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Nonlinear Elliptic Equations Under Weak Monotonicity Conditions in Anisotropic Orlicz Space

Mohamed Mabdaoui[1], Badr El Haji[2], Lakbir Essafi[3]

[1]Faculty Polydisciplinary Sidi benour, University Chouaib Doukkali, Morocco
[2]Department of Mathematics, Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tetouan, Morocco
[3]Cadi Ayyad University, Polydisciplinary faculty, Modeling and Combinatorics Laboratory, Department of Mathematics and Computer Science B.P. 4162, Safi, Morocco

Correspondence should be addressed to Badr El Haji: b.elhaji@uae.ac.ma

Email(s): mohamed.mabdaoui@gmail.com (Mabdaoui), b.elhaji@uae.ac.ma (Haji), lakbir.essafi@uca.ac.ma (Essafi)

Abstract
In the present paper, we discuss the solvability of nonlinear elliptic problems in the setting anisotropic Orlicz space, with the presence of a lower order term and a non polynomial growth which is described by an N$-\nu$plet of N-functions not satisfying the Δ_2-condition.

Keywords: Musielak-Orlicz-Sobolev spaces, entropy solutions, truncations.

Introduction

In this work, we discuss the existence of an entropy solution for a strongly nonlinear elliptic problem with Dirichlet boundary value conditions:

$$\begin{cases} A(w) - \sum_{i=1}^{N} \partial_i(\Phi_i(x,w)) = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where:

- Ω is a bounded subset of $\mathbb{R}^N$;
- The operator $A(w) = -\sum_{i=1}^{N} \partial_i\big(a_i(x,\partial_i w)\big)$ is a Leray-Lions operator defined on a subset of $W_0^1 L_\psi(\Omega)$ where ψ is an N$-\nu$plet of N-functions Orlicz;
- Φ is a Carathéodory function which satisfies the growth condition;
- The second term $f \in L^1(\Omega)$.

It is well known that anisotropic Orlicz spaces include many spaces as special cases, such as Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces, and Orlicz spaces; see [26]. Especially, in recent decades,

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variable exponent function spaces, such as Lebesgue, Sobolev spaces with a variable exponent, were introduced in [11]. These spaces have several applications in various fields such as image restoration, electrorheological fluids; (see, [12, 14, 21]).

The main contribution of this article is to deal with a class of problems for which the classical methods of monotone operators and Lions [22] in the case $W_0^{1,p}(\Omega)$) do not apply. The reason for this is that $a(x, w, \nabla w)$ does not need to satisfy the strict monotonicity condition, but only a condition of large monotonicity:

$$\Big(a(x, s, \eta) - a(x, s, \eta')\Big)(\eta - \eta') \geq 0, \text{ for all } \eta, \eta' \in \mathbb{R}^N, \ (\eta \neq \eta').$$

Our objective is to demonstrate the existence of entropy solutions for (1.1) under the weakest assumption of large monotony, without relying on the convergence almost everywhere of the gradients of the approximate equations, since this is impossible to prove in our framework. Our proof is based on a version of Minty's lemma (G. J. Minty [23]).

Other results for the existence of finite energy solutions can be found in [3–10]. These works contribute to the understanding of finite energy solutions for various classes of nonlinear elliptic problems with different assumptions and conditions.

The paper is organized as follows: In section , we introduce some basic definitions and properties in Orlicz-Sobolev spaces and anisotropic Orlicz space as well as an abstract theorem and prepare some auxiliary results. In Section , we give basic assumptions, and the definition of entropy solution is given as well as our main theorem. Section is devoted to the proof of our main result.

Definitions and preliminary tools

Let's initiate by revisiting a few definitions and properties associated with Orlicz spaces [1, 18]. Following this, we proceed to introduce the concept of anisotropic Orlicz-Sobolev spaces.

N-functions:

In an Orlicz normed space, the fundamental concept revolves around the N function.

Definition 2.1. An N-function is defined as a mapping $G : \mathbb{R} \to \mathbb{R}$ that satisfies the following conditions:
i) G is convex in $\mathbb{R} : G(\lambda s_1 + (1 - \lambda)s_2) \leq \lambda G(s_1) + (1 - \lambda)G(s_2)$, for all $s_1, s_2 \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.
ii) G is an even function: $G(s) = G(-s)$ for all $s \in \mathbb{R}$.
iii) $G(0) = 0$ and $G(s) > 0$ for all $s \in \mathbb{R}$.
iv) $\frac{G(s)}{s} \to 0$ as $s \to 0$ and $\frac{G(s)}{s} \to +\infty$ as $s \to +\infty$.

An N-function G is called to satisfy the Δ_2-condition for all $s \in \mathbb{R}$ if, for some $k > 0$,

$$G(2s) \leq kG(s) \quad \text{or all } s \in \mathbb{R}.$$

G is said to fulfill the Δ_2-condition for s large if there exist $s_0 \geq 0$ and $k > 0$ such that

$$G(2s) \leq kG(s) \text{ for all } s \geq s_0.$$

Another way to define an N-function [18], is as a function M that can be represented as

$$G(s) = \int_0^{|s|} m(\sigma)\mathrm{d}\sigma,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$is a non-decreasing and right-continuous function, $m(s) > 0$ for all $s > 0$ and $m(s) \to +\infty$ as $s \to +\infty$.

For an N-function G, the conjugate is defined by

$$\bar{G}(s) = \int_0^{|s|} \bar{m}(\sigma)\mathrm{d}\sigma,$$

where $\bar{m} : \mathbb{R}^+ \to \mathbb{R}^+$is given by

$$\bar{m}(t) = \sup\{s : m(s) \leq t\}.$$

We have the Young's inequality

$$|ts| \leq G(t) + \bar{G}(s) \text{ for all } t, s \in \mathbb{R}.$$

Suppose Ω represents an open set in $\mathbb{R}^d$, where d belongs to the set of natural numbers $\mathbb{N}$. The Orlicz class $\mathbb{L}_G(\Omega)$ (resp. the Orlicz space $L_G(\Omega)$) is characterized as the set of (equivalence classes of) real-valued Lebesgue measurable functions$w$n Ω such that

$$\int_\Omega G(w(x))\mathrm{d}x < +\infty \quad (\text{resp. } \int_\Omega G\left(\frac{w(x)}{\lambda}\right)\mathrm{d}x < +\infty \text{ for some } \lambda > 0 \text{ }).$$

Remark that $L_G(\Omega)$ is established as a Banach space endowed with the Luxemburg norm.

$$\|w\|_G = \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{w(x)}{\lambda}\right)\mathrm{d}x \leq 1\right\}.$$

Morevoer, $\mathbb{L}_G(\Omega)$ is a convex subset of $L_G(\Omega)$, where $L_G(\Omega)$ is defined as the linear hull of $\mathbb{L}_G(\Omega)$.

