ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2024) 2024:5

On a Global Intra-Divergence Distribution Measure: Definition, Theory and Examples

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Abstract
In probability theory and statistics, the analysis of continuous univariate distributions can be done by studying their probability density functions and cumulative distribution functions. This article introduces a measure that quantifies the global divergence between these two probabilistic functions, focusing on distributions with finite support. The aim of this measure is to highlight significant differences between distributions with smooth probability density functions and those with abruptly varying probability density functions. It therefore helps to classify distributions based on these aspects. Numerous examples, exact calculations and visualizations illustrate the use of this measure, with a focus on unit distributions. It is also involved in several mathematical inequalities that can be considered of independent interest.

Keywords: probability; continuous distributions; unit distributions; divergence measures; integral; inequalities.

1 Introduction

In probability theory and statistics, the analysis of continuous univariate distributions is crucial to understanding the behavior of random variables that model numerical measures. These distributions are characterized by their probability density functions (PDFs) and cumulative distribution functions (CDFs). The PDF gives "the likelihood" that a random variable will take a particular value, while the CDF gives the probability that the random variable will be less than or equal to a particular value. Together, these functions provide a complete description of the properties of the distribution. The general and technical details can be found in [1, 2].

In this article, we focus on a continuous univariate distribution with finite support, denoted as (a, b), where $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a < b$. The corresponding CDF is denoted as $F(x)$, and the corresponding PDF is denoted as $f(x)$. At this point, we recall that $f(x)$ is an integrable function satisfying $f(x) \geq 0$ for any $x \in (a, b)$, and $f(x) = 0$ for any $x \notin (a, b)$ and $\int_a^b f(t)dt = 1$, and $F(x) = \int_a^x f(t)dt$ for any $x \in (a, b)$, $F(x) = 0$ for any $x \leq a$, and $F(x) = 1$ for any $x \geq b$. Furthermore, we assume that $f(x)$ is continuous for $x \in (a, b)$. In this classical probability setting, we introduce the following real number:

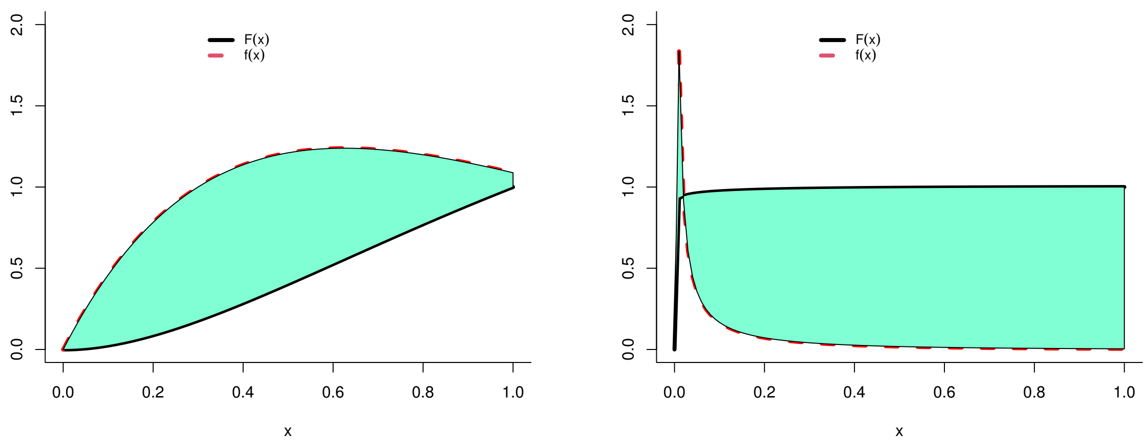
$$D = \int_a^b |F(x) - f(x)| dx. \quad (1)$$

So D is the distance between $F(x)$ and $f(x)$ in the $\mathbb{L}^1$ sense. It is well defined and can be computed using standard tools from calculus and functional analysis. Its main purpose is to measure the global divergence between the CDF and PDF of a distribution. In other words, it quantifies how much the two functions related to

Received: 2024-06-04

the same distribution differ from each other. In this sense, D is a global intra-divergence distribution measure. This difference provides valuable information about the distribution in several ways:

