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On Some Classes of DNA Codes in Finite Alphabet

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Abstract
In this correspondence, lower bound and upper bound on the covering radii of DNA codes in Finite Alphabet with Finite Ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$ by using different distance are given. Some classes bounds for various Repetition DNA codes, Simplex DNA codes of both Type α and Type β and MacDonald DNA codes of both Type α and Type β in R by using covering radii are obtain.

Keywords: DNA code; finite ring; covering radius; different distance.

Introduction

Nucleic acids such as the DNA stores the genetic information of all living beings. DNA is a twisted double stranded helical ladder like structure, which consists of four nitrogenous bases such as Adenine, thiamine, cytosine and guanine, as well as alternating chain of sugars and phosphates. DNA molecules exhibit antiparallel structure, where the dual strands of this helical molecule run in opposite directions of each other. Both these strands are held together by hydrogen bond and they possess distinct 5' and a 3' end and are held in-between the base pairs of nitrogenous bases where adenine pairs with thymine and cytosine pairs with guanine(Watson-Crick[22] complements of each other - A matches by T and C matches by G, also 3' end matches with 5'end).

The problem of designing DNA codes with same length, that satisfy certain combinatorial constraints has applications for reliably storing and retrieving information in synthetic DNA strands. These codes can be used in particular for DNA computing[1] or as molecular bar-codes.

In finite field, there are many researchers doing research on code and the particular, codes over $\mathbb{Z}_4$ received much attention [2, 4–6, 14, 16, 20, 21]. The covering radius of binary linear codes were studied [4, 5]. In 1999, Sole et al gave many upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with different distances. In [6–8], the covering radius of some particular codes over $\mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$ for some block repetition code in $\mathbb{Z}_4$ have been investigated. Recently the covering radius of dna codes over finite ring has been investigated [9].

In this correspondence, a survey of the covering radius of some classes for same type and different type block repetition code in the type α and type β for both Simplex DNA codes and MacDonald DNA codes. In [2], some of the known, lower bond and upper bound have been generalized.

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Preliminary

Coding theory has several applications in Genetics and Bioengineering. The problem of designing DNA codes [length of the codeword is fixed in the the Finite Alphabet $\mathbb{N} = \{A, C, G, T\}$, that satisfy certain combinatorial constraints] has applications for reliably storing and retrieving information in synthetic DNA strands.

Let $(x_1, x_2, \cdots, x_n)$ with $x_i \in \mathbb{N}$ be the set of codeword in DNA code with length n. By using the Watson-Crick complement of a nucleotide, therefore the matches of four nucleotides, A matches by T and C matches by G.

In $\mathbb{N} = \{A, C, G, T\}$ be a alphabet and let DNA code is the sets of codewords of fixed length $n \in \mathbb{N}$. We know that, components of finite ring $\mathbb{Z}_2 + u\mathbb{Z}_2$ with $u^2 = 0$ which is one-one and onto mapping to the nucleotide DNA basis $ACTG$ respectively such that $(A, T \to 0, 2)$ and $(C, G \to 1, u)$. Thus the problem of the DNA codes is corresponding to the problem of the R-linear codes. These transpositions do not affect the GC-weight of the codeword. In this paper, by using the above map in R with different weight for Lee(L), Euclidean(E), Chinese Euclidean(CE) and Bachoc(B)). Obtain the covering radius for some block repetition DNA codes .

Let $d = (d_1, d_2, \cdots, d_n) \in \mathbb{N}^n$, where n is a length of code. If z is an element of $\{A, C, G, T\}$. Define a weight of z at b is $w_z(d) = |\{i|y_i = z\}|$.

A DNA linear code $C \subseteq \mathbb{N}$ is an additive subgroup of $\mathbb{N}^n$. The element of C is said to be a DNA codeword of code. A matrix whose rows generate C is the generator matrix of C.
In, [3, 18] are defined as the weight for Lee(L), Euclidean(E), Chinese Euclidean(CE) and Bachoc(B) to the followings

$\mathbb{N}(R)$	L	E	CE	B
0	0	0	0	0
1	1	1	2	1
u	2	4	4	2
$1+u$	1	1	2	2

Let $\Upsilon : \mathbb{N}(R) \longrightarrow \mathbb{Z}_2^2$ be a gray mapping and define $\Upsilon(l + 2m) = (m, l + m), \forall l + 2m \in \mathbb{N}$. Therefore, the image f a linear cde C in $\mathbb{N}$ of length is n, uing the above map is a binary code of length $2n$ with equal number of elements [20].