The closure in $L_G(\Omega)$ of the set bounded measurable functions with compact support in $\bar{\Omega}$ is symbolized as $E_G(\Omega)$. The equality $E_G(\Omega) = L_G(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all s or for s large according to whether Ω has infinite measure or not. The dual of $E_G(\Omega)$ can be identified with $L_{\bar{G}}(\Omega)$ by means of the duality pairing $\int_\Omega w(x)v(x)\mathrm{d}x$, and the dual norm on $L_{\bar{G}}(\Omega)$ is equivalent to $\|\cdot\|_G$.

In $L_G(\Omega)$, we introduce the Orlicz norm as follows:

$$\|w\|_{(G)} = \sup \int_\Omega w(x)v(x)\mathrm{d}x \tag{2.2}$$

where the supremum is taken over all $v \in E_{\bar{G}(\Omega)}$ such that $\|v\|_{\bar{G}} \leq 1$. Interestingly, the norms $|\cdot|G$ and $|\cdot|(G)$ are found to be equivalent. Indeed, it can be demonstrated that

$$\|w\|_G \leq \|w\|_{(G)} \leq 2\|w\|_G \quad \text{for all } w \in L_G(\Omega). \tag{2.3}$$

Moreover, the Holder inequality holds true.

$$\int_\Omega |w(x)v(x)|\mathrm{d}x \leq \|w\|_G\|v\|_{(G)} \quad \text{for all } w \in L_G(\Omega) \text{ and } v \in L_G(\Omega),$$

and by (2.3)

$$\int_\Omega |w(x)v(x)|\mathrm{d}x \leq 2\|w\|_G\|v\|_G \quad \text{for all } w \in L_G(\Omega) \text{ and } v \in L_G(\Omega).$$

In particular, when Ω possesses finite measure, Holder's inequality yields the continuous inclusion $L_G(\Omega) \subset L^1(\Omega)$.

A crucial inequality in $L_G(\Omega)$ is as follows::

$$\int_\Omega G(w(x))\mathrm{d}x \leq \|w\|_{(G)} \quad \text{for all } w \in L_G(\Omega) \text{ such that } \|w\|_{(G)} \leq 1, \tag{2.4}$$

Hence, we readily conclude

$$\int_\Omega G\left(\frac{w(x)}{\|w\|_{(G)}}\right)\mathrm{d}x \leq 1 \quad \text{for all } w \in L_G(\Omega)\backslash\{0\}. \tag{2.5}$$

Definition 2.2. The sequence $(w_n) \subset L_G(\Omega)$ converges to $w \in L_G(\Omega)$ under the modular convergence in $L_G(\Omega)$ if for some $\lambda > 0$, one has

$$\int_\Omega G\left(\frac{w_n(x) - w(x)}{\lambda}\right) \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Modular convergence is weaker than convergence in the norm of $L_G(\Omega)$. Nonetheless, it suffices for our purposes. The following theorem indicates that modular convergence in L_G implies the convergence in the weak- $*$ topology $\sigma\left(L_G, L_{\bar{G}}\right)$.

Lemma 2.1. *[15]*

Let $(w_n) \subset L_G(\Omega)$, $w \in L_G(\Omega)$ and $v \in L_{\bar{G}}(\Omega)$ such that $w_n \to u$ with respect to the modular convergence. Then

1. *$w_n v \to uv$ strongly in $L^1(\Omega)$. In particular, $\displaystyle\int_\Omega w_n v \to \int_\Omega wv$.*
2. *Furthermore, if $(v_n) \subset L_G(\Omega)$ is such that $v_n \to v$ with respect to the modular convergence, then $w_n v_n \to wv$ strongly in $L^1(\Omega)$.*

Anisotropic Orlicz-Sobolev spaces.

Let Ω be an open subset of $\mathbb{R}^d$, and G_i be an N-function for each $i = 1, \ldots, d$. We write $\underset{\sim}{G} = (G_1, \ldots, G_d)$, $\overline{\underset{\sim}{G}} = \left(\bar{G}_1, \ldots, \bar{G}_d\right)$.

The anisotropic Orlicz space $L_{\underset{\sim}{G}}(\Omega)$ (respectively, $E_{\underset{\sim}{G}}(\Omega)$) is defined by

$$L_{\underset{\sim}{G}}(\Omega) = \prod_{i=1}^{d} L_{G_i}(\Omega) \quad \left(\text{respectively, } E_{\underset{\sim}{G}}(\Omega) = \prod_{i=1}^{d} E_{G_i}(\Omega)\right),$$

endowed with the norm

$$\|w\| = \sum_{i=1}^{d} \|w_i\|_{G_1}, \tag{2.6}$$

In order to introduce the anisotropic Orlicz-Sobolev spaces, it would be pertinent to define the function

$$G_0(s) = \min_{1 \le i \le d} G_i(s). \tag{2.7}$$

Remark 2.1. We can easily verify that:

1. The function G_0 is an N-function.
2. The embedding $L_{G_i}(\Omega) \hookrightarrow L_{G_0}(\Omega)$ is continuous for each $i \in \{1, \ldots, d\}$.

