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An Application of Λ-Cyclic Codes

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Abstract
In this paper, the Λ-cyclic codes of block length $(\alpha_{i+j+k})_{(i,j,k)} \in I$ are investigated, where $\Lambda = M_{000}M_{100}M_{110}M_{111}$ and $I = \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$ in order to search one application of them. By defining a Gray map, we obtain the Gray images of Λ-cyclic codes. Their generator polynomials are given. The structures of separable codes are discussed. The necessary and sufficient conditions of Λ-cyclic codes to be reversible and reversible complement are discussed. We obtain cyclic DNA codes from Λ-cyclic codes.

Keywords: DNA codes; cyclic codes; separable codes.

Introduction

Cyclic codes are error-correcting codes and they have algebraic properties that are suitable for efficient error correction and detection. In addition to studying some features of cyclic codes, studying the application areas of these codes attracted the attention of researchers. One of the application areas of cyclic codes is DNA codes. Some researchers studied cyclic codes over a finite ring or a family of finite rings to obtain DNA codes [2, 5], respectively. Recently, researchers have begun to study cyclic code over a mixed alphabet to obtain DNA code [3]. Motivated by the work [3], we decide to study on $\Lambda = M_{000}M_{100}M_{110}M_{111}$-cyclic codes in order to construct DNA codes, where $M_{000} = F_q$ are finite fields with $q = p^e$ elements, p is a prime, $e \geq 1$ is a positive integer and $M_{100} = F_q[u]/\langle u^2 - \beta_1 u\rangle$, $M_{110} = F_q[u,v]/\langle u^2 - \beta_1 u, v^2 - \beta_2 v, uv - vu\rangle$, $M_{111} = F_q[u,v,w]/\langle u^2 - \beta_1 u, v^2 - \beta_2 v, w^2 - \beta_3 w, uv - vu, uw - wu, vw - wv\rangle$ are finite commutative rings, where $\beta_1, \beta_2, \beta_3 \in M_{000}{}^*$. The aim of this paper is that we hope that to find optimal DNA codes that would not be found in the literature.

The paper is organized as follows. In section 2, some basic knowledge is given. The sets M_{ijk} are introduced for $(i,j,k) \in I$, where I is lexicographic ordered set and it is equal to $\{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$. The structures of the linear codes over M_{ijk}, for $(i,j,k) \in I'$, where $I' = \{(1,0,0), (1,1,0), (1,1,1)\}$ and the Gray maps are given. The structures of the cyclic codes over M_{ijk} are obtained, for $(i,j,k) \in I'$. In section 3, Λ- linear codes are investigated. Λ-cyclic codes are introduced. The Gray map is given, in section 4. The Gray images of Λ-cyclic are obtained. In section 5, the algebraic structures of Λ- cyclic codes are discussed. The separable codes are searched. In section 6, the necessary and sufficient conditions of the Λ- cyclic to be

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reversible and reversible complement are determined. It is obtained the DNA codes, by using a map and Λ-cyclic codes.

Preliminaries

Let F_q denote finite fields of characteristic p with q elements. A k dimensional subspace C of ${F_q}^n$ is called a linear code over F_q of length n. Each element $\mathbf{c} \in C$ is called a codeword. A linear code C over F_q with length n is cyclic code if $\mathbf{c} = (c_0, \ldots, c_{n-1}) \in C$, then its cyclic shift $\sigma(\mathbf{c}) = (c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. The Hamming weight of $\mathbf{c}$ is defined as the number of its non-zero components, it is denoted by $w_H(\mathbf{c})$. The Hamming weight of a code C is defined as the smallest Hamming weight among all its non-zero codewords, it is denoted by $w_H(C)$. The Hamming distance between $\mathbf{c}$ and $\mathbf{e}$ is defined $d_H(\mathbf{c}, \mathbf{e}) = w_H(\mathbf{c} - \mathbf{e})$. The Hamming distance of a code C is defined as $d_H(C) = min\{d_H(\mathbf{c}, \mathbf{e}) | \mathbf{c} \neq \mathbf{e}, \forall \mathbf{c}, \mathbf{e} \in C\}$. The dual of C is defined $C^{\perp} = \{\hat{\mathbf{c}} = (\hat{c}_0, \ldots, \hat{c}_{n-1}) \in {F_q}^n | c_0\hat{c}_0 + \ldots. + c_{n-1}\hat{c}_{n-1} = 0, \text{ all } \mathbf{c} = (c_0, \ldots, c_{n-1}) \in C\}$.

Let R denote any finite commutative ring. A nonempty subset D of R^n is called linear code of length n over R if D forms an R-submodule of R^n. A linear code D over R with length n is cyclic code if $\mathbf{d} = (d_0, \ldots, d_{n-1}) \in D$, then its cyclic shift $\sigma(\mathbf{d}) = (d_{n-1}, d_0, \ldots, d_{n-2}) \in D$. By identifying each codeword $\mathbf{d} = (d_0, \ldots, d_{n-1})$ to a polynomial $d(x) = d_0 + d_1x + \ldots + d_{n-1}x^{n-1}$ in $R[x]/\langle x^n - 1\rangle$, a linear code D is a cyclic code if and only if it is an ideal of the ring $R[x]/\langle x^n - 1\rangle$.

Let

$$M_{ijk} = \left\{ \begin{array}{c} \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} u^i v^j w^k a_{ijk} | a_{ijk} \in M_{000}, u^2 = \beta_1 u, v^2 = \beta_2 v, \\ w^2 = \beta_3 w, \beta_1, \beta_2, \beta_3 \in {M_{000}}^* \end{array} \right\}$$

be commutative rings, where $(i, j, k) \in I' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ and $M_{000} = F_q$ are finite fields with $q = p^e$ elements, p is a prime, $e \geq 1$ is a positive integer. For $(i, j, k) = (1, 0, 0)$, then the finite ring $M_{100} = F_q[u]/\left\langle u^2 - \beta_1 u \right\rangle$. For $(i, j, k) = (1, 1, 0)$, then the finite ring $M_{110} = F_q[u, v]/\left\langle u^2 - \beta_1 u, v^2 - \beta_2 v, uv - vu \right\rangle$. For $(i, j, k) = (1, 1, 1)$, then the finite ring
$M_{111} = F_q[u, v, w]/\left\langle u^2 - \beta_1 u, v^2 - \beta_2 v, w^2 - \beta_3 w, uv - vu, uw - wu, vw - wv \right\rangle$, where $\beta_1, \beta_2, \beta_3 \in {M_{000}}^*$. They have q^2, q^4, q^8 elements, respectively.

For $(i, j, k) \in I'$, an arbitrary element $m_{ijk} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} u^i v^j w^k a_{ijk} \in M_{ijk}$ can be uniquely

$$\sum_{(r,s,t)\in T} \eta_{rst} b_{\alpha_{i+j+k}, rst}$$

where $b_{\alpha_{i+j+k}, rst} \in M_{000}$ and the set T is lexicographic ordered on the cartesian product $A \times B \times C$. Since $A = \{0, 1\}, B = \{0\}, C = \{0\}$, for R_{100}, then $T = \{(0, 0, 0), (1, 0, 0)\}$. Since $A = \{0, 1\}, B = \{0, 1\}, C = \{0\}$, for R_{110}, then $T = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 0)\}$. Since $A = \{0, 1\}, B = \{0, 1\}, C = \{0, 1\}$, for R_{111}, then $T = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$.

Moreover, η_{rst} are idempotent generators, where $(r, s, t) \in T$, for $(i, j, k) \in I'$. That is, $\sum_{(r,s,t)\in T} \eta_{rst} = 1$, $\eta_{rst}^2 = \eta_{rst}$ and $\eta_{rst}\eta_{lzn} = 0$, where $\eta_{rst} \neq \eta_{lzn}$, all $(r, s, t), (l, z, n) \in T$, for $(i, j, k) \in I'$.