Any DNA linear code $C \subseteq \mathbb{N}$ is equivalent to a code with, the Generator Matrix(GM). Then the following,

$$GM = \begin{bmatrix} I_{k_0} & A' & B' \\ 0 & 2I_{k_1} & 2D' \end{bmatrix}, \text{ here } A', B', D \text{ are matrices in } \mathbb{N}.$$

Therefore, a DNA code C contain all DNA codewords $[c_0, c_1]\,GM$, here c_0 is a codeword of length k_1 in $\mathbb{N}$ and c_1 is a codeword of length k_2 in $\mathbb{Z}_2$. So, C contains a total number of $2^{2k_1+k_2}$ codewords. Thus, the parameters of C is an $\left[n, 2^{2k_1+k_2}, d\right]$ code, here d is the minimum different distance of C.

A DNA linear code $C \subseteq \mathbb{N}^n$, where n is a length of codeword in code, dimension of code is k, and finally the minimum distance of C is d, then the code is said to be a $\left[n, k, d_\sharp\right]$ code, where $\sharp \in \{L, E, CE, B\}$ and it is denoted by $[n, k, d]$ code.

DNA Repetition Codes with Covering Radii

The covering radius of DNA Repetition Codes C :

$r_d(C) = \max_{u \in \mathbb{N}^n} \{\min_{c \in C}\{d(c, u)\}\}$, where d is an various distance of the code in $\mathbb{N}^n$.

Let C be an q-ary repetition code C over a finite field $F_q = \left\{0, 1, \gamma_2, \gamma_3, \cdots, \gamma_{q-1}\right\}$ is an $[n, 1, n]$ code, where $C = \left\{\bar{\gamma} \mid \gamma \in F_q\right\}$ and $\bar{\gamma} = (\gamma, \gamma, \cdots, \gamma)$. In [17], the covering radius of C is $\left\lceil \frac{n(q-1)}{q} \right\rceil$. By using above, it can be found that the covering radius of block of size n repetition code of the parameter:$[n(q-1), 1, n(q-1)]$ is generated by

$$GM = [\overbrace{11\cdots 1}^{n}\overbrace{\gamma_2\gamma_2\cdots\gamma_2}^{n}\overbrace{\gamma_3\gamma_3\cdots\gamma_3}^{n}\cdots\overbrace{\gamma_{q-1}\gamma_{q-1}\cdots\gamma_{q-1}}^{n}]$$

is $\left\lceil \frac{n(q-1)^2}{q} \right\rceil$, since it will be equivalent to a repetition code of length $(q-1)n$.
Consider the repetition dna code over $\mathbb{N}$. There are two type

- **cytosine**

  The parameter of cytosine repetition code $C_\beta : [n, 1, n, n, n, n]$ is generator matrix $GM_\beta = [\overbrace{C\ C\cdots C}^{n}]$

- **thymine**

  The parameter of thymine repetition code $C_\alpha : (n, 2, 2n, 2n, 4n, 2n)$ is generator matrix $GM_\alpha = [\overbrace{T\ T\cdots T}^{n}]$, where n is a length of the code.

Theorem 3.1. *Let C_β and C_α be the dna code of type β and α type in generator matrices GM_β and GM_α. Then,*

1. $\left\lfloor \frac{n}{2} \right\rfloor \le r_L(C_\alpha) \le 2n$ *and* $n \le r_L(C_\beta) \le \frac{5n}{3}$,
2. $4\left\lfloor \frac{n}{2} \right\rfloor \le r_E(C_\alpha) \le 2n$ *and* $r_E(C_\beta) \le \frac{3n}{2}$,
3. $4\left\lfloor \frac{n}{2} \right\rfloor \le r_{CE}(C_\alpha) \le 2n$ *and* $r_{CE}(C_\beta) = 2n$,
4. $2\left\lfloor \frac{n}{2} \right\rfloor \le r_B(C_\alpha) \le 2n$ *and* $n \le r_B(C_\beta) \le 2n$.