We define the anisotropic Orlicz-Sobolev spaces as follows:

$$W^1 L_{\underset{\sim}{G}}(\Omega) = \left\{w \in L_{G_0}(\Omega) : \partial_i w \in L_{G_i}(\Omega), i = 1, \ldots, d\right\},$$
$$W^1 E_{\underset{\sim}{G}}(\Omega) = \left\{w \in E_{G_0}(\Omega) : \partial_i w \in E_{G_i}(\Omega), i = 1, \ldots, d\right\},$$

These spaces are Banach spaces under the norm

$$\|w\|_{1,\underset{\sim}{G}} = \|w\|_{G_0} + \sum_{i=1}^{d} \|\partial_i \upsilon\|_{G_i} \tag{2.8}$$

Both spaces, $W^1 L_{\underset{\cdot}{G}}(\Omega)$ and $W^1 E_{\underset{\cdot}{G}}(\Omega)$, can be identified as subspaces of the product space $\Pi = L_{G_0}(\Omega) \times L_{\underset{\cdot}{G}}(\Omega)$. Then, the predual space of Π, $\widehat{\Pi}$, is $\widehat{\Pi} = E_{\bar{M}_0}(\Omega) \times E_{\overline{\underset{\cdot}{G}}}(\Omega)$. We will use the weak-* topology $\sigma(\Pi, \widehat{\Pi})$. Let $D(\Omega)$ be the space of functions in $C^\infty(\Omega)$ with compact support in Ω. The space $W_0^1 E_{\underset{\cdot}{G}}(\Omega)$ is defined as the (norm) closure of the space $D(\Omega)$ in $W^1 E_{\underset{\cdot}{G}}(\Omega)$, and the space $W_0^1 L_{\underset{\cdot}{G}}(\Omega)$ as the $\sigma(\Pi, \widehat{\Pi})$-closure of $D(\Omega)$ in $W^1 L_{\underset{\cdot}{G}}(\Omega)$.

Lemma 2.2. *[25] Let Ω be a bounded and open set in $\mathbb{R}^d$. Assume that $m_i(t) \geq t$ for all $t \geq 0$ and all $i = 1, \ldots, d$. Then the following continuous embeddings hold for $i = 1, \ldots, d$:*

$$L_{M_i}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{M}_i}(\Omega).$$

In particular, $W_0^1 L_{\underset{\cdot}{G}}(\Omega) \hookrightarrow H_0^1(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow W^{-1} L_{\underset{\cdot}{G}}(\Omega)$.

Remark 1. Assume that, for each $i = 1, \ldots, d$, one has $m_i(t) \geq t$ for all $t \geq 0$. Then

$$\int_\Omega v^2 \, \mathrm{d}x \leq 2 \int_\Omega M_i(v) \mathrm{d}x, \qquad \text{for all } v \in \mathbb{L}_{G_i}(\Omega).$$

Theorem 2.3. *[27] Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with locally Lipschitz boundary. Then the embedding $W^1 L_G(\Omega) \hookrightarrow E_G(\Omega)$ is compact. Furthermore, the compact imbedding $W_0^1 L_G(\Omega) \hookrightarrow E_G(\Omega)$ holds without the locally Lipschitz boundary assumption.*

Corollary 2.4. *Let Ω be an open and bounded set in $\mathbb{R}^d$ and G_0 the N-function defined in (2.7). Then, the embedding $W_0^1 L_{\underset{\cdot}{G}}(\Omega) \hookrightarrow E_{G_0}(\Omega)$ is compact. Poincaré's inequality in $W_0^1 L_{\underset{\cdot}{G}}(\Omega)$ also holds.*

Lemma 2.5. *[15] Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. Then, there exist constants κ_0 and $\kappa_1 = \kappa_1(\Omega)$ such that*

$$\int_\Omega G_0(_\nu \mathrm{d}x \leq \kappa_0 \sum_{i=1}^d \int_\Omega G_i\left(\kappa_1 \partial \nu_i\right) \mathrm{d}x \quad \text{for all } \nu \in W_0^1 L_{\underset{\cdot}{G}}(\Omega).$$

Corollary 2.6. *The seminorm $\nu \in W^1 L_{\underset{\cdot}{G}}(\Omega) \mapsto \sum_{i=1}^d \|\partial_i \nu\|_{G_i}$ is a norm in $W_0^1 L_{\underset{\cdot}{G}}(\Omega)$ and it is equivalent to the norm $\|\cdot\|_{1,\underset{\cdot}{G}}$ given in (2.8).*

Since the elements of the space $W_0^1 L_{\underset{\cdot}{G}}(\Omega)$ have been defined as the weak-* limit of convergent sequences in $D(\Omega)$, the following result states that, for certain domains Ω, $D(\Omega)$ is 'dense' in $W_0^1 L_{\underset{\cdot}{G}}(\Omega)$ with respect to the modular convergence as well.

Definition 2.3. A bounded domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the segment property, if there exist a locally finite open covering $\{U_i\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\} \subset \mathbb{R}^d$ such that for all $x \in \bar{\Omega} \cap U_i$ and any $\mu \in (0, 1)$ one has $x + \mu y_i \in \Omega$.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set satisfying the segment property and $\nu \in W_0^1 L_{\underset{\cdot}{G}}(\Omega)$. Then there exists a sequence $(\nu_n) \subset D(\Omega)$ such that $\nu_n \longrightarrow u$ with respect to the modular convergence in $W^1 L_{\underset{\cdot}{G}}(\Omega)$; that is, there exists $\lambda > 0$ such that*

$$\int_\Omega G_0\left(\left(\nu_n - \nu\right)/\lambda\right) + \sum_{i=1}^d \int_\Omega M_i\left(\left(\partial_i \nu_n - \partial_i \nu\right)/\lambda\right) \to 0 \quad \text{as } n \longrightarrow \infty.$$

The proof of the above lemma is a straightforward adaptation of [[16], Theorem 4] for isotropic Orlicz-Sobolev spaces. Finally, we introduce the following dual spaces

Theorem 2.8. *[27] Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with locally Lipschitz boundary. Then the embedding $W^1 L_G(\Omega) \hookrightarrow E_G(\Omega)$ is compact. Furthermore, the compact imbedding $W_0^1 L_G(\Omega) \hookrightarrow E_G(\Omega)$ holds without the locally Lipschitz boundary assumption.*

Corollary 2.9. *Let Ω be an open and bounded set in $\mathbb{R}^d$ and G_0 the N-function defined in (2.7). Then, the embedding $W_0^1 L_{\underset{\sim}{G}}(\Omega) \hookrightarrow E_{G_0}(\Omega)$ is compact. Poincaré's inequality in $W_0^1 L_{\underset{\sim}{G}}(\Omega)$ also holds.*

[15] Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. Then, there exist constants κ_0 and $\kappa_1 = \kappa_1(\Omega)$ such that

$$\int_\Omega G_0(u)\mathrm{d}x \le \kappa_0 \sum_{i=1}^{d} \int_\Omega M_i\left(\kappa_1 \partial v_i\right)\mathrm{d}x \quad \text{for all } v \in W_0^1 L_{\underset{\sim}{G}}(\Omega).$$

Corollary 2.10.