- For the distributions where the PDF is smooth and monotonic (increasing or decreasing) at a slow rate, one might expect the difference between $F(x)$ and $f(x)$ to be small, resulting in a smaller value of D. Conversely, for distributions with sharp variations in the PDF, D may be larger, indicating a more significant deviation between the CDF and the PDF. This is particularly true for abrupt right-skewed distributions with values concentrated in the interval $(a, a+\epsilon)$ with relatively small ϵ. In this case, the peak associated with $f(x)$ is maximal, while $F(x)$ is close to 0, which means that D can be maximal (equal to an identifiable bound, as shown later). Also, distributions with thin tails or multimodal behavior may have larger divergences between the CDF and the PDF, leading to higher values of D. Examples are given in Figure 1, where two different unit distributions, i.e., distributions with support $(0, 1)$, are considered. In it, D is represented by the coloured area between the CDF $F(x)$ and the PDF $f(x)$.



(a) (b)

Figure 1: Plots of the coloured area measures by D for (a) a smooth distribution with $D \approx 0.5700264$ and (b) a right-skewed thin-tailed distribution with $D \approx 1.843216$

In this example, it is visually clear that the value of D reflects the abrupt change around 0 and the resulting thin tail of the second distribution.

- When comparing several distributions, D can help to classify the similar and the different. More precisely, by calculating D for each candidate distribution and comparing their values, we can consider two families of distributions: those that have abrupt variations in the PDF in the broad sense, characterized by high values of D, and those that do not.

In addition, some mathematical facts about D are attractive. In particular, we can find sharp lower and upper bounds of D under mild assumptions on $f(x)$ and $F(x)$, which can help to establish an understable criterion. We can also compute it for a wide range of distributions. Several concrete examples are given in this study. We concentrate mainly on the unit distributions because they are particularly numerous and useful in modern applications. See [3–18]. More specifically, we consider mainly the following ones: the $(0, 1)$-truncated normal distribution, the power distribution, the transmuted uniform distribution over $(0, 1)$, the $(0, 1)$-truncated exponential distribution, the special exponential distribution, the $(0, 1)$-truncated sine distribution, the simple

(0, 1)-truncated Lomax distribution, the (0, 1)-truncated tangent distribution, and the special inverse exponential distribution. In addition to the main use of D, some technical inequalities involving the CDF, PDF and D are established, which may be of independent interest.

The following sections structure the rest of the article: Section 2 focuses on some bounds and inequalities satisfied by D. A possible distribution criterion is also discussed. Exact computations of D are given in Section 3. A conclusion is given in Section 4.

2 Properties

Here and after, the measure D is defined by Equation (1). In this section, some properties of D are examined. These are mainly inequalities involving D, which aim to determine its natural bounds, and to show how D appears in different inequality settings.

2.1 Bounds and inequalities

The proposition below gives simple bounds on D, depending only on a and b.

Proposition 2.1. *The following lower and upper bounds of D independent of the PDF and CDF are true:*

$$e^{a-b} \leq D \leq 1+b-a.$$

Proof. For the upper bound, by applying the triangle inequality, and using $F(x) \leq 1$ for any $x \in (a, b)$ and $\int_a^b f(x)dx = 1$, we have

$$\begin{aligned} D &= \int_a^b |F(x) - f(x)|dx \leq \int_a^b [|F(x)| + |f(x)|]\, dx = \int_a^b F(x)dx + \int_a^b f(x)dx \\ &= \int_a^b F(x)dx + 1 \leq \int_a^b dx + 1 = 1 + b - a. \end{aligned}$$

For the lower bound, by introducing the exponential function e^x, and using standard differentiation rules, the Jensen inequality for the integral (applied with the basic convex function: the absolute value function), $F(a) = 0$ and $F(b) = 1$, we obtain

$$\begin{aligned} D &= \int_a^b |F(x) - f(x)|dx = \int_a^b e^x |F(x)e^{-x} - f(x)e^{-x}|dx = \int_a^b e^x \left|(e^{-x}F(x))'\right| dx \\ &\geq e^a \int_a^b \left|(e^{-x}F(x))'\right| dx \geq e^a \left| \int_a^b (e^{-x}F(x))'\, dx \right| = e^a \left| e^{-b}F(b) - e^{-a}F(a) \right| \\ &= \left| e^{a-b} \times 1 - 0 \right| = e^{a-b}. \end{aligned}$$

The stated bounds are established. □

In particular, in the case of unit distributions, we have $a = 0$ and $b = 1$, implying that

$$0.3678794 \approx e^{-1} \leq D \leq 2.$$

The result below gives alternative bounds of D, but they depend on the CDF.