Proof. The code of $C = \{A\ A\cdots A, T\ T\cdots T\}$ is generated by $[T\ T\cdots T]$ is an $[n, 1, 2n]$ code. Then, $d_L(x, AA\cdots A) = w_L(x - AA\cdots A) = \left\lceil \frac{n}{2} \right\rceil$ and $d_L(x, TT\cdots T) = wt_L(x - TT\cdots T) = \left\lfloor \frac{n}{2} \right\rfloor$. Therefore $d_L(x, C_\alpha) = \min\left\{\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, where $x = \overbrace{T\ T\cdots T}^{\lfloor \frac{n}{2} \rfloor}\overbrace{A\ A\cdots A}^{\lceil \frac{n}{2} \rceil} \in \mathbb{N}^n$.
Using by definition,

$$r_L(C_\alpha) \ge \left\lfloor \frac{n}{2} \right\rfloor \tag{3.1}$$

Let x be the codewords in $\mathbb{N}^n$ and let us take x has ω_0' coordinates as 0's, ω_1' coordinates as 1's, ω_2' coordinates as 2's, ω_3' coordinates as 3's, so $\omega_0' + \omega_1' + \omega_2' + \omega_3' = n$. Since $C_\alpha = \{AA\cdots A, TT\cdots T\}$ and lee weight of $\mathbb{N} : A$ is 0, C and G is 1 and T is 2. Therefore, $d_L(x, 00\cdots 0) = n - \omega_0' + \omega_1' + \omega_2' + \omega_3'$ and $d_L(x, TT\cdots T) = n - \omega_2' + \omega_1' + \omega_0' + \omega_3'$.
Thus $d_L(x, C_\alpha) = \min\left\{n - \omega_0' + \omega_1' + \omega_2' + \omega_3', n - \omega_2' + \omega_1' + \omega_{0+}'\omega_3'\right\}$ and hence,

$$d_L(x, C_\alpha) \le n + n = 2n. \tag{3.2}$$

By using, (3.1) and (3.2), so $\left\lfloor \frac{n}{2} \right\rfloor \le r_L(C_\alpha) \le 2n$.
In C_β code, with lee weight. Then $d_L(x, AA\cdots A) = n - \omega_0' + \omega_1' + 3\omega_2' + \omega_{3,}' d_L(x, CC\cdots C) = n - \omega_1' + \omega_0' + \omega_2' + \omega_{3,}' d_L(x, TT\cdots T) = n - \omega_2' + \omega_1' + \omega_3'$ and $d_L(x, GG\cdots G) = n - \omega_3' + \omega_1' + \omega_0' + \omega_2'$, here $x \in \mathbb{N}^n$.
Therefore, $d_L(x, C_\beta) = \min\{n - \omega_0' + \omega_1' + \omega_2' + \omega_{3,}' n - \omega_1' + \omega_0' + \omega_2' + \omega_3', n - \omega_2' + \omega_1' + \omega_3', n - \omega_3' + \omega_1' + \omega_0' + \omega_2'\} \le \frac{5n}{3}$. So, $r(C_\beta) \le \frac{5n}{3}$.

Let $x = \overbrace{AA\cdots A}^{t}\overbrace{CC\cdots C}^{t}\overbrace{TT\cdots T}^{t}\overbrace{GG\cdots G}^{n-3t}$, where $t = \left\lfloor \frac{n}{4} \right\rfloor$,
then $d_L(x, AA\cdots A) = n$, $d_L(x, CC\cdots C) = 2n - 4t$, $d_L(x, TT\cdots T) = n$ and $d_L(x, GG\cdots G) = 4t$.
Therefore $r_L(C_\beta) \ge \min\{2n, 2n - 4t, 4t\} \ge n$.
The above arguments are follows for the remaining weights. □