For the finite ring M_{100}, then

$$\eta_{000} = u/\beta_1, \eta_{100} = (\beta_1 - u)/\beta_1.$$

For the finite ring M_{110}, then

$$\eta_{000} = (uv)/\beta_1\beta_2, \eta_{010} = (u\beta_2 - uv)/\beta_1\beta_2,$$

$$\eta_{100} = (v\beta_1 - uv)/\beta_1\beta_2, \eta_{110} = (\beta_1\beta_2 - \beta_1 v - u\beta_2 + uv)/\beta_1\beta_2.$$

For the finite ring M_{111}, then

$$\eta_{000} = (wuv)/\beta_1\beta_2\beta_3, \eta_{001} = (uv\beta_3 - uvw)/\beta_1\beta_2\beta_3,$$

$$\eta_{010} = (uw\beta_2 - uvw)/\beta_1\beta_2\beta_3, \eta_{011} = (u\beta_2\beta_3 - uw\beta_2 - vu\beta_3 + uvw)/\beta_1\beta_2\beta_3,$$

$$\eta_{100} = (vw\beta_1 - uvw)/\beta_1\beta_2\beta_3, \eta_{101} = (v\beta_1\beta_3 - vw\beta_1 - uv\beta_3 + uvw)/\beta_1\beta_2\beta_3,$$

$$\eta_{110} = (\beta_1\beta_2 w - \beta_1 vw - uw\beta_2 + uvw)/\beta_1\beta_2\beta_3,$$

$$\eta_{111} = (\beta_1\beta_3\beta_2 - w\beta_1\beta_2 - \beta_1\beta_3 v + vw\beta_1 - u\beta_2\beta_3 + uw\beta_2 + uv\beta_3 - uvw)/\beta_1\beta_2\beta_3.$$

The Gray map is defined as follows;

$$\begin{array}{rcl} \psi_{ijk} & : & M_{ijk} \longrightarrow M_{000}^{2^{i+j+k}} \\ \sum_{(r,s,t)\in T} \eta_{rst} b_{\alpha_{i+j+k},rst} & \longmapsto & (f_l)_{l\in U} \end{array}$$

where $f_l = b_{\alpha_{i+j+k},rst}$, for $(r,s,t) \in T$ and $U = \{1, 2, 3, ..., 2^{i+j+k}\}$ for $(i,j,k) \in I'$. For example;

$$\begin{array}{rcl} \psi_{110} & : & M_{110} \longrightarrow M_{000}^4 \\ m_{110} & \longmapsto & (b_{\alpha_2,000}, b_{\alpha_2,010}, b_{\alpha_2,100}, b_{\alpha_2,110}). \end{array}$$

We define the Lee weight of $m_{ijk} \in M_{ijk}$ as ;

$$w_L(m_{ijk}) = w_H(\psi_{ijk}(m_{ijk}))$$

where $(i,j,k) \in I'$ and w_H denotes the Hamming weight over M_{000}.

This map can be extended from $M_{ijk}^{\alpha_{i+j+k}}$ to $M_{000}^{\alpha_{i+j+k}2^{i+j+k}}$ as follows,

$$\psi_{ijk} : M_{ijk}^{\alpha_{i+j+k}} \longrightarrow M_{000}^{\alpha_{i+j+k}2^{i+j+k}}$$

$$\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V} \longmapsto (b^a_{\alpha_{i+j+k},rst})_{(r,s,t)\in T, a\in V}$$

where $(i,j,k) \in I'$. The elements $\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V} \in M_{ijk}^{\alpha_{i+j+k}}$ denotes $\mathbf{m_{ijk}} = (m_{ijk}^0, ..., m_{ijk}^{\alpha_{i+j+k}-1})$. The Lee weight of $\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V} \in M_{ijk}^{\alpha_{i+j+k}}$ where $V = \{0, 1, ..., \alpha_{i+j+k} - 1\}$ for $(i,j,k) \in I'$ is $w_L(\mathbf{m_{ijk}}) = \sum_{a=0}^{\alpha_{i+j+k}-1} w_L(m_{ijk}^a)$. The Lee distance between any two elements $\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V}$,$\hat{\mathbf{m_{ijk}}} = (\hat{m}_{ijk}^a)_{a\in V} \in M_{ijk}^{\alpha_{i+j+k}}$, where $V = \{0, 1, 2, ..., \alpha_{i+j+k} - 1\}$ for $(i,j,k) \in I'$ is defined as

$$d_L(\mathbf{m_{ijk}}, \hat{\mathbf{m_{ijk}}}) = w_H(\psi_{ijk}(\mathbf{m_{ijk}} - \hat{\mathbf{m_{ijk}}})).$$

It is easily seen that the Gray map ψ_{ijk} is an M_{000}-linear and distance preserving map from $M_{ijk}^{\alpha_{i+j+k}}$(Lee distance) to $M_{000}^{\alpha_{i+j+k}2^{i+j+k}}$ (Hamming Distance), where $(i,j,k) \in I'$.

Let $C_{\alpha_{i+j+k}}$ be a linear code of length α_{i+j+k} over M_{ijk} for $(i,j,k) \in I'$. We define

$$C_{\alpha_{i+j+k},oln} = \left\{ \begin{array}{c} \mathbf{b}_{\alpha_{i+j+k},oln} = (b^a_{\alpha_{i+j+k},oln})_{a\in V} \in M_{000}^{\alpha_{i+j+k}} : \sum_{(r,s,t)\in T} \eta_{rst}\mathbf{b}_{\alpha_{i+j+k},rst} \\ \in C_{\alpha_{i+j+k}} \text{ for some } \mathbf{b}_{\alpha_{i+j+k},rst} \neq \mathbf{b}_{\alpha_{i+j+k},oln} \end{array} \right\}$$

where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Then $C_{\alpha_{i+j+k},oln}$ are linear codes of length α_{i+j+k} over M_{000} for $(i,j,k) \in I'$ and $C_{\alpha_{i+j+k}}$ can be uniquely written as;

$$C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$$

for $(i,j,k) \in I'$. Then $|C_{\alpha_{i+j+k}}| = \Pi_{(o,l,n)\in T}|C_{\alpha_{i+j+k},oln}|$, for $(i,j,k) \in I'$.

Since ψ_{ijk} is distance preserving map and $\psi_{ijk}(C_{\alpha_{i+j+k}}) = \otimes_{(o,l,n)\in T}C_{\alpha_{i+j+k},oln}$, for $(i,j,k) \in I'$, we get

$$d_L(C_{\alpha_{i+j+k}}) = min\{d_H(C_{\alpha_{i+j+k},oln})|(o,l,n) \in T\}$$

Example 1 *If C_{α_2} is a linear code of length α_2 over M_{110}, then $C_{\alpha_2} = \eta_{000}C_{\alpha_2,000} + \eta_{010}C_{\alpha_2,010} + \eta_{100}C_{\alpha_2,100} + \eta_{110}C_{\alpha_2,110}$, where*

$C_{\alpha_2,000} = \{\mathbf{b}_{\alpha_2,000} \in M_{000}^{\alpha_2} | \eta_{000}\mathbf{b}_{\alpha_2,000} + \eta_{010}\mathbf{b}_{\alpha_2,010} + \eta_{100}\mathbf{b}_{\alpha_2,100} + \eta_{110}\mathbf{b}_{\alpha_2,110} \in C_{\alpha_2}$ *for some* $\mathbf{b}_{\alpha_2,000} \neq \mathbf{b}_{\alpha_2,rst}\}$,
$C_{\alpha_2,010} = \{\mathbf{b}_{\alpha_2,010} \in M_{000}^{\alpha_2} | \eta_{000}\mathbf{b}_{\alpha_2,000} + \eta_{010}\mathbf{b}_{\alpha_2,010} + \eta_{100}\mathbf{b}_{\alpha_2,100} + \eta_{110}\mathbf{b}_{\alpha_2,110} \in C_{\alpha_2}$ *for some* $\mathbf{b}_{\alpha_2,010} \neq \mathbf{b}_{\alpha_2,rst}\}$,
$C_{\alpha_2,100} = \{\mathbf{b}_{\alpha_2,100} \in M_{000}^{\alpha_2} | \eta_{000}\mathbf{b}_{\alpha_2,000} + \eta_{010}\mathbf{b}_{\alpha_2,010} + \eta_{100}\mathbf{b}_{\alpha_2,100} + \eta_{110}\mathbf{b}_{\alpha_2,110} \in C_{\alpha_2}$ *for some* $\mathbf{b}_{\alpha_2,100} \neq \mathbf{b}_{\alpha_2,rst}\}$,
$C_{\alpha_2,110} = \{\mathbf{b}_{\alpha_2,110} \in M_{000}^{\alpha_2} | \eta_{000}\mathbf{b}_{\alpha_2,000} + \eta_{010}\mathbf{b}_{\alpha_2,010} + \eta_{100}\mathbf{b}_{\alpha_2,100} + \eta_{110}\mathbf{b}_{\alpha_2,110} \in C_{\alpha_2}$ *for some* $\mathbf{b}_{\alpha_2,110} \neq \mathbf{b}_{\alpha_2,rst}\}$.