Proposition 2.2. *Let us set*

$$E = \int_a^b F(x)dx. \tag{2}$$

The following lower and upper bounds of D dependent of E are true:

$$|E - 1| \leq D \leq E + 1.$$

Proof. For the upper bound, applying the triangle inequality and $\int_a^b f(x)dx = 1$, we immediately obtain

$$D = \int_a^b |F(x) - f(x)|dx \leq \int_a^b [|F(x)| + |f(x)|]\, dx = \int_a^b F(x)dx + \int_a^b f(x)dx = E + 1.$$

For the lower bound, by using the Jensen inequality for the integral and $\int_a^b f(x)dx = 1$, we get

$$D = \int_a^b |F(x) - f(x)|dx \geq \left|\int_a^b [F(x) - f(x)]dx\right| = \left|\int_a^b F(x)dx - \int_a^b f(x)dx\right| = |E - 1|.$$

The desired bounds are obtained. □

This result shows how finding E can help to understand the range of values for D. Note that the following inequality holds: $D \leq E + 1 \leq 1 + b - a$, which means that the upper bound in Proposition 2.2 is sharper than that in Proposition 2.1.

Under certain assumptions on $f(x)$, a and b, the term E is the main component of the expression of D. This is formalized in the result below.

Proposition 2.3. *Let E be defined as in Equation (2). If $f(x)$ is increasing on (a, b) and $b \leq a + 1$, we have*

$$D = 1 - E.$$

Proof. If $f(x)$ is increasing on (a, b), for any $x \in (a, b)$, we have

$$F(x) = \int_a^x f(t)dt \leq f(x) \int_a^x dt = f(x)(x - a).$$

Hence, since $b \leq a + 1$ and $f(x) \geq 0$, we have

$$F(x) - f(x) \leq f(x)(x - a - 1) \leq f(x)(b - a - 1) \leq 0.$$

As a result, by using $\int_a^b f(x)dx = 1$, we have

$$\begin{aligned} D &= \int_a^b |F(x) - f(x)|dx = \int_a^b [f(x) - F(x)]dx \\ &= \int_a^b f(x)dx - \int_a^b F(x)dx = 1 - E. \end{aligned}$$

The stated formula is established. □

So if $f(x)$ is increasing on (a, b) and $b \leq a + 1$, we always have $f(x) \geq F(x)$ for any $x \in (a, b)$, and the expression for D is simplified. On the other hand, since $F(x) \in (0, 1)$ for any $x \in (a, b)$, we have the following upper bound of D: $D \leq 1$, so that

$$e^{a-b} \leq D \leq 1. \tag{3}$$

The result below is about a technical inequality where D appears as a component of a lower bound of a sum of integrated squares of the CDF and PDF.

Proposition 2.4. *Under the assumption that $\int_a^b [f(x)]^2 dx$ exists, we have*

$$\int_a^b [F(x)]^2 dx + \int_a^b [f(x)]^2 dx \geq 1 + \frac{D^2}{b - a}.$$

Proof. We have

$$\begin{aligned}\int_a^b [f(x)-F(x)]^2 dx &= \int_a^b \{[f(x)]^2 + [F(x)]^2 - 2f(x)F(x)\} dx \\ &= \int_a^b [f(x)]^2 dx + \int_a^b [F(x)]^2 dx - 2\int_a^b f(x)F(x) dx \\ &= \int_a^b [f(x)]^2 dx + \int_a^b [F(x)]^2 dx - \{[F(x)]^2\}_a^b \\ &= \int_a^b [f(x)]^2 dx + \int_a^b [F(x)]^2 dx - [F(b)]^2 + [F(a)]^2 \\ &= \int_a^b [f(x)]^2 dx + \int_a^b [F(x)]^2 dx - 1.\end{aligned}$$