Block Repetition Code

Let $GM = [\overbrace{C\,C\cdots C}^{n}\overbrace{T\,T\cdots T}^{n}\overbrace{G\,G\cdots G}^{n}]$ be a generator matrix of $\mathbb{N}$ in each block of repetition code. Therefore, the code of $BRC = \{c_0 = A\cdots AA\cdots AA\cdots A$,
$c_1 = C\cdots CT\cdots TG\cdots G, c_2 = T\cdots TA\cdots AT\cdots T, c_3 = G\cdots GT\cdots TC\cdots C\}$. Then, the parametrs of Block Repetition Code(BRC)

$\mathbb{N}(R)$	L	E	CE	B
length	$3n$	$3n$	$3n$	$3n$
dimension	1	1	1	1
weight(distance)	$4n$	$6n$	$8n$	$4n$

Remark:
The same length of a block repetition code has permanent lee weight is $4n$. Obtain, the following

Theorem 3.2. *1.* $3n \leq r_L\left(BRC^{3n}\right) \leq \frac{7n}{2}$

2. $r_E\left(BRC^{3n}\right) = 5n$.

3. $4\left\lfloor\frac{n}{2}\right\rfloor + 4n \leq r_{CE}\left(BRC^{3n}\right) \leq 6n$,

4. $2\lfloor\frac{n}{2}\rfloor + 2n \leq r_B\left(BRC^{3n}\right) \leq 4n$.

Proof. Let $y' = AA\cdots A \in \mathbb{N}^{3n}$. Then, $d(x, BRC^{3n}) = 3n$ and hence, $r_L\left(BRC^{3n}\right) \geq 3n$.

If $y' = (\gamma'|\delta'|\zeta') \in \mathbb{N}^{3n}$, here γ', δ' and ζ' have compositions $(\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3)$, $(\upsilon_0, \upsilon_1, \upsilon_2, \upsilon_3)$ and $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ respectively such that $\sum_{i=0}^{3}\varsigma_i = n$, $\sum_{i=0}^{3}\upsilon_i = n$ and $\sum_{i=0}^{3}\varphi_i = n$, then $d_L(y', c_0) = 3n - \varsigma_0 + \varsigma_1 + \varsigma_2 + \varsigma_3 - \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 - \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3$, $d_L(y', c_1) = 3n - \varsigma_1 + \varsigma_0 + \varsigma_2 + \varsigma_3 - \upsilon_2 + \upsilon_0 + \upsilon_1 + \upsilon_0 - \varphi_3 + \varphi_0 + \varphi_1 + \varphi_2$, $d_L(y', c_2) = 3n - \varsigma_2 + \varsigma_1 + \varsigma_0 + \varsigma_3 - \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 - \varphi_2 + \varphi_0 + \varphi_1 + \varphi_3$ and $d_L(y', c_3) = 3n - \varsigma_3 + \varsigma_1 + \varsigma_0 + \varsigma_2 - \upsilon_2 + \upsilon_0 + \upsilon_1 + \upsilon_3 - \varphi_1 + \varphi_3 + \varphi_0 + \varphi_2$.

Thus, $d_L(y', BRC^{3n}) = \min\{3n - \varsigma_0 + \varsigma_1 + \varsigma_2 + \varsigma_3 - \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 - \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3, 3n - \varsigma_1 + \varsigma_0 + \varsigma_2 + \varsigma_3 - \upsilon_2 + \upsilon_0 + \upsilon_1 + \upsilon_3 - \varphi_3 + t_0' + \varphi_1 + \varphi_2, 3n - \varsigma_2 + \varsigma_1 + \varsigma_0 + \varsigma_3 - \upsilon_0 + \upsilon_1 + \upsilon_2 + \upsilon_3 - \varphi_2 + \varphi_0 + \varphi_1 + \varphi_3, 3n - \varsigma_3 + \varsigma_1 + \varsigma_0 + \varsigma_2 - \upsilon_2 + \upsilon_0 + \upsilon_1 + \upsilon_3 - \varphi_1 + \varphi_3 + \varphi_0 + \varphi_2\} \leq \frac{7n}{2}$ and hence, $r_L\left(BRC^{3n}\right) \leq \frac{7n}{2}$. Similar proof of the all other weight. □

Define a two block repetition dna code over $\mathbb{N}$ of each of length is n and the parameters of two block repetition code BRC:

$\mathbb{N}(R)$	L	E	CE	B
length	$2n$	$2n$	$2n$	$2n$
dimension	1	1	1	1
weight(distance)	$2n$	$2n$	$4n$	$2n$

are generated by $G = [\overbrace{CC\cdots C}^{n}\overbrace{TT\cdots T}^{n}]$. Use the above and obtain a following

Theorem 3.3. *1.* $2n \leq r_L\left(BRC^{2n}\right) \leq \frac{11n}{3}$

2. $r_E\left(BRC^{2n}\right) = \frac{7n}{2}$.

