{-s[-\log(x)]}dx = \int_{[0,1]} x^s dx = 1/(s+1)$, we have

$$-\frac{1}{s}e^{1/s}\,\mathrm{Ei}\left(-\frac{1}{s}\right) \geq \frac{1}{1+s},$$

which is equivalent to

$$\mathrm{Ei}\left(-\frac{1}{s}\right) \leq -\frac{s}{1+s}e^{-1/s}.$$

Both proofs converge to the same result. □

The bound in Proposition 5.21 is observed in Figure 3, with the consideration of the following function:

$$k(s) = \mathrm{Ei}\left(-\frac{1}{s}\right) + \frac{s}{1+s}e^{-1/s}$$

for $s \in (0, 1)$. It is logically negative, as demonstrated.

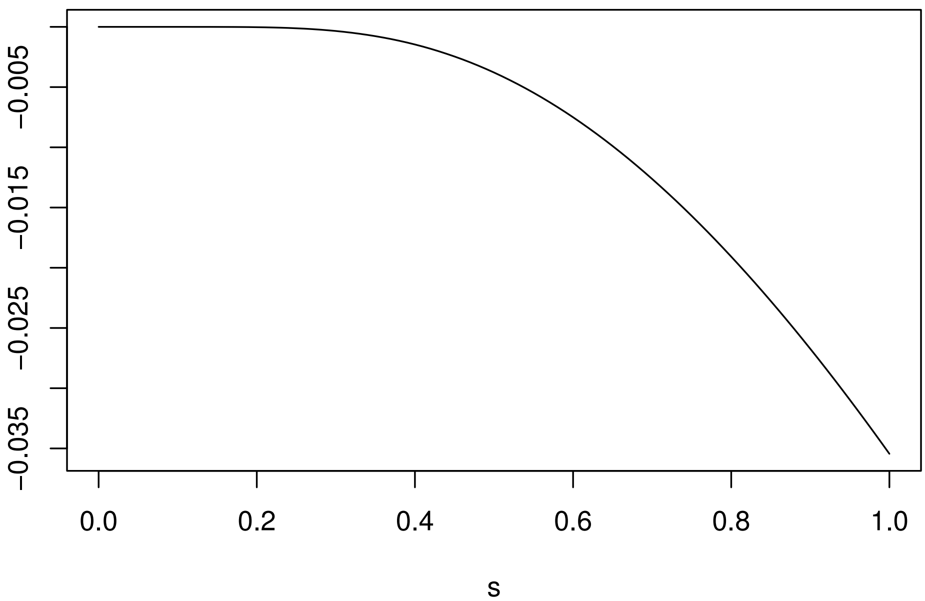

Figure 3: Plots of the function $k(s)$ for $s \in (0, 1)$.

The result below is about an inequality involving the polylogarithmic function in a subtle way.

Proposition 5.22 *Let n be a positive integer. Then, for any $|s| < 1$ and $s \neq 0$, we have*

$$\sum_{k=0}^{\infty} (-1)^k \frac{s^k}{(k+1)^n} = -\frac{1}{s} \operatorname{Li}_n(-s) \geq \frac{2^n}{2^n + s}.$$

Proof. For $f(\mathbf{x}) = \prod_{i=1}^{n} x_i$, $\mathbf{x} \in \Omega$, we have

$$T(f)(s) = \int_{\Omega} \frac{1}{1 + s\prod_{i=1}^{n} x_i} d\mathbf{x} = -\frac{1}{s} \operatorname{Li}_n(-s),$$

with $|s| < 1$ and $s \neq 0$. Based on item 1 of Proposition 3.11, we obtain

$$T(f)(s) \geq \frac{1}{1 + s\int_{\Omega} f(\mathbf{x}) d\mathbf{x}}.$$

Since $\int_{\Omega} f(\mathbf{x}) d\mathbf{x} = \int_{[0,1]^n} \left(\prod_{i=1}^{n} x_i\right) d\mathbf{x} = \prod_{i=1}^{n} \left(\int_{[0,1]} x_i dx_i\right) = 1/2^n$, we have

$$-\frac{1}{s} \operatorname{Li}_n(-s) \geq \frac{1}{1 + s/2^n} = \frac{2^n}{2^n + s}.$$

This ends the proof. □

For $n = 1$, for any $|s| < 1$ with $s \neq 0$, since $\operatorname{Li}_1(s) = -\log(1 - s)$, we refind

$$-\frac{1}{s} \operatorname{Li}_n(-s) = \frac{1}{s} \log(1 + s) \geq \frac{2}{2 + s}.$$

In this sense, Proposition 5.22 can be viewed as a generalization of this logarithmic inequality under the assumption $|s| < 1$ with $s \neq 0$.