Theorem 2 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$ be a linear code of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I'$. Then $C_{\alpha_{i+j+k}}$ is a cyclic code of length α_{i+j+k} if and only if $C_{\alpha_{i+j+k},oln}$ are cyclic codes of length α_{i+j+k} over M_{000}, where $(o,l,n) \in T$ for $(i,j,k) \in I'$.*

Proof. Let $(b^a_{\alpha_{i+j+k},oln})_{a\in V} \in C_{\alpha_{i+j+k},oln}$, where $V = \{0, 1, 2, ..., \alpha_{i+j+k} - 1\}$ and $(o,l,n) \in T$ for $(i,j,k) \in I'$. Then $\mathbf{c} = (c_a)_{a\in V} \in C_{\alpha_{i+j+k}}$, where $V = \{0, 1, 2, ..., \alpha_{i+j+k} - 1\}$ and $c_d = \sum_{(o,l,n)\in T} \eta_{oln} b^d_{\alpha_{i+j+k},oln}$, where $d \in V$ for $(i,j,k) \in I'$. Suppose $C_{\alpha_{i+j+k}}$ is a cyclic code of length α_{i+j+k} over M_{ijk} for $(i,j,k) \in I'$, the $\sigma(\mathbf{c}) \in C_{\alpha_{i+j+k}}$, where $\sigma(\mathbf{c}) = \sum_{(o,l,n)\in T} \eta_{oln}\sigma(b^a_{\alpha_{i+j+k},oln})_{a\in V}$. So $\sigma(b^a_{\alpha_{i+j+k},oln})_{a\in V} \in C_{\alpha_{i+j+k},oln}$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Therefore $C_{\alpha_{i+j+k},oln}$ are cyclic codes over M_{000}, where $(o,l,n) \in T$ for $(i,j,k) \in I'$.

Conversely, let $\mathbf{s} = (s_a)_{a\in V} \in C_{\alpha_{i+j+k}}$, where $V = \{0, 1, 2, ..., \alpha_{i+j+k} - 1\}$ such that $s_t = \sum_{(o,l,n)\in T} \eta_{oln} b^t_{\alpha_{i+j+k},oln}$, where $t \in V$. Then $(b^t_{\alpha_{i+j+k},oln})_{t\in V} \in C_{\alpha_{i+j+k},oln}$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Suppose that $C_{\alpha_{i+j+k},oln}$ are cyclic codes of length α_{i+j+k} over M_{000}, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Then $\sigma(b^t_{\alpha_{i+j+k},oln})_{t\in V} \in C_{\alpha_{i+j+k},oln}$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Since $\sum_{(o,l,n)\in T} \eta_{oln}\sigma(b^t_{\alpha_{i+j+k},oln})_{t\in M} = \sigma(\mathbf{s}) \in C_{\alpha_{i+j+k}}$, we get $C_{\alpha_{i+j+k}}$ is a cyclic code. ■

Corollary 3 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$ be a linear code of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I'$. Then its dual $C^{\perp}_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C^{\perp}_{\alpha_{i+j+k},oln}$ is also cyclic code of length α_{i+j+k} over M_{ijk} if and only if $C^{\perp}_{\alpha_{i+j+k},oln}$ are cyclic codes of length α_{i+j+k} over M_{000}, where $(o,l,n) \in T$ for $(i,j,k) \in I'$.*

Theorem 4 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$ be a cyclic code of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I'$, and $m_{\alpha_{i+j+k},oln}(x)$ be the generator monic polynomial of the cyclic code $C_{\alpha_{i+j+k},oln}$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Then*

$$C_{\alpha_{i+j+k}} = \left\{ \begin{array}{c} \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x) : \\ s_{\alpha_{i+j+k},oln}(x) \in M_{ijk}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle \end{array} \right\}$$

for $(i,j,k) \in I'$. Moreover,

$$C_{\alpha_{i+j+k}} = \left\langle m_{\alpha_{i+j+k}}(x) \right\rangle$$

where $m_{\alpha_{i+j+k}}(x) = \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x)$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$ such that $m_{\alpha_{i+j+k}}(x) | x^{\alpha_{i+j+k}} - 1$.

Proof. If $C_{\alpha_{i+j+k}}$ is a cyclic code of length α_{i+j+k} over M_{ijk}, then $C_{\alpha_{i+j+k},oln}$ is cyclic code of length α_{i+j+k} over M_{000}, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. So $C_{\alpha_{i+j+k},oln} = \left\langle m_{\alpha_{i+j+k},oln}(x) \right\rangle \subseteq M_{000}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle$, where $(o,l,n) \in T$ for $(i,j,k) \in I'$. Since $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$, then $C_{\alpha_{i+j+k}} = \{m_{\alpha_{i+j+k}}(x) | m_{\alpha_{i+j+k}}(x) = \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x)$ for $m_{\alpha_{i+j+k},oln}(x) \in M_{000}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle\}$. So

$$C_{\alpha_{i+j+k}} \subseteq \{ \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x) | s_{\alpha_{i+j+k},oln}(x) \in MY\},$$

where $MY = M_{ijk}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle$ and $(o, l, n) \in T$ for $(i, j, k) \in I'$. The other part is easily seen. That is,

$$\left\{\sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x) | s_{\alpha_{i+j+k},oln}(x) \in MY\right\} \subseteq C_{\alpha_{i+j+k}},$$

where $MY = M_{ijk}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle$. So

$$\left\{\sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x) | s_{\alpha_{i+j+k},oln}(x) \in MY\right\} = C_{\alpha_{i+j+k}},$$

where $MY = M_{ijk}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle$.

Let $m_{\alpha_{i+j+k},oln}(x)$ be the generator monic polynomial of the cyclic code $C_{\alpha_{i+j+k},oln}$ where $(o, l, n) \in T$ for $(i, j, k) \in I'$. We know that $C_{\alpha_{i+j+k}} = \{\sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x) | s_{\alpha_{i+j+k},oln}(x) \in M_{ijk}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle\}$ for $(i, j, k) \in I'$.

Let $C' = \langle m_{\alpha_{i+j+k}}(x)\rangle$, where $m_{\alpha_{i+j+k}}(x) = \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x)$. Then $m_{\alpha_{i+j+k}}(x) \in C_{\alpha_{i+j+k}}$. Therefore $C' \subseteq C_{\alpha_{i+j+k}}$. By multiplying with η_{abc}, we get $\eta_{abc}(\sum_{(o,l,n)\in T} \eta_{pln}\ m_{\alpha_{i+j+k},oln}(x) = \eta_{abc} m_{\alpha_{i+j+k},abc}(x)$, where $(a, b, c) \in T$ for $(i, j, k) \in I'$. So $C_{\alpha_{i+j+k}} \subseteq C'$. Therefore $C' = C_{\alpha_{i+j+k}} = \langle m_{\alpha_{i+j+k}}(x)\rangle$, where $m_{\alpha_{i+j+k}}(x) = \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x)$. Since $C_{\alpha_{i+j+k},oln} = \left\langle m_{\alpha_{i+j+k},oln}(x)\right\rangle$ there exists $f_{\alpha_{i+j+k},oln}(x) \in M_{000}[x]/\langle x^{\alpha_{i+j+k}} - 1\rangle$, where $x^{\alpha_{i+j+k}} - 1 = m_{\alpha_{i+j+k},oln}(x) f_{\alpha_{i+j+k},oln}(x)$. So $(\sum_{(o,l,n)\in T} \eta_{oln})(x^{\alpha_{i+j+k}} - 1) = (\sum_{(o,l,n)\in T} \eta_{oln} f_{\alpha_{i+j+k},oln}(x))$ $(\sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x)) = (\sum_{(o,l,n)\in T} \eta_{oln} f_{\alpha_{i+j+k},oln}(x)) m_{\alpha_{i+j+k}}(x)$. Therefore $m_{\alpha_{i+j+k}}(x) | x^{\alpha_{i+j+k}} - 1$. ■

Corollary 5 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T} \eta_{oln} C_{\alpha_{i+j+k},oln}$ be a linear code of length α_{i+j+k} over M_{ijk}, for $(i, j, k) \in I'$. Suppose $m_{\alpha_{i+j+k},oln}(x)$ are the generator monic polynomial of the cyclic code $C_{\alpha_{i+j+k},oln}$ and $s^*_{\alpha_{i+j+k},oln}(x)$ are the reciprocal polynomial of $s_{\alpha_{i+j+k},oln}(x)$ such that $x^{\alpha_{i+j+k}} - 1 = m_{\alpha_{i+j+k},oln}(x) s_{\alpha_{i+j+k},oln}(x)$, where $(o, l, n) \in T$ for $(i, j, k) \in I'$. Then $C^{\perp}_{\alpha_{i+j+k}} = \left\langle d_{\alpha_{i+j+k}}(x)\right\rangle$, where $d_{\alpha_{i+j+k}}(x) = \sum_{(o,l,n)\in T} \eta_{oln} s^*_{\alpha_{i+j+k},oln}(x)$, where $(o, l, n) \in T$ for $(i, j, k) \in I'$.*