3. $4\left\lfloor\frac{n}{2}\right\rfloor + 2n \leq r_{CE}\left(BRC^{2n}\right) \leq 4n$,

4. $2\lfloor \frac{n}{2} \rfloor + n \leq r_B\left(BRC^{2n}\right) \leq 4n$.

Let $GM = [\overbrace{C\ C \cdots C}^{m}\ \overbrace{T\ T \cdots T}^{n}]$ be the generalized generator matrix for two different block repetition dna code of length are m and n respectively. In the parameters of two different block repetition code(BRC^{m+n}) are

$\mathbb{N}(R)$	L	E	CE	B
length	$m+n$	$m+n$	$m+n$	$m+n$
dimension	1	1	1	1
weight(distance)	$w*$	$w*$	$min\{4m, 3m+3n\}$	$w*$

here $w* = min\{2m, m+n\}$

and Theorem 3.3 can be easily generalized for two alternating length by using alike arguments to the following.

Theorem 3.4. 1. $m+n \leq r_L\left(BRC^{m+n}\right) \leq 2m + \frac{5n}{3}$,

2. $r_E\left(BRC^{m+n}\right) \leq \frac{3m}{2} + 2n$

3. $2m + 4\left\lfloor \frac{n}{2} \right\rfloor \leq r_{CE}\left(BRC^{m+n}\right) \leq 2m+2n$,

4. $m + 2\lfloor \frac{n}{2} \rfloor + \leq r_B\left(BRC^{2n}\right) \leq 2m+2n$.

Type α and Type β of Simplex DNA Code over $\mathbb{N}$

The Quaternary Simplex codes of type α and type β have been studied[4]. Here S_k^α is an Type α Simplex code of a linear DNA code $\mathbb{N}$ and its parameters $\left[4^k, k\right]$ and an inductively generator matrix is

$$GM_k^\alpha = \left[\begin{array}{c|c|c|c} A\cdots A & C\cdots C & T\cdots T & G\cdots G \\ \hline GM_{k-1}^\alpha & GM_{k-1}^\alpha & GM_{k-1}^\alpha & GM_{k-1}^\alpha \end{array}\right] \tag{4.3}$$

with $GM_1^\alpha = [A\ C\ T\ G]$.

Type α simplex code S_k^β is a pricked versioning of S_k^α with parameterized $[2^{k-1}, \left(2^k - 1\right), k]$ and an inductive generator matrix given by

$$GM_2^\beta = \left[\begin{array}{cccc|c|c} C & C & C & C & A & T \\ \hline A & C & T & G & C & C \end{array}\right] \tag{4.4}$$

$$GM_k^\beta = \left[\begin{array}{c|c|c} CC\cdots C & AA\cdots A & TT\cdots T \\ \hline GM_{k-1}^\alpha & GM_{k-1}^\beta & GM_{k-1}^\beta \end{array}\right] \tag{4.5}$$

and for $k > 2$, where GM_{k-1}^α is a type α simplex cde of the generator matrix of S_{k-1}^α and ref. to [4]. The minimum weight of Type α code with various weight(such as, L, E, CE and B) are 4, 8, 4 and 4 respectively for the $[A\ C\ T\ G]$.

Theorem 4.5. 1. $r_L(S_k^\alpha) \leq 2^{2k} + 1$,

2. $r_E(S_k^\alpha) \leq \frac{5\cdot 4^k + 5}{3}$,

3. $r_{CE}\left(S_k^\alpha\right) \leq 2^{2k+1} - 3$,

4. $r_B(S_k^\alpha) \leq \frac{2^{2(k+1)}-1}{3}$.