Corollary 2.10. *The seminorm $v \in W^1 L_{\underset{\sim}{G}}(\Omega) \mapsto \sum_{i=1}^{d} \|\partial_i v\|_{M_i}$ is a norm in $W_0^1 L_{\underset{\sim}{G}}(\Omega)$ and it is equivalent to the norm $\|\cdot\|_{1,\underset{\sim}{G}}$ given in (2.8).*

Since the elements of the space $W_0^1 L_{\underset{\sim}{G}}(\Omega)$ have been defined as the weak-* limit of convergent sequences in $D(\Omega)$, the following result states that, for certain domains Ω, $D(\Omega)$ is 'dense' in $W_0^1 L_{\underset{\sim}{G}}(\Omega)$ with respect to the modular convergence as well.

Definition 2.4. A bounded domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the segment property, if there exist a locally finite open covering $\{U_i\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\} \subset \mathbb{R}^d$ such that for all $x \in \bar{\Omega} \cap U_i$ and any $\mu \in (0,1)$ one has $x + \mu y_i \in \Omega$.

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set satisfying the segment property and $v \in W_0^1 L_{\underset{\sim}{G}}(\Omega)$. Then there exists a sequence $(v_n) \subset D(\Omega)$ such that $v_n \to u$ with respect to the modular convergence in $W^1 L_{\underset{\sim}{G}}(\Omega)$; that is, there exists $\lambda > 0$ such that*

$$\int_\Omega G_0\left(\left(v_n - v\right)/\lambda\right) + \sum_{i=1}^{d} \int_\Omega G_i\left(\left(\partial_i v_n - \partial_i v\right)/\lambda\right) \to 0 \quad \text{as } n \to \infty.$$

The proof of the above lemma is a straightforward adaptation of [[16], Theorem 4] for isotropic Orlicz-Sobolev spaces. Finally, we introduce the following dual spaces

$$W^{-1} L_{\underset{\sim}{G}}(\Omega) = \left\{ f \in D'(\Omega) : f = \sum_{i=1}^{d} \partial_i f_i \text{ with } f_i \in L_{G_i}(\Omega), \text{ for all } i,\ 1 \le i \le d \right\}$$

$$W^{-1} E_{\overline{\underset{\sim}{G}}}(\Omega) = \left\{ f \in D'(\Omega) : f = \sum_{i=1}^{d} \partial_i f_i \text{ with } f_i \in E_{\bar{M}_i}(\Omega), \text{ for all } i,\ 1 \le i \le d \right\}$$

These spaces are equipped by their usual quotient norms.

Assumptions and main results

Hypotheses

We now state the assumptions on the differential operator in divergence form given by $A : \overset{\circ}{W}{}^1_{\overrightarrow{G}}(\Omega) \to W^{-1} L_{\overrightarrow{G}}(\Omega)$

$$A(w) = -\sum_{i=1}^{N} \partial_i\big(a_i(x, \partial_i w)\big)$$

(H1) For each $i = 1, \ldots, N$ the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $a_i = a_i(x, z)$ satisfies the Caratheodory conditions, that it is measurable in x for each fixed $z \in \mathbb{R}$ and continuous in z for all x in Ω.

(H2) There exist N-functions G_i, $i = 1, \ldots, N$, a function $\kappa(.) \in E_{G_i^*}$ and positive constants k and γ such that a.e. x in Ω, for all $z_i \in \mathbb{R}$ and for $i = 1, \ldots, N$

$$|a_i(x, z)| \leq \eta \left(\kappa(x) + G_i^{*-1}(G_i(K_1|z|))\right). \tag{3.9}$$

(H3) For a.e. $x \in \Omega$, for all $z, z' \in \mathbb{R}^N$ with $z \neq z'$, $z = (z_1, \ldots, z_N)$, $z' = (z'_1, \ldots, z'_N)$ we have

$$\sum_{i=1}^{N} [a_i(x, z_i) - a_i(x, z'_i)](z_i - z'_i) \geq 0 \tag{3.10}$$

(H4) There exists $w_0 > 0$ such that a.e. x in Ω, for all $(z_1, \ldots, z_N) \in \mathbb{R}^N$ we have

$$\sum_{i=1}^{N} a_i(x, z_i) z_i \geq \alpha \sum_{i=1}^{N} G_i\Big(\frac{z_i}{w_0}\Big) \tag{3.11}$$

(H5) For each $i = 1, \ldots, N$ the function $\Phi_i : \Omega \times \mathbb{R} \to \mathbb{R}$, is a Carathéodory function which satisfies the following growth condition for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,

$$|\Phi_i(x, s)| \leq \gamma(x) B^{*-1}(B(\beta|s|)); \text{ for each } i = 1, \ldots, N \tag{3.12}$$

where $B << G_0$ and $\gamma \in L^\infty(\Omega)$.

Consider the following problem:

$$\begin{cases} A(w) - \sum\limits_{i=1}^{N} \partial_i(\Phi_i(x, w)) = f & \text{in } \Omega \\ \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

Definition of an entropy solution

Definition 3.5. A measurable function $w : \Omega \to \mathbb{R}$ is called entropy solution of (3.13) if the following conditions holds:

(C1) $T_k(w) \in \overset{\circ}{W}{}^1_{\vec{G}}(\Omega)$ and $a_i(x, \partial_i w) \in L_{G_i^*}(\Omega)$, for $i = 1, \ldots, N$;

(C2) $\displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i w)) \partial_i T_k(w - \upsilon)\, dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k(w - \upsilon)\, dx \leq \int_\Omega f T_k(w - \upsilon)\, dx$

for every $\upsilon \in \overset{\circ}{W}{}^1_{\vec{G}}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$.