Hence, we have

$$\int_a^b [f(x)]^2 dx + \int_a^b [F(x)]^2 dx = 1 + \int_a^b [f(x)-F(x)]^2 dx. \tag{4}$$

On the other hand, by applying the Cauchy-Schwarz inequality, we get

$$D = \int_a^b |F(x)-f(x)| dx \le \sqrt{\int_a^b [f(x)-F(x)]^2 dx}\sqrt{b-a},$$

from which we derive

$$\int_a^b [f(x)-F(x)]^2 dx \ge \frac{D^2}{b-a}. \tag{5}$$

It follows from Equations (4) and (5) the desired inequality, i.e.,

$$\int_a^b [F(x)]^2 dx + \int_a^b [f(x)]^2 dx \ge 1 + \frac{D^2}{b-a}.$$

We get the desired inequality. □

Combining Propositions 2.1 and 2.4, we obtain the following inequalities:

$$\int_a^b [F(x)]^2 dx + \int_a^b [f(x)]^2 dx \ge 1 + \frac{D^2}{b-a} \ge 1 + \frac{e^{2(a-b)}}{b-a}.$$

In particular, in the case of unit distributions, we have

$$\int_0^1 [F(x)]^2 dx + \int_0^1 [f(x)]^2 dx \ge 1 + D^2 \ge 1 + e^{-2} \approx 1.135335.$$

We also get the following lower bounds of $\int_0^1 [F(x)]^2 dx$:

$$\int_0^1 [F(x)]^2 dx \ge \max\left[0, 1 - \int_0^1 [f(x)]^2 dx + D^2\right].$$

Similarly, we have

$$\int_0^1 [f(x)]^2 dx \ge 1 - \int_0^1 [F(x)]^2 dx + D^2 \ge 0.$$

Here, the positivity is because $F(x) \le 1$ for any $x \in (a,b)$. These bounds may be of interest for purposes beyond this article. For example, the main component of differential extropy is the term $\int_a^b [f(x)]^2 dx$. See [19]. This term also appears as a special case of certain differential entropy measures. See [20]. Therefore, based on the above inequalities, we can determine some bounds of them. We will leave this aspect for another more specific work.

2.2 On a possible distribution criterion

In this part, we discuss how D can be used to establish a discrimination rule between distributions.

Description

A suitable criterion can be derived from Equation (3). Indeed, we have shown that if the PDF $f(x)$ is increasing, we have $e^{a-b} \leq D \leq 1$, which means that 1 is the maximum value. In this particular monotonic case, the abrupt variation is characterized by a gradual increase around the axis $x = 1$. With this in mind, we can consider the following real number using a weighted mean rule that favours the limit 1:

$$\theta = \frac{1}{3}(2 + e^{a-b}).$$

Then, beyond the increasing case, a simple and direct criterion can be established by the following two states:

State I: If $D < \theta$, then the corresponding distribution has a PDF with smooth or moderate variations,

State II: If $D \geq \theta$, then the corresponding distribution has a PDF with significant or abrupt or extreme variations.

Of course, there is still a place for subjectivity in interpretation, especially when values are close to θ. But empirical tests tend to make these states acceptable. We also insist on the global nature of the interpretation: some shapes of the PDF can be abrupt in many ways, D does not capture the exact nature of this.

On the other hand, the following is noted: If $D > 1$, then we know directly that the PDF is not increasing. Based on Proposition 2.3, if D is close to $1 - b - a$, then we know that there are abrupt changes for the PDF.

Note that, in the case of the unit distributions, we have

$$\theta = \frac{1}{3}(2 + e^{-1}) \approx 0.7892931 \approx 0.8.$$

In this setting, let us illustrate this criterion with an example.