Proof. Let $x = CC\cdots C \in \mathbb{N}^n$. The generator matrix of type α(4.3), Proposition [10] and using Theorem 3.2, then

$$\begin{aligned} r_L\left(S_k^\alpha\right) &\leq r_L\left(S_{k-1}^\alpha\right) + r_L(<\overbrace{CC\cdots C}^{4^{(k-1)}}\overbrace{TT\cdots T}^{4^{(k-1)}}\overbrace{GG\cdots G}^{4^{(k-1)}}>) \\ &= r_L\left(S_{k-1}^\alpha\right) + 3.4^{(k-1)} \\ &= 3.4^{(k-1)} + 3.4^{(k-2)} + 3.4^{(k-3)} + \ldots\ldots + 3.4^1 + r_L\left(S_1^\alpha\right) \\ r_L\left(S_k^\alpha\right) &\leq 2^{2k} + 1 \left(\text{since } r_L\left(S_1^\alpha\right) = 5\right). \end{aligned}$$

The proof of remaining part is alike way of same for the above. □

Theorem 4.6. 1. $r_L(S_k^\beta) \leq 2^k\left(2^k - 1\right) - 1$,

2. $r_E(S_k^\beta) \leq \frac{5\cdot 4^k - 6\cdot 2^k - 8}{6}$,

3. $r_{CE}\left(S_k^\beta\right) \leq 2^{2k} - 2^k - 7$,

4. $r_B(S_k^\beta) \leq \frac{4^{k+1} + 3\cdot 4^{k-1} - 9\cdot 2^{k-2} - 20}{3}$

Proof. In the generator matrix of type β(4.5), Proposition [10] and Theorem 3.4, thus

$$\begin{aligned} r_L\left(S_k^\beta\right) &\leq r_L\left(S_{k-1}^\beta\right) + r_L(<\overbrace{CC\cdots C}^{2^{2(k-1)}}\ \overbrace{TT\cdots T}^{2^{(2k-3)}-2^{(k-2)}}\ >) \\ &= r_L\left(S_{k-1}^\beta\right) + 2^{(2k-2)} + 2^{(2k-3)} - 2^{(k-2)} \end{aligned}$$

$$\begin{aligned} &\leq 2\left(4^{(k-1)} + 4^{(k-2)} + \ldots + 4^2\right) + \left(4^{(2k-3)} + 4^{(2k-5)} + \ldots + 4^2\right) - \\ &\quad \left(4^{(k-2)} + 4^{(k-3)} + \ldots + 4\right) + r_L\left(S_2^\beta\right) \\ r_L\left(S_k^\beta\right) &\leq 2^{k-1}\left(2^k - 1\right) - 1\left(\text{since } r_L\left(S_2^\beta\right) = 5\right). \end{aligned}$$

The remaining part of proof is alike above way with various weights. □

Type α and Type β for MacDonald DNA Code Over $\mathbb{N}$

In [15], $M_{k,t}(q)$ be the q-ary MacDonald code Over the Finite Field F_q and its parameter$\left[\frac{q^k-q^t}{q-1}, k, q^{k-1} - q^{t-1}\right]$ code, but every non-zero codeword of code has weight is either q^{k-1} or $q^{k-1} - q^{t-1}$. The author has studied the covering radius of MacDonald codes over a finite field and also many exact values for smaller dimension are givan[17].

Define a MacDonald codes over a ring by us ing the generator matrices of Simplex Codes[12]. Then the generator matrices of MacDonald Code of Type α and Type β are

$$GM_{k,t}^\alpha = \left[GM_k^\alpha \backslash \frac{0}{GM_t^\alpha}\right]_{(k-t)\times 2^{2t}} \tag{5.6}$$

and

$$GM_{k,t}^\beta = \left[GM_k^\beta \backslash \frac{0}{GM_t^\beta}\right]_{(k-t)\times 2^{t-1}(2^t-1)}, \tag{5.7}$$

here the symbol " \ " is denoted by deleting corresponding columns for both matrix and $2 \leq t \leq k$. Therefore, Type α Macdonald code:

Let $GM_{k,t}^{\alpha}$ i a generator matrix, that generator by codes and its parameter $\left[4^k - 4^t, k\right]$ and

Type β Macdonald code:

Let $GM_{k,t}^{\beta}$ i a generator matrix, that generator by codes and its parameter $\left[\left(2^{k-1} - 2^{t-1}\right)\left(2^k + 2^t - 1\right), k\right]$ code over $\mathbb{N}$.