Lemma 3.12. *Let u be a measurable function such that $T_k(w)$ belongs to $\overset{\circ}{W}{}^1_{\vec{G}}(\Omega)$ for every $k > 0$. Then*

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i \upsilon) \partial_i T_k(w - \upsilon)\, dx \leq \int_\Omega f T_k(w - \upsilon)\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k(w - \upsilon)\, dx \tag{3.14}$$

is equivalent to

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i w) \partial_i T_k(w - \upsilon)\, dx = \int_\Omega f T_k(w - \upsilon)\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k(w - \upsilon)\, dx \tag{3.15}$$

for every υ in $\overset{\circ}{W}{}^1_{\vec{G}} \cap L^\infty(\Omega)$, and for every $k > 0$.

Proof. **Proof of** (3.15) $\Rightarrow$ (3.14) : Let $k > 0$ and u be a measurable function such that $T_k(w)$ belongs to $\overset{\circ}{W}{}^1_{\vec{G}}(\Omega)$, where v in $\overset{\circ}{W}{}^1_{\vec{G}}(\Omega) \cap L^\infty(\Omega)$ and assume (3.15). By adding and subtracting the term $\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i v)\, \partial_i T_k(w - v)\, dx$, we obtain

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, \partial_i w) - a_i(x, \partial_i v))\, \partial_i T_k(w - v)\, dx = \int_\Omega f T_k(w - v)\, dx$$
$$- \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k(w - v)\, dx - \sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i v) \partial_i T_k(w - v)\, dx.$$

From (3.10), we get:

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i v) \partial_i T_k(w - v) dx \le \int_\Omega f T_k(w - v) dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k(w - v) dx.$$

Proof of (3.14) $\Rightarrow$ (3.15) : Let h and k be positive real numbers, and let $\lambda_0 \in [-1, 1]$. For v in $\overset{\circ}{W}{}^1_{\vec{G}}(\Omega) \cap L^\infty(\Omega)$, we take $\phi = T_h(u - \lambda_0 T_k(w - v))$ as a test function in (3.14), yielding

$$I_1(h, k) + I_2(h, k) \le I_3(h, k) \tag{3.16}$$

with

$$\begin{cases} I_1(h, k) = \sum_{i=1}^{N} \int_\Omega a_i\, [x, \partial_i T_h(w - \lambda_0 T_k(w - v))]\, \partial_i T_k\, (w - T_h(w - \lambda_0 T_k(w - v)))\, dx; \\ I_2(h, k) = \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w) \partial_i T_k\, (w - T_h(w - \lambda_0 T_k(w - v)))\, dx; \\ \text{and} \\ I_3(h, k) = \int_\Omega f T_k(w - T_h(w - \lambda_0 T_k(w - v))) dx. \end{cases}$$

Put

$$E_1(h, k) = \{x \in \Omega;\ |w - T_h(w - \lambda_0 T_k(w - v))| \le k\},$$

and

$$E_2(h, k) = \{x \in \Omega;\ |(w - \lambda_0 T_k(w - v))| \le h\}.$$

Then we obtain

$$I_1(h, k) = \sum_{i=1}^{N} \int_{E_1} a_i\, [x, \partial_i T_h(w - \lambda_0 T_k(w - v))]\, \partial_i T_k\, (w - T_h(w - \lambda_0 T_k(w - v)))\, dx$$
$$+ \sum_{i=1}^{N} \int_{E_1^C} a_i\, [x, \partial_i T_h(w - \lambda_0 T_k(w - v))]\, \partial_i T_k\, (w - T_h(w - \lambda_0 T_k(w - v)))\, dx.$$

Since $\partial_i T_k(w - T_h(w - \lambda_0 T_k(w - v)) = 0$ in E_1^C, we have

$$\sum_{i=1}^{N} \int_{E_1^C} a_i\, [x, \partial_i T_h(w - \lambda_0 T_k(w - v))]\, \partial_i T_k\, (w - T_h(w - \lambda_0 T_k(w - v)))\, dx \quad = 0,$$

$$\begin{aligned} I_1(h,k) &= \sum_{i=1}^{N} \int_{E_1} a_i \left[x, \partial_i T_h(w-\lambda_0 T_k(w-v))\right] \partial_i \left(w - T_h(w-\lambda_0 T_k(w-v))\right) dx \\ &= \sum_{i=1}^{N} \int_{E_1\cap E_2} a_i \left[x, \partial_i T_h(w-\lambda_0 T_k(w-v))\right] \partial_i \left(w - T_h(w-\lambda_0 T_k(w-v))\right) dx \\ &\quad + \sum_{i=1}^{N} \int_{E_1\cap E_2^C} a_i \left[x, \partial_i T_h(w-\lambda_0 T_k(w-v))\right] \partial_i \left(w - T_h(w-\lambda_0 T_k(w-v))\right) dx \end{aligned}$$

Since $\partial_i T_h(w-\lambda_0 T_k(w-v)) = 0$ on E_2^C and $\partial_i \left(w - T_h(w-\lambda_0 T_k(w-v))\right) = \lambda_0 \partial_i \left(T_k(w-v)\right)$ on E_2, we obtain

$$\sum_{i=1}^{N} \int_{E_1\cap E_2^C} a_i \left[x, \partial_i T_h(w-\lambda_0 T_k(w-v))\right] \partial_i \left(w - T_h(w-\lambda_0 T_k(w-v))\right) dx = 0,$$

and

$$I_1(h,k) = \lambda_0 \sum_{i=1}^{N} \int_{E_1\cap E_2} a_i \left[x, \partial_i T_h(w-\lambda_0 T_k(w-v))\right] \partial_i \left(T_k(w-v)\right) dx.$$

Letting $h \to +\infty$ and $|\theta| \le 1$, we have $E_1 \cap E_2 \to \Omega$.