Example

We consider the (0, 1)-truncated normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, defined by the following CDF:

$$F(x) = \frac{\Phi(x) - \Phi(0)}{\Phi(1) - \Phi(0)}, \quad x \in (0, 1),$$

$F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, where $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$, and $\phi(t)$ is the PDF of the normal distribution with parameters μ and σ defined by

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/(2\sigma^2)}, \quad t \in \mathbb{R}.$$

Note that, in the standardized case, i.e., $\mu = 0$ and $\sigma = 1$, we have $\Phi(0) = 1/2$ and $\Phi(1) \approx 0.8413447$. The PDF corresponding to $F(x)$ is given as

$$f(x) = \frac{\phi(x)}{\Phi(1) - \Phi(0)}, \quad x \in (0, 1),$$

and $f(x) = 0$ for $x \notin (0, 1)$. For this distribution, D can not be expressed in a closed form, but it can be studied numerically and graphically without any problem. To support this claim, in Figure 2, we present some cases corresponding to the two states, i.e., State I characterized by $D < 0.8$ and State II characterized by $D \geq 0.8$, of the proposed criterion (with two examples for each state).

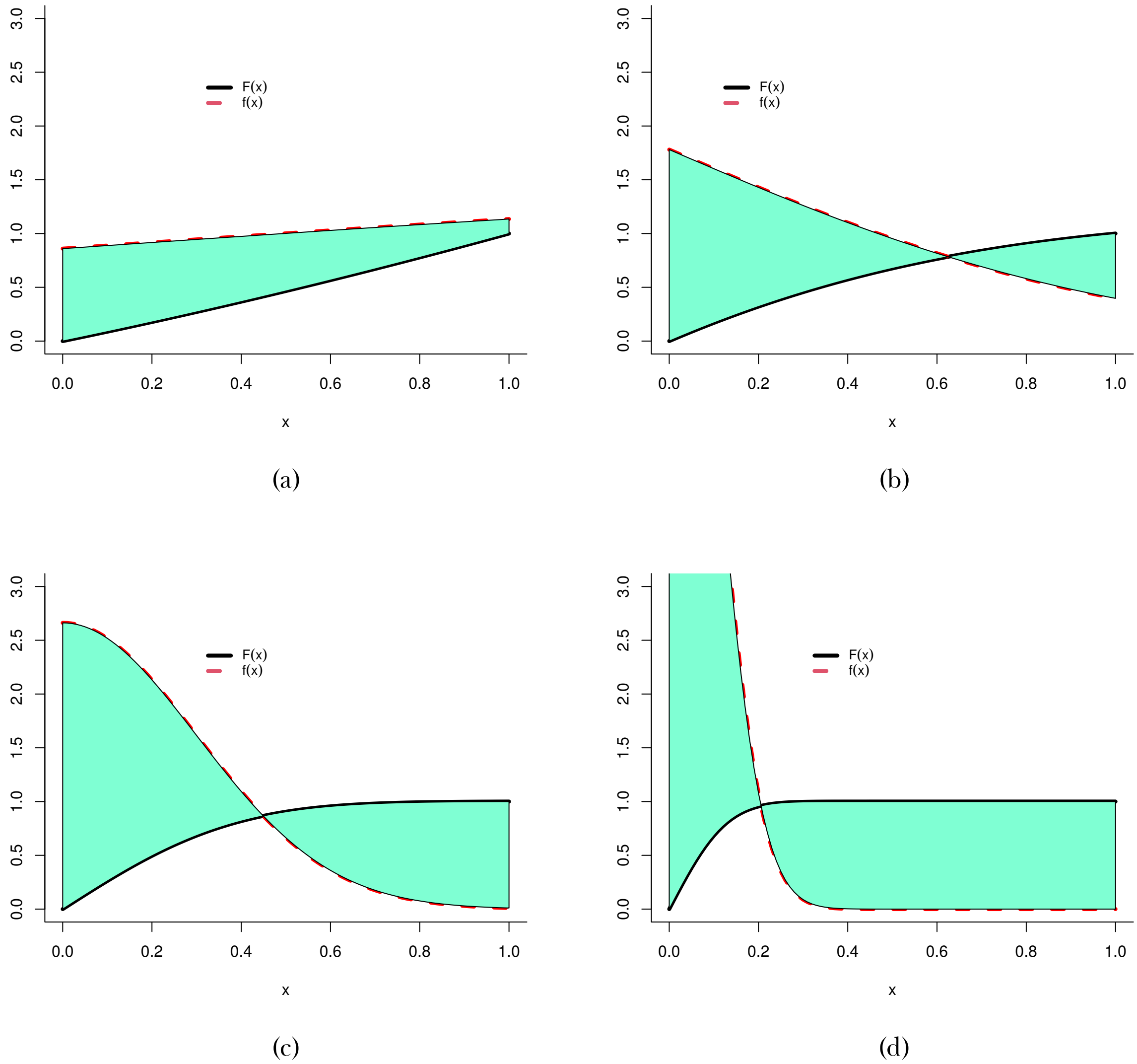


Figure 2: Plots of the coloured area measures by D for (a) $\mu = 3$ and $\sigma = 3$ for which we get $D \approx 0.5230331$ corresponding to State I, (b) $\mu = -1$ and $\sigma = 1$, which gives $D \approx 0.6296511$, also corresponding to State I, (c) $\mu = 0$ and $\sigma = 0.3$ for which we get $D \approx 1.038787$ corresponding to State II, and (d) $\mu = 0$ and $\sigma = 0.1$ for which we get $D \approx 1.586105$, also corresponding to State II

Note that the y axis is truncated to $[0, 3]$ for comparison; the curve for the PDF and the corresponding coloured area of D in case (d) are also truncated. We can thus see how D discriminates the associated distributions based, in a sense, on the degree of abruptness of the change in the shapes of the PDF. It is a relevant global measure in this respect.

2.3 Estimation