The bound in Proposition 5.22 is illustrated in Figure 4, with the consideration of the following function:

$$\ell_n(x) = \frac{1}{s} \operatorname{Li}_n(-s) + \frac{2^n}{2^n + s}$$

for $n = 1, 2, 3$ and 4 and $s \in (0, 1)$. It is logically negative, as demonstrated.

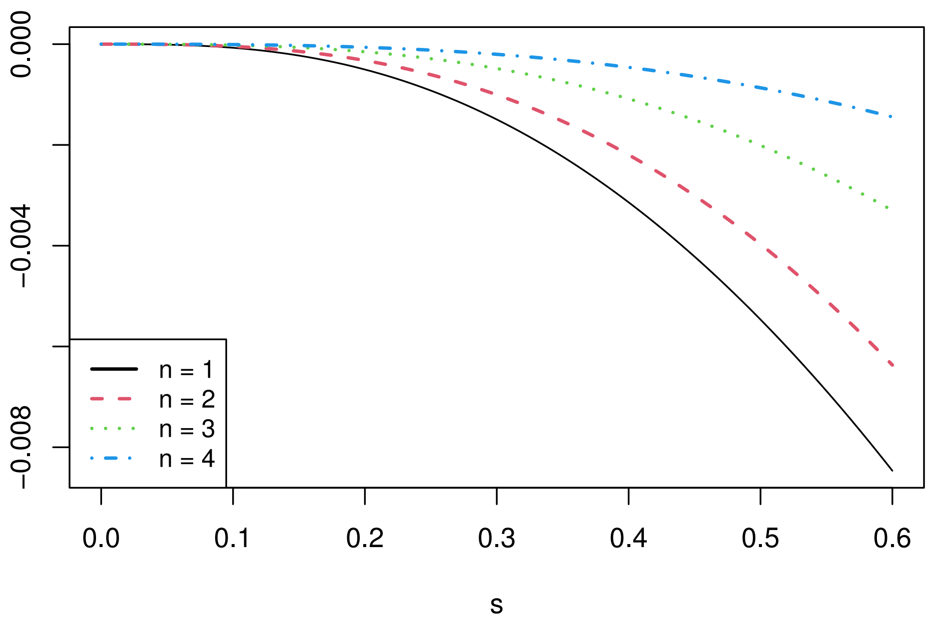

Figure 4: Plots of the function $\ell_n(s)$ for $n = 1, 2, 3$ and 4 and $s \in (0, 0.6)$.

The inequalities presented in this section are just a short sample of what can be derived from the findings about the IR operator. More can be done in this direction, beyond the existing inequalities in the literature.

Conclusion

In conclusion, this article focused on the introduction of a new multivariate integral operator mainly based on a one-parameter ratio transformation. This operator is notable for its nonlinear nature and its numerous comprehensive mathematical properties, including original functional equations, various integral and series representations, and a large repertoire of inequalities. Precise arguments employing a variety of mathematical techniques assisted in understanding these aspects. Finally, through the presentation of concrete examples, the results were applied to concrete inequalities involving standard and special functions. Certain existing inequalities were revisited, and new ones were established. In a sense, this article contributes to a broader understanding of multivariate nonlinear integral operators, highlighting their usefulness and versatility in mathematical modeling and analysis.

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Appendix

Weierstrass product inequality: The Weierstrass product inequality can be formulated as follows: let m be a positive integer and $x_1, \ldots, x_m$ such that $x_j \in [0, 1]$ for any $j = 1, \ldots, m$. Then we have

$$\prod_{j=1}^{m}(1 + x_j) \geq 1 + \sum_{j=1}^{m} x_j.$$

Continuous version of the Chebyshev sum inequality: The continuous version of the Chebyshev sum inequality can be formulated as follows: let $(a, b) \in \mathbb{R}^2$ with $b > a$, and $f(x)$ and $g(x)$, $x \in [a, b]$, be integrable functions, both non-increasing or both non-decreasing. Then we have

$$\frac{1}{b-a}\int_{[a,b]} f(x)g(x)\, dx \geq \left[\frac{1}{b-a}\int_{[a,b]} f(x)dx\right]\left[\frac{1}{b-a}\int_{[a,b]} g(x)dx\right].$$

(The reversed inequality holds if one function is non-increasing and the other is non-decreasing).