By using the Lebesgue theorem, we get

$$\lim_{h\to+\infty} I_1(h,k) = \lambda_0 \sum_{i=1}^{N} \int_{\Omega} a_i \left[x, \partial_i (w-\lambda_0 T_k(w-v))\right] \partial_i \left(T_k(w-v)\right) dx. \tag{3.17}$$

Using the same techniques, we show that

$$\lim_{h\to+\infty} I_2(h,k) = \lambda_0 \sum_{i=1}^{N} \int_{\Omega} \Phi_i \left(x, w\right) \partial_i \left(T_k(w-v)\right) dx \tag{3.18}$$

and

$$\lim_{h\to+\infty} I_3(h,k) = \lambda_0 \int_{\Omega} f(x) T_k(w-v) dx. \tag{3.19}$$

From (3.16)-(3.19), and passing to the limit, we get

$$\begin{aligned} &\lambda_0 \sum_{i=1}^{N} \int_{\Omega} a_i \left[x, \partial_i (w-\lambda_0 T_k(w-v))\right] \partial_i \left(T_k(w-v)\right) dx \\ &\le \lambda_0 \left[\sum_{i=1}^{N} \int_{\Omega} \Phi_i \left(x, w\right) \partial_i \left(T_k(w-v)\right) dx + \int_{\Omega} f(x) T_k(w-v) dx \right] \end{aligned}$$

For $\lambda_0 > 0$ and letting $\lambda_0 \to 0$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} a_i \left(x, \partial_i w\right) \partial_i \left(T_k(w-v)\right) dx \\ &\le \sum_{i=1}^{N} \int_{\Omega} \Phi_i \left(x, w\right) \partial_i \left(T_k(w-v)\right) dx + \int_{\Omega} f(x) T_k(w-v) dx \end{aligned} \tag{3.20}$$

Choosing $\lambda_0 < 0$ and letting λ_0 tend to zero, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} a_i \left(x, \partial_i w\right) \partial_i \left(T_k(w-v)\right) dx \\ &\ge \sum_{i=1}^{N} \int_{\Omega} \Phi_i \left(x, w\right) \partial_i \left(T_k(w-v)\right) dx + \int_{\Omega} f(x) T_k(w-v) dx \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we conclude

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,\partial_i w\right)\partial_i\left(T_k(w-v)\right)dx\\&=\sum_{i=1}^{N}\int_{\Omega}\Phi_i\left(x,w\right)\partial_i\left(T_k(w-v)\right)dx+\int_{\Omega}f(x)T_k(w-v)dx\end{aligned} \tag{3.22}$$

□

Existence results

Theorem 3.13. *Let the hypotheses **(H1)-(H5)** hold, and let f be in $L^1(\Omega)$, then there exists an entropy solution w of the problem (3.13).*

Proof of Theorem 3.13

- **Approximate problem and a priori estimate:**

Let $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$. We consider the problem

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(a_i(x,\partial_i w_n)\right)-\sum_{i=1}^{N}\partial_i(\Phi_{i,n}(x,w_n))=f_n & \text{in } \Omega\\ w_n=0 & \text{on } \partial\Omega\end{cases} \tag{4.23}$$

with $\Phi_{i,n}(x,s)=\Phi_i(x,T_n(s))$. We define the operator $\mathbb{B}_n : \overset{\circ}{W}{}^1_{\vec{G}}(\Omega) \to W^{-1}L_{\vec{G}}(\Omega)$ by

$$\mathbb{B}_n(w_n)=\sum_{i=1}^{N}\partial_i\left(a_i(x,\partial_i w_n)\right)+\sum_{i=1}^{N}\partial_i\left(\Phi_{i,n}(x,w_n)\right).$$

We prove that $b_{i,n}(x,w_n,\nabla w_n)=a_i(x,\partial_i w_n)+\Phi_{i,n}(x,w_n)$ satisfies the assumptions (A_1), (A_2), (A_3), and (A_4) mentioned in [17]. From the assumptions (3.9), (3.10), (3.11), and (3.12), it's quite easy to see that $b_n=(b_{i,n}(x,w_n,\nabla w_n))_i$ satisfies (A_1), (A_2), and (A_3). It remains to show (A_4).

Let $w_n \in \overset{\circ}{W}{}^1_{\vec{G}}(\Omega)$, using (3.12) and Young's inequality, we get

$$\begin{aligned}\left|\sum_{i=1}^{N}\Phi_{i,n}(x,w_n)\partial_i w_n\right| \le{}& \frac{\alpha}{2}G_0\left(\frac{2}{\alpha}\left|\gamma(x)B^{*-1}B(\beta|T_n(w_n)|)\right|\right)\\&+\frac{\alpha}{2}\sum_{i=1}^{N}G_0\left(|\partial_i w_n|\right),\end{aligned} \tag{4.24}$$

which implies

$$\left|\sum_{i=1}^{N}\Phi_{i,n}(x,w_n)\partial_i w_n\right| \le C_n(x)+\frac{\alpha}{2}\sum_{i=1}^{N}G_0\left(|\partial_i w_n|\right), \tag{4.25}$$

where

$$C_n(x)=\frac{\alpha}{2}G_0^*\left[\frac{2}{\alpha}|\gamma(x)|\times B^{*-1}B(n\beta)\right].$$

Thus, we obtain

$$\sum_{i=1}^{N}\Phi_{i,n}(x,w_n)\partial_i w_n \ge -C_n(x)-\frac{\alpha}{2}\sum_{i=1}^{N}G_0\left(|\partial_i w_n|\right). \tag{4.26}$$

From (3.11) and (4.26), we deduce

$$\sum_{i=1}^{N}\left[a_i(x,\partial_i w_n)+\Phi_{i,n}(x,w_n)\right]\partial_i w_n \geq -C_n(x)+\frac{\alpha}{2}\sum_{i=1}^{N}G_0\left(|\partial_i w_n|\right), \tag{4.27}$$

then the hypothesis (A_4) in [17] is verified. Consequently, the approximate problem (4.23) admits a weak solution $w_n \in \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega)$.

Lemma 4.14. *Assume that the hypotheses (3.9)-(3.12) hold true, and let w_n be a solution of the approximate problem (4.23). Then we have*

a) For $k>0$,

$$\sum_{i=1}^{N}\int_\Omega G_i\left(\frac{\partial_i T_k(w_n)}{\mu_0}\right)\leq C_k,$$

where C_k is a positive constant independent of n.

b) $\lim_{k\to+\infty} mes\,\{x\in\Omega\,;\ |w_n|>k\}=0$.