Λ-Linear Codes

The set

$$\Lambda = M_{000} M_{100} M_{110} M_{111} = \{(m_{ijk})_{(i,j,k)\in I} | m_{ijk} \in M_{ijk}\}$$

forms an M_{111} module under the componentwise addition and the following multiplication. For any elements $m = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} u^i v^j w^k a_{ijk} \in M_{111}$ and $(m_{ijk})_{(i,j,k)\in I} \in \Lambda$, the multiplication is defined as

$$\begin{aligned} . \quad : \quad & \Lambda \longrightarrow \Lambda \\ (m, (m_{ijk})_{(i,j,k)\in I}) \quad \mapsto \quad & (s_{ijk})_{(i,j,k)\in I} \end{aligned}$$

where $s_{ijk} = m.m_{ijk}$ for $(i, j, k) = (1, 1, 1)$ and $s_{ijk} = \pi_{ijk}(m) m_{ijk}$ where the maps

$$\begin{aligned} \pi_{ijk} \quad : \quad & M_{111} \longrightarrow M_{ijk} \\ & m = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} u^i v^j w^k a_{ijk} \mapsto p_{ijk} \end{aligned}$$

are projection maps for $(i, j, k) \in I'' = \{(0, 0, 0), (1, 0, 0), (1, 1, 0)\}$. They are also ring homomorphisms. For the finite ring M_{000}, then $p_{ijk} = a_{000}$, for the finite ring M_{100}, then $p_{ijk} = \sum_{i=0}^{1} u^i a_{i00}$, for the finite ring M_{110}, then $p_{ijk} = \sum_{i=0}^{1} \sum_{j=0}^{1} u^i v^j a_{ij0}$.

This multiplication can be extended componentwise on $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3} = M_{000}^{\alpha_0} \times M_{100}^{\alpha_1} \times M_{110}^{\alpha_2} \times M_{111}^{\alpha_3}$ as follows for any $m \in M_{111}$ and $((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$,

$$\begin{aligned} . \quad : \quad & M_{111} \times \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3} \longrightarrow \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3} \\ (m, ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I}) \quad \mapsto \quad & (h_{ijk})_{(i,j,k)\in I} = (h_{000}, h_{100}, h_{110}, h_{111}) \end{aligned}$$

where $h_{ijk} = m(m^a_{ijk})_{a\in V}$ for $(i,j,k) = (1,1,1)$ and $h_{ijk} = (\pi_{ijk}(m))(m^a_{ijk})_{a\in V}$ and $V = \{0,1,2,\ldots,\alpha_{i+j+k}-1\}$ for $(i,j,k) \in I''$. The element $((m^a_{ijk})_{a\in V})_{(i,j,k)\in I}$ denotes the element $(m^0_{000},\ldots,m^{\alpha_0-1}_{000},m^0_{100},\ldots,m^{\alpha_1-1}_{100},m^0_{110},\ldots,m^{\alpha_2-1}_{110},m^0_{111},\ldots,m^{\alpha_3-1}_{111})$. So $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ is an M_{111}-module.

Definition 6 *A non empty subset* $\mathbf{C}$ *of* $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ *is called* Λ*-linear code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$, *if* $\mathbf{C}$ *is an* M_{111}*- submodule of* $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$, *where* $I = \{(0,0,0),(1,0,0)\ (1,1,0),(1,1,1)\}$.

An inner product between $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} = (\mathbf{m_{ijk}})_{(i,j,k)\in I}$ and $\hat{\mathbf{m}} = ((\hat{m}^a_{ijk})_{a\in V})_{(i,j,k)\in I} = (\hat{\mathbf{m}}_{\mathbf{ijk}})_{(i,j,k)\in I}$ is defined as

$$\mathbf{m}.\hat{\mathbf{m}} = \sum_{(i,j,k)\in I}\left(\sum_{s=0}^{\alpha_{i+j+k}-1} m^s_{ijk}\hat{m}^s_{ijk}\right) = \sum_{(i,j,k)\in I}\mathbf{m}_{ijk}.\hat{\mathbf{m}}_{\mathbf{ijk}}$$

where $V = \{0,1,2,\ldots,\alpha_{i+j+k}-1\}$. If $\mathbf{C}$ is an Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$, then the dual code of $\mathbf{C}$ is defined as

$$\mathbf{C}^\perp = \{\mathbf{c}' \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3} | \mathbf{c}.\mathbf{c}' = 0, \forall \mathbf{c} \in \mathbf{C}\}.$$

Definition 7 *A* Λ*-linear code* $\mathbf{C}$ *of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$ *is said to be a* Λ*-cyclic code if its cyclic shift* $\sigma(\mathbf{c}) = (\sigma(m^a_{ijk})_{a\in V})_{(i,j,k)\in I} = (m^{\alpha_0-1}_{000}, m^0_{000},\ldots,m^{\alpha_0-2}_{000}, m^{\alpha_1-1}_{100},m^0_{100},\ldots,m^{\alpha_1-2}_{100},m^{\alpha_2-1}_{110},m^0_{110},\ldots,m^{\alpha_2-2}_{110},m^{\alpha_3-1}_{111},m^0_{111},\ldots,m^{\alpha_3-2}_{111}) \in \mathbf{C}$, *for any* $\mathbf{c} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$,*where* $V = \{0,1,2,\ldots,\alpha_{i+j+k}-1\}$.

For the polynomial representation:

Let $m_{ijk}(x) = \sum_{t=0}^{\alpha_{i+j+k}-1} m^t_{ijk}x^t$ where $(i,j,k) \in I$. Then any element $\mathbf{c} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$, where $V = \{0,1,2,\ldots,\alpha_{i+j+k}-1\}$ can be identified with

$$c(x) = (m_{ijk}(x))_{(i,j,k)\in I}$$

where $m_{ijk}(x) = m^0_{ijk} + m^1_{ijk}x + \ldots + m^{\alpha_{i+j+k}-1}_{ijk}x^{\alpha_{i+j+k}-1}$ for $(i,j,k) \in I$. Let $\Delta = \Pi_{(i,j,k)\in I}(M_{ijk}[x]/< x^{\alpha_{i+j+k}} - 1 >)$. There is a one to one correspondence between $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ and Δ. The set Δ is an $M_{111}[x]$-module under the following multiplication. For any $m(x) = m_0 + m_1x + \ldots + m_ex^e \in M_{111}[x]$ and $c(x) \in \Delta$

$$m(x)c(x) = (y_{ijk}(x))_{(i,j,k)\in I}$$

where $y_{ijk} = m(x)m_{ijk}(x)$ for $(i,j,k) = (1,1,1)$ and $y_{ijk} = \pi_{ijk}(m(x))m_{ijk}(x)$ where $\pi_{ijk}(m(x)) = \pi_{ijk}(m_0) + \pi_{ijk}(m_1)x + \ldots + \pi_{ijk}(m_e)x^e \in M_{ijk}[x]$ for $(i,j,k) \in I''$.

Theorem 8 *A code* $\mathbf{C}$ *is called* Λ *-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$, *if and only if* $\mathbf{C}$ *is an* $M_{111}[x]$*-submodule of* Δ.

Proof. Let $\mathbf{C}$ be an Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$ for $(i,j,k) \in I$. Since every element $\mathbf{c} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$ corresponds to the $c(x) = (m_{ijk}(x))_{(i,j,k)\in I}$, the polynomial $x.c(x)$ corresponds to the elements $\sigma(\mathbf{c})$. As $\mathbf{C}$ is a Λ-cyclic code, we have $\sigma(\mathbf{c}) \in \mathbf{C}$. By using the fact that $\mathbf{C}$ is a linear, the polynomial $f(x).c(x)$ corresponds to element $\hat{\mathbf{c}} = ((\hat{m}^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, for every $f(x) \in M_{111}[x]$. So $\mathbf{C}$ is an $M_{111}[x]$-submodule of Δ. The other part is seen from the definition. ■

Theorem 9 *Let* $\mathbf{C}$ *be an* Λ*-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$. *Then its dual* $\mathbf{C}^\perp$ *is also an* Λ*-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$.