In fact, these codes are punctured code of S_k^{α} and S_k^{β} respectively and obtain the following,

Theorem 5.7. *1.* $r_L\left(M_{k,t}^{\alpha}\right) \leq 2^{2k} - 2^{2r} + r\left(M_{r,t}^{\alpha}\right)$, *for* $t < r \leq k$

2. $r_E\left(M_{k,t}^{\alpha}\right) \leq \frac{5}{3}(4^k - 4^r) + r\left(M_{r,t}^{\alpha}\right)$, *for* $t < r \leq k$

3. $r_{CE}\left(M_{k,t}^{\alpha}\right) \leq 2^{2k+1} - 2^{2r+1} + r\left(M_{r,t}^{\alpha}\right)$, *for* $t < r \leq k$.

4. $r_B\left(M_{k,t}^{\alpha}\right) \leq \frac{4^{k+1}-4^{r+1}}{3} + r\left(M_{r,t}^{\alpha}\right)$, *for* $t < r \leq k$.

Proof. In equation(5.6), Proposition [10] and Theorem 3.2, thus

$$r_L\left(M_{k,t}^{\alpha}\right) \leq r_L(<\overbrace{CC\cdots C}^{4^{(k-1)}}\overbrace{TT\cdots T}^{4^{(k-1)}}\overbrace{GG\cdots G}^{4^{(k-1)}}>) + r_L\left(M_{r,t}^{\alpha}\right)$$

$$= 3.4^{k-1} + r_L\left(M_{k-1,t}^{\alpha}\right), \text{ for } k \geq r > t.$$
$$\leq 3.4^{k-1} + 3.4^{k-2} + \cdots + 3.4^r + r_L\left(M_{r,t}^{\alpha}\right) \text{ for } k \geq r > t$$

$$r_L\left(M_{k,t}^{\alpha}\right) \leq 2^{2k} - 2^{2r} + r_L\left(M_{r,t}^{\alpha}\right), \text{ for} k \geq r > t.$$

The proof of the other part is same arguments to 1. ◻

Theorem 5.8. *1.* $r_L\left(M_{k,t}^{\beta}\right) \leq 2^{2k-1} - 2^{k-1} + 2^{r-1} - 2^{2r-1} + r_L\left(M_{r,t}^{\beta}\right)$, *for* $t < r \leq k$

2. $r_E\left(M_{k,t}^{\beta}\right) \leq \frac{2^k(5\cdot 2^k-6)+2^r(6-5\cdot 2^r)}{6} + r_E\left(M_{r,t}^{\beta}\right)$, *for* $t < r \leq k$.

3. $r_{CE}\left(M_{k,t}^{\beta}\right) \leq 2^k\left(2^k - 1\right) + 2^r\left(1 - 2^r\right) + r_{CE}\left(M_{r,t}^{\beta}\right)$, *for* $t < r \leq k$.

4. $r_B\left(M_{k,t}^{\beta}\right) \leq \frac{4^{k+1}-4^{r+1}+3(4^{k-1}-4^{r-1})+9(2^{r-1}-2^{k-1})}{6} + r_B\left(M_{r,t}^{\beta}\right)$, *for* $t < r \leq k$.

Proof. Using Proposition [10], Theorem 3.4 and in equation(5.7), obtain

$$r_L\left(M_{k,t}^{\beta}\right) \leq r_L(<\overbrace{CC\cdots C}^{4^{(k-1)}}\ \overbrace{TT\cdots T}^{4^{(k-1)-1}-2^{(k-2)}}\ >) + r_L\left(M_{k-1,t}^{\beta}\right)$$
$$r_L\left(M_{k,t}^{\beta}\right) \leq 2^{2k-1} - 2^{k-1} + 2^{r-1} - 2^{2r-1} + r_L\left(M_{r,t}^{\beta}\right), \text{ for } t < r \leq k.$$

The remaining part of the proof is similar idea to 1. ◻

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