Proof. **Show a):** We take $T_k(w_n)$ as a test function in (4.23), we get

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\partial_i w_n)\partial_i T_k(w_n)dx+\sum_{i=1}^{N}\int_\Omega \Phi_{i,n}(x,w_n)\partial_i T_k(w_n)dx=\int_\Omega f_n T_k(w_n)dx$$

which implies that,

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\partial_i T_k(w_n))\partial_i T_k(w_n)dx+\sum_{i=1}^{N}\int_\Omega \Phi_i(x,T_k(w_n))\partial_i T_k(w_n)dx=\int_\Omega f_n T_k(w_n)dx \tag{4.28}$$

While $B<<G_0$, we have, for all $\varepsilon>0$, there exists a constant c_ε such that

$$B(t)\leq G_0(\varepsilon t)+c_\varepsilon,\quad \forall t\geq 0. \tag{4.29}$$

Using the Poincaré inequality, there exist δ_{p_0} and c_{p_0} two strictly positive constants such that

$$\int_\Omega G_0\left(\delta_{p_0}|v|\right)dx\leq \int_\Omega c_{p_0}G_0\left(|\partial_i v|\right)dx\quad \forall v\in \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega). \tag{4.30}$$

From (4.30) and (4.29), we get

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega \Phi_i(x,T_k(w_n))\partial_i T_k(w_n)dx &\leq \sum_{i=1}^{N}\int_\Omega |\gamma(x)|B^{*-1}B(\beta|T_k(w_n)|)|\partial_i T_k(w_n)|dx\\ &\leq \sum_{i=1}^{N}\int_\Omega |\gamma(x)|B^{*-1}G_0((\varepsilon\beta|T_k(w_n)|)+c_\varepsilon)|\partial_i T_k(w_n)|dx\\ &\leq \sum_{i=1}^{N}\int_\Omega |\gamma(x)|\left[G_0(\varepsilon\beta|T_k(w_n)|)+c_\varepsilon)+B(|\partial_i T_k(w_n)|)\right]dx\\ &\leq \sum_{i=1}^{N}\int_\Omega |\gamma(x)|\left[c_{p_0}G_0(\frac{\varepsilon\beta}{\delta_{p_0}}|\partial_i T_k(w_n)|)+G_0\left(\varepsilon|\partial_i T_k(w_n)|\right)+2c_\varepsilon\right]dx,\end{aligned} \tag{4.31}$$

According to (3.11), (4.28), and (4.31), for $\varepsilon = \frac{\alpha\lambda_0\delta_p}{4(1+\alpha)(1+c_{p_0})(1+\delta_{p_0})(1+\mu_0)(1+\beta)(1+||\gamma||_\infty)}$, we get

$$\left(\frac{\alpha}{2}\right)\sum_{i=1}^{N}\int_\Omega G_i\left(\frac{|\partial_i T_k(w_n)|}{\mu_0}\right)dx \quad \leq d||\gamma||_\infty\left(2c_\varepsilon \mathrm{mes}(\Omega)\right)+k||f||_1, \tag{4.32}$$

hence

$$\sum_{i=1}^{N}\int_\Omega G_i\left(\frac{|\partial_i T_k(w_n)|}{\mu_0}\right)dx \quad \leq C_k, \tag{4.33}$$

where $C_k = \left(\frac{2d||\gamma||_\infty(2c_\varepsilon \mathrm{mes}(\Omega))+2k||f||_1}{\alpha}\right)$.
Show b): Using the Poincaré inequality (4.30), we get

$$\begin{aligned} G_0\left(\frac{\delta_{p_0}k}{\mu_0}\right)\mathrm{meas}\left(\{|w_n|>k\}\right) &= \int_{\{|w_n|>k\}} G_0\left(\frac{\delta_{p_0}|T_k(w_n)|}{\lambda_0}\right)dx \\ &\leq \int_\Omega c_{p_0}G_0\left(\frac{|\partial_i T_k(w_n)|}{\mu_0}\right)dx \\ &\leq \sum_{i=1}^{N}\int_\Omega G_i\left(\frac{|\partial_i T_k(w_n)|}{\mu_0}\right)dx \leq C_k. \end{aligned} \tag{4.34}$$

This implies that

$$\mathrm{meas}\left(\{|w_n|>k\}\right) \quad \leq \frac{C_k}{G_0\left(\frac{\delta_{p_0}k}{\mu_0}\right)}. \tag{4.35}$$

By passing to the limit we get

$$\lim_{k\to+\infty}\mathrm{meas}\left(\{|w_n|>k\}\right) \quad = \lim_{k\to+\infty}\frac{C_k}{G_0\left(\frac{\delta_p k}{\mu_0}\right)} = 0. \tag{4.36}$$

□

For every fixed $\mu>0$ and every real positive k, we have

$$\begin{aligned} \mathrm{meas}(\{\,|w_p-w_q|>\mu\}) \leq \mathrm{meas}(\{|w_p|>k\}) + \mathrm{meas}(\{|w_q|>k\}) \\ +\mathrm{meas}(\{\,|T_k(w_p)-T_k(w_q)|>\mu\}). \end{aligned} \tag{4.37}$$

Given that $(T_k(w_n))_n$ is bounded in $\overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega)$ for every $k>0$, there exists a certain $\xi_k \in \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega)$ such that

$$T_k(w_n) \rightharpoonup \xi_k \text{ weakly in } \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega). \tag{4.38}$$

Hence, we can conclude that $(T_k(w_n))_n$ forms a Cauchy sequence in measure in Ω.
Then for all $\epsilon>0$, Using (4.37) and **Lemma 4.14**, there exists $N_0(\mu,\epsilon)>0$ such that

$$\mathrm{meas}(\{\,|w_p-w_q|>\mu\}) \leq \epsilon, \text{ for all } p,q \geq N_0((\mu,\epsilon)), \tag{4.39}$$

consequently $(w_n)_n$ is a Cauchy sequence in measure in Ω. It follows that there exists a subsequence denoted by $(w_n)_n$ which converges almost everywhere to some u. Then

$$T_k(w_n) \rightharpoonup T_k(w) \text{ weakly in } \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega), \tag{4.40}$$

which implies that

$$\xi_k = T_k(w). \tag{4.41}$$

- **Modular convergence of $(\Phi_i(x,T_k(w_n)))_n$:**

Lemma 4.15. *Let w_n be a solution of the approximate problem (4.23). Then for almost every x in Ω, the sequence $(\Phi_i(x, T_k(w_n)))_n$ converges to $\Phi_i(x, T_k(w))$ with respect to the modular convergence in $L_{B^*}(\Omega)$.*

Proof. Let us show that there exists $\mu > 0$ such that

$$\int_\Omega B^*\left(\frac{|\Phi_i(x, T_k(w_n)) - \Phi_i(x, T_k(w))|}{\mu}\right) dx \to 0 \quad \text{as} \quad n \to \infty.$$