Proof. For every $\mathbf{d} = ((d^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}^\perp$, we will show that $\sigma(\mathbf{d}) \in \mathbf{C}^\perp$. Let $\mathbf{C}$ be an Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Let $\mathbf{c} = ((c^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, where $V = \{0,1,\ldots,\alpha_{i+j+k}\}$ for $(i,j,k) \in I$. Take $lcm(\alpha_0,\alpha_1,\alpha_2,\alpha_3) = s$. As $\mathbf{C}$ is cyclic code, we have $\sigma^{s-1}(\mathbf{c}) \in \mathbf{C}$. So $\sigma^{s-1}(\mathbf{c})\mathbf{d} = 0$. Since $\sigma^{s-1}(\mathbf{c})\mathbf{d} = \mathbf{c}\sigma(\mathbf{d})$, we get $\mathbf{c}\sigma(\mathbf{d}) = 0$. Therefore $\mathbf{C}^\perp$ is cyclic code. ■

The Gray map over Λ

For any element $(m_{ijk})_{(i,j,k)\in I}$, the Gray map defined also as follows

$$\begin{aligned} \Psi \quad &: \quad \Lambda \longrightarrow M_{000}^{15} \\ (m_{ijk})_{(i,j,k)\in I} \quad &\longmapsto \quad (m_{000}, (\psi_{ijk}(m_{ijk}))_{(i,j,k)\in I'}). \end{aligned}$$

This can be extended on $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$. It goes from $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ to $M_{000}^{\sum_{(i,j,k)\in I}\alpha_{i+j+k}2^{i+j+k}}$. The Lee weight of any element $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I}$ is defined as

$$w_L(\mathbf{m}) = w_H((m^a_{000})_{a\in V}) + \sum_{(i,j,k)\in I'}(\sum_{t=0}^{\alpha_{i+j+k}-1} w_L(m^t_{ijk}))$$

where w_H denotes the Hamming weight over M_{000}. The Lee distance between $\mathbf{m}$ and $\hat{\mathbf{m}} \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ is defined as $d_L(\mathbf{m}, \hat{\mathbf{m}}) = w_L(\mathbf{m} - \hat{\mathbf{m}}) = w_H(\Psi(\mathbf{m} - \hat{\mathbf{m}}))$.

Proposition 10 *Ψ is an M_{000}- linear map which preserves distance from $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$ (Lee distance) to $M_{000}^{\sum_{(i,j,k)\in I}\alpha_{i+j+k}2^{i+j+k}}$ (Hamming distance). Moreover if $\mathbf{C}$ is Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$ with d_L, then $\Psi(\mathbf{C})$ is a linear code of length $\sum_{(i,j,k)\in I}\alpha_{i+j+k}2^{i+j+k}$ over M_{000} with d_H, where $d_L = d_H$.*

Let $\mathbf{C}$ be an Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. For $(o, l, n) \in T$ and all $(i, j, k) \in I'$, we define $C_{\alpha_0} = \{(m^a_{\alpha_0,000})_{a\in V} \in M_{000}^{\alpha_0} | ((m^a_{\alpha_0,000})_{a\in V}, ((\sum_{(r,s,t)\in T}\eta_{rst}b^a_{\alpha_{i+j+k},rst})_{a\in V})_{(i,j,k)\in I'}) \in \mathbf{C}$ for some $(b^a_{\alpha_{i+j+k},oln})_{a\in V} \in M_{000}^{\alpha_{i+j+k}}\}$. $C_{\alpha_{i+j+k},oln} = \{(b^a_{\alpha_{i+j+k},oln})_{a\in V} \in M_{000}^{\alpha_{i+j+k}} | ((m^a_{\alpha_0,000})_{a\in V}, (\sum_{(r,s,t)\in T}\eta_{rst}b^a_{\alpha_{i+j+k},rst})_{a\in V})_{(i,j,k)\in I'}) \in \mathbf{C}$, for some $(m^a_{\alpha_0,000})_{a\in V} \in M_{000}^{\alpha_0}$, $(b^a_{\alpha_{i+j+k},rst})_{a\in V} \in M_{000}^{\alpha_{i+j+k}}$, $(b^a_{\alpha_{i+j+k},rst})_{a\in V} \neq (b^a_{\alpha_{i+j+k},oln})_{a\in V}\}$, where $V = \{0, 1, \dots, \alpha^{i+j+k} - 1\}$.

Lemma 11 *Let $\mathbf{C}$ be an Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Then*

$$\Psi(\mathbf{C}) = C_{\alpha_0} \otimes_{(o,l,n)\in T} C_{\alpha_{i+j+k},oln}$$

for $(i, j, k) \in I'$. Moreover $|\Psi(\mathbf{C})| = |C_{\alpha_0}|\Pi_{(o,l,n)\in T}|C_{\alpha_{i+j+k},oln}|$, where $(o, l, n) \in T$ for all $(i, j, k) \in I'$.

Lemma 12 *Let $\mathbf{C}$ be an Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Then C_{α_0} and all $C_{\alpha_{i+j+k,oln}}$ are cyclic codes over M_{000} with length α_{i+j+k}, where $(o, l, n) \in T$ for all $(i, j, k) \in I'$.*

Definition 13 *Define*

$$\kappa \quad : \quad F_q^{sn} \longrightarrow F_q^{sn}$$

by $\kappa(\mathbf{d_1}, \dots, \mathbf{d_n}) = (\sigma(\mathbf{d_1}), \dots, \sigma(\mathbf{d_n}))$, where $\mathbf{d} = (\mathbf{d_1}, \dots, \mathbf{d_n}) \in F_q^{sn}$ and σ is the cyclic shift from F_q^s and F_q^s. Then a code C is called a quasi-cyclic code of index n, if $\kappa(C) = C$.

Define

$$\kappa_g \quad : \quad F_q^{s_1} \times \dots \times F_q^{s_n} \longrightarrow F_q^{s_1} \times \dots \times F_q^{s_n}$$

by $\kappa_g(\hat{\mathbf{d_1}}, \dots, \hat{\mathbf{d_n}}) = (\sigma(\hat{\mathbf{d_1}}), \dots, \sigma(\hat{\mathbf{d_n}}))$, where $\hat{\mathbf{d}} = (\hat{\mathbf{d_1}}, \dots, \hat{\mathbf{d_n}}) \in F_q^{s_1} \times \dots \times F_q^{s_n}$ and σ is the cyclic shift from $F_q^{s_i}$ and $F_q^{s_i}$, for $i = 1, \dots, n$. Then a code C is called a generalized quasi-cyclic code of index n, if $\kappa_g(C) = C$.

Proposition 14 *If κ_g and Ψ are as above and σ is a cyclic shift over $\Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$, then we have $\Psi\sigma = \kappa_g\Psi$.*

Proof. For every $\mathbf{c} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$, where $V = \{0, 1, \dots, \alpha_{i+j+k} - 1\}$, it is easily seen that $\Psi(\sigma(\mathbf{c})) = \kappa_g(\Psi(\mathbf{c}))$. So $\Psi\sigma = \kappa_g\Psi$. ■

Theorem 15 *Let* $\mathbf{C}$ *be an* Λ*-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$*. Then the Gray image of* Λ*-cyclic code is a generalized quasi-cyclic code of index 15 over* M_{000}*. If* $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$*, then the Gray image of* Λ*-cyclic code is a quasi-cyclic code of index 15 over* M_{000}*.*

Proof. Let $\mathbf{C}$ be an Λ-cyclic code. So $\sigma(\mathbf{C}) = \mathbf{C}$. By applying Ψ, we have $\Psi(\sigma(\mathbf{C}) = \Psi(\mathbf{C})$. By using Proposition 14, we get So $\Psi\sigma(\mathbf{C}) = \kappa_g\Psi(\mathbf{C}) = \Psi(\mathbf{C})$. Therefore $\Psi(\mathbf{C})$ is a generalized quasi-cyclic code. ■

The Structures of Λ-Cyclic codes

Proposition 16 *Let* $\mathbf{C}$ *be an* Λ*-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$*. Then* $\xi_{ijk}(\mathbf{C}) = C_{\alpha_{i+j+k}}$ *is a cyclic code of length* α_{i+j+k} *over* M_{ijk}*, where*

$$\begin{array}{rcl} \xi_{ijk} & : & \Delta \longrightarrow M_{ijk}[x]/\langle x^{\alpha_{i+j+k}}-1\rangle \\ c(x) = (m_{ijk}(x))_{(i,j,k)\in I} & \longmapsto & (m_{ijk})(x) \end{array}$$

for $(i, j, k) \in I$*.*

Theorem 17 *Let* $\mathbf{C}$ *be an* Λ*-cyclic code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$*. Then*

$$\mathbf{C} = \left\langle (m_{\alpha_0}(x)|0|0|0), (0|m_{\alpha_1}(x)|0|0), (0|0|m_{\alpha_2}(x)|0), (f_0(x)|f_1(x)|f_2(x)|m_{\alpha_3}(x)) \right\rangle,$$

where $m_{\alpha_{i+j+k}}(x)|x^{\alpha_{i+j+k}}-1$ *for* $(i, j, k) \in I$ *and* $f_{i+j+k}(x) \in M_{ijk}[x]/\langle x^{\alpha_{i+j+k}}-1\rangle$ *, where* $(i, j, k) \in I''$*.*

Proof. From Proposition 16, we have $\xi_{ijk}(\mathbf{C}) = C_{\alpha_{i+j+k}}$ for $(i, j, k) \in I$. So $C_{\alpha_{i+j+k}} = \left\langle m_{\alpha_{i+j+k}(x)} \right\rangle$ such that $m_{\alpha_{i+j+k}}(x)|x^{\alpha_{i+j+k}}-1$ for $(i, j, k) \in I$. Hence the proof follows from Theorem 3.1 of [6]. ■

Definition 18 *Let* $\mathbf{C}$ *be an* Λ*-cyclic codes of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$ *and* $C_{\alpha_{i+j+k}}$ *be the canonical projection of* $\mathbf{c}$ *on the* α_{i+j+k} *coordinates. The code* $\mathbf{C}$ *is separable if* $\mathbf{C}$ *is the direct product of* $C_{\alpha_{i+j+k}}$*,where* $(i, j, k) \in I$*. i.e.,* $\mathbf{C} = \otimes_{(i,j,k)\in I} C_{\alpha_{i+j+k}}$*.*

Theorem 19 *Let* $\mathbf{C} = \otimes_{(i,j,k)\in I} C_{\alpha_{i+j+k}}$ *be an* Λ*-cyclic codes of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$*. Then* $\mathbf{C}$ *is separable* Λ*-cyclic codes if and only only if* $C_{\alpha_{i+j+k}}$ *are cyclic codes over* M_{ijk} *with length* α_{i+j+k}*, for* $(i, j, k) \in I$*, respectively.*

Let $C_{\alpha_{i+j+k}}$ be cyclic codes of length α_{i+j+k}, where $(i, j, k) \in I$, respectively. If $\mathbf{C}$ is separable, then

$$\mathbf{C} = \left\langle (m_{\alpha_0}(x)|0|0|0), (0|m_{\alpha_1}(x)|0|0), (0|0|m_{\alpha_2}(x)|0), (0|0|0|m_{\alpha_3}(x)) \right\rangle,$$

where $C_{\alpha_{i+j+k}} = \left\langle m_{\alpha_{i+j+k}}(x) \right\rangle$, where $m_{\alpha_{i+j+k}}(x)|x^{\alpha_{i+j+k}}-1$.

DNA Codes

In this chapter, some basic definitions and details about cyclic DNA codes over M_{000} in literature will be given. Later the necessary and sufficient conditions cyclic codes over M_{ijk} for $(i, j, k) \in I'$ and separable Λ cyclic codes to be reversible and reversible complement will be discussed. In this section, we take $q = 4$,$\beta_1 = \xi$, $\beta_2 = \xi^2$, $\beta_3 = \xi$, where $M_{000} = F_4 = \{0, 1, \xi, \xi^2\}$.

It is well known that DNA has two strands that are linked by Watson-Crick pairing, every A is linked with a T and every C is linked with a G, and vice versa, where A, T, C and G are the four bases of DNA sequences. i.e. one writes $\overline{A} = T$, $\overline{T} = A$, $\overline{C} = G$ and $\overline{G} = C$. The $\overline{A}$ denotes complement of A.

Let M be a finite commutative ring and D be a linear code of length n over M. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a codeword in D. The reverse of $\mathbf{a}$ is $\mathbf{a}^r = (a_n, a_{n-1}, \ldots, a_1)$. The complement of $\mathbf{a}$ is $\mathbf{a}^c = (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)$. The reverse complement of $\mathbf{a}$ is $\mathbf{a}^{rc} = (\overline{a}_n, \overline{a}_{n-1}, \ldots, \overline{a}_1)$, where $\overline{a_i}$ denotes complement of a_i, for $i = 1, \ldots, n$.

Definition 20 *Let D be a linear code of length n over M. Then D is called reversible if $\mathbf{a}^r \in D$, for any $\mathbf{a} \in D$, D is called complement if $\mathbf{a}^c \in D$, for any $\mathbf{a} \in D$ and D is called reversible complement if $\mathbf{a}^{rc} \in D$, for any $\mathbf{a} \in D$.*

Definition 21 *Let D be a linear code of length n over M. Then D is said to be cyclic DNA codes if D is a cyclic and reversible complement.*

Definition 22 *For any polynomial $s(x) = s_0 + s_1x + \ldots + s_tx^t \in M[x]$, with $s_t \neq 0$, the reciprocal polynomial of $s(x)$ is defined as $s^*(x) = x^t s(1/x)$. If $s^*(x) = s(x)$, then $s(x)$ is called self reciprocal.*

With the map ζ from $M_{000}^{\alpha_0}$ to $M_{000}[x]/\langle x^{\alpha_0} - 1\rangle$, to any element $\mathbf{m_{000}} = (m_{000}^0, m_{000}^1, \ldots, m_{000}^{\alpha_0-1}) \in M_{000}^{\alpha_0}$ corresponds to the elements $m_{000}(x) = m_{000}^0 + m_{000}^1x + \ldots + m_{000}^{\alpha_0-1}x^{\alpha_0-1} \in M_{000}[x]/\langle x^{\alpha_0} - 1\rangle$. If C_{α_0} is a cyclic code over M_{000} of length α_0, then $\zeta(C_{\alpha_0})$ is an ideal in $M_{000}[x]/\langle x^{\alpha_0} - 1\rangle$.
Shortly, we denotes $\zeta(C_{\alpha_0})$ as C_{α_0}.

Theorem 23 *[1] Let C_{α_0} be a cyclic code of length α_0 over M_{000}. Then there exists a unique monic polynomial $f_{000}(x) \in M_{000}[x]/\langle x^{\alpha_0} - 1\rangle$ such that $C_{\alpha_0} = \langle f_{000}(x)\rangle$ and $f_{000}(x)$ divides $x^{\alpha_0} - 1$. Moreover C_{α_0} has 4^{k_1} codewords, where $k_1 = \alpha_0 - deg f_{000}(x)$ and the set $\{f_{000}(x), xf_{000}(x), \ldots, x^{k_1-1}f_{000}(x)\}$ forms a basis of C_{α_0}.*

Lemma 24 *[4] Let $C_{\alpha_0} = \langle f_{000}(x)\rangle$ be a cyclic code of length α_0 over M_{000}. Then C_{α_0} is reversible if and only if $f_{000}(x)$ is self reciprocal.*

Lemma 25 *[1] Let $C_{\alpha_0} = \langle f_{000}(x)\rangle$be a cyclic code of length α_0 over M_{000}. Then C_{α_0} is complement if and only if $f_{000}(x)$ is not divisible by $x - 1$.*

Theorem 26 *[3] Let $C_{\alpha_0} = \langle f_{000}(x)\rangle$be a cyclic code of length α_0 over M_{000}. Then C_{α_0} is reversible complement if and only if $f_{000}(x)$ is self reciprocal and $f_{000}(x)$ is not divisible by $x - 1$.*

In [1], they studied cyclic DNA code over M_{000} and used the bijection map γ_0 between the set of DNA alphabet $S_{D_4} = \{A, T, C, G\}$ and $M_{000} = \{0, 1, \xi, \xi^2\}$, with $0 \mapsto A$, $1 \mapsto T$, $\xi \mapsto C$, $\xi^2 \mapsto G$.