First of all, the sequence $(T_k(w_n))$ converges nearly everywhere to $T_k(w)$, and $B^*(0) = 0$. Since B^* is a convex continuous function, and υ is a Carathéodory function, then

$$B^*\left(\frac{|\Phi_i(x, T_k(w_n)) - \Phi_i(x, T_k(w))|}{\mu}\right) \to 0 \text{ a.e. as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned} |\Phi_i(x, T_k(w_n)) - \Phi_i(x, T_k(w))| &\leq |\Phi_i(x, T_k(w_n))| + |\Phi_i(x, T_k(w))| \\ &\leq 2||\gamma(x)||_\infty B^{*-1}(B(\beta k)) \end{aligned}$$

so, choosing μ such that $\frac{2||\gamma(x)||_\infty}{\mu} < 1$ and using the convexity of B^*, we get

$$\begin{aligned} B^*\left(\frac{|\Phi_i(x,T_k(w_n))-\Phi_i(x,T_k(w))|}{\mu}\right) &\leq B^*\left(\frac{2||\gamma(x)||_\infty B^{*-1}(B(\beta k))}{\mu}\right) \\ &\leq \left(\frac{2||\gamma(x)||_\infty}{\mu}\right) B(\beta k) = g_k(x). \end{aligned}$$

Then, by Lebesgue's dominated convergence theorem, we obtain

$$\int_\Omega B^*\left(\frac{|\Phi_i(x, T_k(w_n)) - \Phi_i(x, T_k(w))|}{\mu}\right) dx \to 0 \quad \text{as} \quad n \to \infty,$$

□

- **Minty's inequality:**

Lemma 4.16. *Let w_n be a solution of the approximate problem (4.23). Then for $\upsilon \in \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$ we have:*

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i \upsilon)\partial_i T_k(w_n - \upsilon)\, dx \leq \int_\Omega f_n T_k(w_n - \upsilon)\, dx \\ - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w_n)\partial_i T_k(w_n - \upsilon)\, dx \end{aligned} \tag{4.42}$$

Proof. Let $\upsilon \in \overset{\circ}{W}{}^{1}_{\vec{G}}(\Omega) \cap L^\infty(\Omega)$, by taking $T_k(w_n - \upsilon)$ as a test function in (4.23), we get

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i w_n)\partial_i T_k(w_n - \upsilon)\, dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, T_k(w_n))\partial_i T_k(w_n - \upsilon)\, dx \\ = \int_\Omega f_n T_k(w_n - \upsilon))dx, \end{aligned}$$

by subtracting the term $\sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i \upsilon)\partial_i T_k(w_n - \upsilon)\, dx$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega [a_i(x, \partial_i w_n) - a_i(x, \partial_i \upsilon)]\, \partial_i T_k(w_n - \upsilon)\, dx \\ &= \int_\Omega f_n T_k(w_n - \upsilon)\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, w_n)\partial_i T_k(w_n - \upsilon)\, dx \\ &\quad - \sum_{i=1}^{N} \int_\Omega a_i(x, \partial_i \upsilon)\partial_i T_k(w_n - \upsilon)\, dx. \end{aligned}$$

According to (3.10) and using the definition of truncation function, we may get

$$\sum_{i=1}^{N}\int_{\Omega}\left[a_i(x,\partial_i w_n)-a_i(x,\partial_i v)\right]\partial_i T_k\left(w_n-v\right)dx\geq 0,$$

we deduce from the above that

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,\partial_i v)\partial_i T_k\left(w_n-v\right)dx\leq\int_{\Omega}fT_k\left(w_n-v\right)dx-\sum_{i=1}^{N}\int_{\Omega}\Phi_i(x,w_n)\partial_i T_k\left(w_n-v\right)dx$$

□

- **Passing to the limit:** We shall prove that for $v\in W_0^1L_{G_i}(\Omega)\cap L^\infty(\Omega)$, $i=0,\dots,N$, we have

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,\partial_i v)\partial_i T_k\left(w-v\right)dx\leq\int_{\Omega}fT_k\left(w-v\right)dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}\Phi_i(x,w)\partial_i T_k\left(w-v\right)dx.$$

Let $M=k+||v||_\infty$. From (4.14), we have $T_M(w_n)\rightharpoonup T_G(w)$ weakly in $W_0^1L_{G_i}(\Omega)$. Then we have

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,\partial_i v)\partial_i T_k\left(w_n-v\right)dx\to\sum_{i=1}^{N}\int_{\Omega}a_i(x,\partial_i v)\partial_i T_k\left(w-v\right)dx. \tag{4.43}$$

Again using the fact that $T_k\left(w_n-v\right)\rightharpoonup T_k\left(w-v\right)$ weakly in $W_0^1L_{G_i}(\Omega)$ and that $((\Phi_i\left(x,T_k(w_n)\right))_n$ converges to $\Phi_i\left(x,T_k(w)\right)$ with respect to the modular convergence, then we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\Phi_i\left(x,T_k(w_n)\right)\partial_i T_k\left(w_n-v\right)dx\to\sum_{i=1}^{N}\int_{\Omega}\Phi_i\left(x,T_k(w)\right)\partial_i T_k\left(w-v\right)dx. \tag{4.44}$$

Since $f_n\to f$ and $T_k(w_n-v)\to T_k(w-v)$ almost everywhere in Ω then $f_nT_k(w_n-v)\to fT_k(w-v)$ almost everywhere in Ω. On the other hand we have $|f_nT_k(w_n-v)|\leq kf$, so by using Vitali's theorem, we obtain

$$\int_{\Omega}f_nT_k\left(w_n-v\right)dx\to\int_{\Omega}fT_k\left(w-v\right)dx. \tag{4.45}$$

From (4.43), (4.44), (4.45) and by passing to the limit in (4.42), we deduce

$$\sum_{i=1}^{N}\int_{\Omega}a_i\left(x,\partial_i v\right)\partial_i T_k\left(w-v\right)dx\leq\int_{\Omega}fT_k\left(w-v\right)dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}\Phi_i\left(x,w\right)\partial_i T_k\left(w-v\right)dx.$$

In view of **Lemma 3.12**, we conclude that w is an entropy solution of the problem (3.13). This achieves the proof of **Theorem 3.13**.

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