We extend the map from M_{ijk} to $S_{D_4}^{2^{i+j+k}}$, by using the Gray map ψ_{ijk}. For the finite ring M_{100},

$m_{100} \in M_{100}$	Gray Images $\psi_{100}(m_{100})$	Codon $\gamma_1(m_{100})$
0	$(0, 0)$	AA
u	$(\xi, 0)$	CA
$u\xi$	$(\xi^2, 0)$	GA
$u\xi^2$	$(1, 0)$	TA
1	$(1, 1)$	TT
$1 + u$	$(\xi^2, 1)$	GT
$1 + u\xi^2$	$(0, 1)$	AT
$1 + u\xi$	$(\xi, 1)$	CT
ξ	(ξ, ξ)	CC
$u\xi + \xi$	$(1, \xi)$	TC
$u\xi^2 + \xi$	(ξ^2, ξ)	GC
$u + \xi$	$(0, \xi)$	AC
$\xi^2 + u\xi^2$	(ξ, ξ^2)	CG
$\xi^2 + u$	$(1, \xi^2)$	TG
$\xi^2 + u\xi$	$(0, \xi^2)$	AG
ξ^2	(ξ^2, ξ^2)	GG

Similarly, we define a bijection map γ_2 between M_{110} to $S_{D_4}^4$ as follows, by considering the Gray images of elements of M_{110}.

$m_{110} \in M_{110}$	Gray Images $\psi_{110}(m_{110})$	Codon $\gamma_2(m_{110})$
0	$(0, 0, 0, 0)$	$AAAA$
u	$(\xi, \xi, 0, 0)$	$CCAA$
$1+u$	$(\xi^2, \xi^2, 1, 1)$	$GGTT$
1	$(1, 1, 1, 1)$	$TTTT$
$\vdots$		

Similarly, we define a bijection map γ_3 between M_{111} to $S_{D_4}^8$ as follows, by considering the Gray images of elements of M_{111}.

$m_{111} \in M_{111}$	Gray Images $\psi_{111}(m_{111})$	Codon $\gamma_3(m_{111})$
0	$(0, 0, 0, 0, 0, 0, 0, 0)$	$AAAAAAAA$
u	$(\xi, \xi, \xi, \xi, 0, 0, 0, 0)$	$CCCCAAAA$
$1+u$	$(\xi^2, \xi^2\xi^2, \xi^2, 1, 1, 1, 1)$	$GGGGTTTT$
1	$(1, 1, 1, 1, 1, 1, 1, 1)$	$TTTTTTTT$
$\vdots$		

Let $C_{\alpha_{i+j+k}}$ be a linear code of length α_{i+j+k} over M_{ijk} and $\mathbf{m_{ijk}} = (m_{ijk}^0, \ldots, m_{ijk}^{\alpha_{i+j+k}-1}) \in C_{\alpha_{i+j+k}}$ for $(i, j, k) \in I$. By using the table, the map $\Gamma_{\alpha_{i+j+k}}$ is defined as follows,

$$\begin{array}{rcl} \Gamma_{\alpha_{i+j+k}} & : & C_{\alpha_{i+j+k}} \longrightarrow S_{D_4}^{2^{i+j+k}\alpha_{i+j+k}} \\ \mathbf{m_{ijk}} = (m_{ijk}^0, \ldots, m_{ijk}^{\alpha_{i+j+k}-1}) & \mapsto & \Gamma_{\alpha_{i+j+k}}(\mathbf{m_{ijk}}) = (\gamma_{i+j+k}(m_{ijk}^0), \ldots, \gamma_{i+j+k}(m_{ijk}^{\alpha_{i+j+k}-1})). \end{array}$$

Theorem 27 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$ be a cyclic code of length α_{i+j+k} over M_{ijk}, where $(o, l, n) \in T$ for $(i, j, k) \in I'$. Then $C_{\alpha_{i+j+k}}$ is reversible over M_{ijk} if and only if $C_{\alpha_{i+j+k},oln}$ are reversible over M_{000}, where all $(o, l, n) \in T$ for $(i, j, k) \in I'$.*

Proof. Let $C_{\alpha_{i+j+k}}$ is reversible code over M_{ijk} for $(i, j, k) \in I'$. So for every $\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V} \in C_{\alpha_{i+j+k}}$, we have $\mathbf{m_{ijk}}^r = \sum_{(o,l,n)\in T}\eta_{oln}(b_{\alpha_{i+j+k},oln}^a)_{a\in V}^r \in C_{\alpha_{i+j+k}}$. Since $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$, we get $(b_{\alpha_{i+j+k},oln}^a)^r \in C_{\alpha_{i+j+k},oln}$, where all $(o, l, n) \in T$ for $(i, j, k) \in I'$. So $C_{\alpha_{i+j+k},oln}$ are reversible. Conversely, suppose that $C_{\alpha_{i+j+k},oln}$ are reversible, where all $(o, l, n) \in T$ for $(i, j, k) \in I'$. Let $\mathbf{m_{ijk}} \in C_{\alpha_{i+j+k}}$. Since $\mathbf{m_{ijk}}^r = \sum_{(o,l,n)\in T}\eta_{oln}(b_{\alpha_{i+j+k},oln}^a)_{a\in V}^r \in C_{\alpha_{i+j+k}}$, we have $C_{\alpha_{i+j+k}}$ are reversible. ■

Lemma 28 *For any $m_{ijk} \in M_{ijk}$ where $(i, j, k) \in I'$, $\overline{m_{ijk}} + m_{ijk} = 1$.*

Proof. For $v_1 = a + bu \in M_{100}$, $\psi_{100}(v_1) = (b\xi + a, a)$. It corresponds $SE \in S_{D_4}^2$, where $S, E \in S_{D_4}$. For $v_1 + 1 \in M_{000}$, $\psi_{100}(v_1 + 1) = (a + 1 + b\xi, a + 1)$. It corresponds $\overline{S}\,\overline{E}$. Since $\psi_{110}(v_2) = (d + a + b\xi + c\xi^2, a + b\xi, a + c\xi^2, a)$ for $v_2 = a + bu + cv + duv \in M_{110}$ and $\psi_{111}(v_3) = (g + e + a + h\xi + c\xi^2 + f\xi^2 + d\xi + b\xi, e + a + c\xi^2 + b\xi, a + b\xi + d\xi + f\xi^2, a + b\xi, g + c\xi^2 + d\xi + a, c\xi^2 + a, d\xi + a, a)$ for $v_3 = a + bu + cv + dw + euv + fuw + gvw + huvw \in M_{111}$, it is seen similarly. ■

Corollary 29 *Let $C_{\alpha_{i+j+k}} = \left\langle m_{\alpha_{i+j+k}(x)} \right\rangle = \left\langle \sum_{(o,l,n)\in T}\eta_{oln}m_{\alpha_{i+j+k},oln}(x) \right\rangle$ be a cyclic code. Then $C_{\alpha_{i+j+k}}$ is reversible code if and only if $m_{\alpha_{i+j+k},oln}(x) \in M_{000}[x]$ are self reciprocal polynomial where all $(o, l, n) \in T$ for $(i, j, k) \in I'$.*

Theorem 30 *Let $C_{\alpha_{i+j+k}} = \oplus_{(o,l,n)\in T}\eta_{oln}C_{\alpha_{i+j+k},oln}$ be a cyclic code of length α_{i+j+k} over M_{ijk}, where $(o, l, n) \in T$ for $(i, j, k) \in I'$. Then $C_{\alpha_{i+j+k}}$ is reversible complement over M_{ijk} if and only if $C_{\alpha_{i+j+k}}$ is reversible and $(\overline{0}, \ldots, .\overline{0}) \in C_{\alpha_{i+j+k}}$.*

Proof. Let $C_{\alpha_{i+j+k}}$ be reversible complement, for $(i, j, k) \in I'$. So we have $\mathbf{m_{ijk}} = ([m_{ijk}^a]^{rc})_{a\in V} \in C_{\alpha_{i+j+k}}$, for every $\mathbf{m_{ijk}} = (m_{ijk}^a)_{a\in V} \in C_{\alpha_{i+j+k}}$. Since $C_{\alpha_{i+j+k}}$ is a linear code, so $(0, \ldots, 0) \in C_{\alpha_{i+j+k}}$. By using the fact that $C_{\alpha_{i+j+k}}$ is reversible complement, we get $(\overline{0}^a)_{a\in V}$. By using Lemma 28, we get $\mathbf{m_{ijk}}^r = ([m_{ijk}^a]^{rc})_{a\in V} + (\overline{0}^a)_{a\in V}$.

So $\mathbf{m_{ijk}}^r \in C_{\alpha_{i+j+k}}$. Therefore $C_{\alpha_{i+j+k}}$ is reversible over M_{ijk} for $(i,j,k) \in I'$. Conversely, let $(\overline{0}^a)_{a\in V} \in C_{\alpha_{i+j+k}}$ and $C_{\alpha_{i+j+k}}$ be reversible. Then for any $\mathbf{m_{ijk}} \in C_{\alpha_{i+j+k}}$, we have $\mathbf{m_{ijk}}^r \in C_{\alpha_{i+j+k}}$. By using the Lemma 28 and by using the fact that $C_{\alpha_{i+j+k}}$ is linear, we have $\mathbf{m_{ijk}}^{rc} = ([m^a_{ijk}]^r)_{a\in V} + (\overline{0}^a)_{a\in V} \in C_{\alpha_{i+j+k}}$. Hence $C_{\alpha_{i+j+k}}$ is reversible complement over M_{ijk} for $(i,j,k) \in I'$. ■

Corollary 31 *Let $C_{\alpha_{i+j+k}} = \left\langle m_{\alpha_{i+j+k}(x)} \right\rangle = \left\langle \sum_{(o,l,n)\in T} \eta_{oln} m_{\alpha_{i+j+k},oln}(x) \right\rangle$ be a cyclic code. Then $C_{\alpha_{i+j+k}}$ is reversible complement code if and only if all $m_{\alpha_{i+j+k},oln}(x) \in M_{000}[x]$ are self reciprocal polynomial where $(o,l,n) \in T$ for $(i,j,k) \in I'$ and $(1^a)_{a\in V} \in C_{\alpha_{i+j+k}}$.*

Theorem 32 *Let $C_{\alpha_{i+j+k}}$ be a cyclic DNA code of length α_{i+j+k} over the ring M_{ijk} where $(i,j,k) \in I'$ and minimum Hamming distance d_H. Then, $\Gamma_{\alpha_{i+j+k}}(C_{\alpha_{i+j+k}})$ is a code of length $2^{i+j+k}\alpha_{i+j+k}$ over the alphabet $\{A,T,G,C\}$ with minimum Hamming distance at least d_H for $(i,j,k) \in I'$.*

Definition 33 *Let $\mathbf{C}$ be an Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Then $\mathbf{C}$ is said to be reversible, if $\mathbf{m}^r = ([(m^a_{ijk})_{a\in V}]^r)_{(i,j,k)\in I} \in \mathbf{C}$, for every element $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, where $V = \{0,1,\ldots,\alpha_{i+j+k}-1\}$, for $(i,j,k) \in I$.*

Definition 34 *Let $\mathbf{C}$ be an Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Then $\mathbf{C}$ is said to be complement, if $\mathbf{m}^c = ([(m^a_{ijk})_{a\in V}]^c)_{(i,j,k)\in I} \in \mathbf{C}$, for every element $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, where $V = \{0,1,\ldots,\alpha_{i+j+k}-1\}$, for $(i,j,k) \in I$.*

Definition 35 *Let $\mathbf{C}$ be an Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$. Then $\mathbf{C}$ is said to be reversible complement, if $\mathbf{m}^{rc} = ([(m^a_{ijk})_{a\in V}]^{rc})_{(i,j,k)\in I} \in \mathbf{C}$, for every element $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, where $V = \{0,1,\ldots,\alpha_{i+j+k}-1\}$, for $(i,j,k) \in I$.*

Theorem 36 *Let $\mathbf{C} = \otimes_{(i,j,k)\in I} C_{\alpha_{i+j+k}}$ be separable Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$, where $C_{\alpha_{i+j+k}}$ are cyclic codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$. Then $\mathbf{C}$ is reversible if and only if all $C_{\alpha_{i+j+k}}$ are reversible codes over M_{ijk}, for $(i,j,k) \in I$.*

Proof. Let $\mathbf{C}$ be a reversible code. Take $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$ where $\mathbf{m_{ijk}} = (m^a_{ijk})_{a\in V}$ where $V = \{0,1,\ldots,\alpha_{i+j+k}-1\}$, for $(i,j,k) \in I$. By using the fact that $\mathbf{C}$ is a reversible, we have $\mathbf{m}^r = (\mathbf{m_{ijk}}^r)_{(i,j,k)\in I} \in \mathbf{C}$. So this shows that $\mathbf{m_{ijk}}^r \in C_{\alpha_{i+j+k}}$, where $(i,j,k) \in I$. Therefore $C_{\alpha_{i+j+k}}$ are reversible codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$. Conversely, let $C_{\alpha_{i+j+k}}$ be reversible codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$ and take $\mathbf{m} = (\mathbf{m_{ijk}})_{(i,j,k)\in I}$, for $(i,j,k) \in I$. By using the fact that $C_{\alpha_{i+j+k}}$ are reversible, then $\mathbf{m_{ijk}}^r \in C_{\alpha_{i+j+k}}$ for $(i,j,k) \in I$. So $\mathbf{m}^r = (\mathbf{m_{ijk}}^r)_{(i,j,k)\in I} \in \mathbf{C}$. Hence $\mathbf{C}$ is reversible. ■

Theorem 37 *Let $\mathbf{C} = \otimes_{(i,j,k)\in I} C_{\alpha_{i+j+k}}$ be separable Λ-cyclic code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$, where $C_{\alpha_{i+j+k}}$ are cyclic codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$. Then $\mathbf{C}$ is a reversible complement code if and only if $C_{\alpha_{i+j+k}}$ are reversible complement codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$.*

Proof. Let $\mathbf{C}$ be a reversible complement code. Take $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \mathbf{C}$, where $\mathbf{m_{ijk}} = (m^a_{ijk})_{a\in V}$, $V = \{0,1,\ldots,\alpha_{i+j+k}-1\}$, for $(i,j,k) \in I$. By using the fact that $\mathbf{C}$ is a reversible complement code, we have $\mathbf{m}^{rc} = (\mathbf{m_{ijk}}^{rc})_{(i,j,k)\in I} \in \mathbf{C}$. So this shows that $\mathbf{m_{ijk}}^{rc} = [(m^a_{ijk})_{a\in V}]^{rc} \in C_{\alpha_{i+j+k}}$, where $(i,j,k) \in I$. Therefore $C_{\alpha_{i+j+k}}$ are reversible complement codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$. Conversely, let $C_{\alpha_{i+j+k}}$ be reversible complement codes of length α_{i+j+k} over M_{ijk}, for $(i,j,k) \in I$ and take $\mathbf{m} = (\mathbf{m_{ijk}})_{(i,j,k)\in I} \in \mathbf{C}$, for $(i,j,k) \in I$. By using the fact that $C_{\alpha_{i+j+k}}$ are reversible complement, then $\mathbf{m_{ijk}}^{rc} \in C_{\alpha_{i+j+k}}$ for $(i,j,k) \in I$. So $\mathbf{m}^{rc} = (\mathbf{m_{ijk}}^{rc})_{(i,j,k)\in I} \in \mathbf{C}$. Hence $\mathbf{C}$ is a reversible complement. ■

Definition 38 *Let $\mathbf{C}$ be an Λ-linear code of block length $(\alpha_{i+j+k})_{(i,j,k)\in I}$, for $(i,j,k) \in I$ and $\mathbf{m} = ((m^a_{ijk})_{a\in V})_{(i,j,k)\in I} \in \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3}$. By using the table, the map Ξ is defined as follows,*

$$\begin{aligned} \Xi \quad &: \quad \Lambda_{\alpha_0\alpha_1\alpha_2\alpha_3} \longrightarrow S_{D_4}^{\sum_{(i,j,k)\in I} \alpha_{i+j+k} 2^{i+j+k}} \\ \mathbf{m} = (\mathbf{m_{ijk}})_{(i,j,k)\in I} \quad &\mapsto \quad \Xi(\mathbf{m}) = (\Gamma_{\alpha_{i+j+k}}(\mathbf{m_{ijk}}))_{(i,j,k)\in I} \end{aligned}$$

Theorem 39 *Let* **C** *be a separable* Λ *cyclic DNA code of block length* $(\alpha_{i+j+k})_{(i,j,k)\in I}$, *for* $(i,j,k) \in I$ *with* $|C| = M$ *and minimum Hamming distance* d_H. *Then,* $\Xi(C)$ *is a DNA code of length* $\sum_{(i,j,k)\in I} \alpha_{i+j+k} 2^{i+j+k}$ *over the alphabet* $\{A, T, G, C\}$ *with minimum Hamming distance at least* d_H.

Conclusion

The structures of the Λ-cyclic codes are obtained. Their generator polynomials are constructed. An inner product is defined. It was shown that if **C** is an Λ-cyclic code, then $\mathbf{C}^{\perp}$ is an Λ-cyclic code. The separable Λ- cyclic codes are introduced. The necessary and sufficient conditions of the separable Λ-cyclic codes to be reversible and reversible complement are determined. It is shown that DNA codes can be constructed from them.

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