In a statistical context, the underlying distribution is unknown, and so is D. In this case, it may be interesting to estimate D based on data. One possible strategy is to estimate $f(x)$ and $F(x)$, and use a substitution approach.

Denoting $\hat{f}(x)$ as the estimate of $f(x)$ and $\hat{F}(x)$ as the estimate of $F(x)$, we thus introduce

$$\hat{D} = \int_a^b |\hat{F}(x) - \hat{f}(x)|dx.$$

For $\hat{f}(x)$, we can consider a semi-parametric approach: We select a possible family of distributions governed by PDFs of the form $f(x) = f(x;\xi)$, where ξ denotes a vector of unknown parameter(s), estimate ξ from the data by the maximum likelihood method, among other methods, yielding $\hat{\xi}$, and then consider $\hat{f}(x) = f(x;\hat{\xi})$. Alternatively, we can consider non-parametric estimation approaches, such as the kernel density method or the projection method, among others. See [21]. For $\hat{F}(x)$, we can consider the integrated version of $\hat{f}(x)$, i.e.,

$$\hat{F}(x) = \int_a^x \hat{f}(t)dt, \quad x \in (a,b).$$

A more classical estimate is the empirical CDF. A simple result on the convergence of $\hat{D}$ is proposed below.

Proposition 2.5. *Let n represent the number of data used to construct $\hat{f}(x)$ and $\hat{F}(x)$, and, by considering the expectation operator $\mathbb{E}$, suppose that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_a^b |\hat{F}(x) - F(x)|dx\right] = 0, \quad \lim_{n\to+\infty} \mathbb{E}\left[\int_a^b |\hat{f}(x) - f(x)|dx\right] = 0.$$

Then $\hat{D}$ is asymptotically unbiased, i.e.,

$$\lim_{n\to+\infty} \mathbb{E}\left(\hat{D}\right) = D.$$

Proof. By the Jensen inequality, we get

$$\left|\mathbb{E}\left(\hat{D}\right) - D\right| = \left|\mathbb{E}\left(\hat{D} - D\right)\right| \leq \mathbb{E}\left(\left|\hat{D} - D\right|\right). \tag{6}$$

On the other hand, by the Jensen and triangle inequalities, we have

$$\begin{aligned}
\left|\hat{D} - D\right| &= \left|\int_a^b |\hat{F}(x) - \hat{f}(x)|dx - \int_a^b |F(x) - f(x)|dx\right| \\
&= \left|\int_a^b \left[|\hat{F}(x) - \hat{f}(x)| - |F(x) - f(x)|\right] dx\right| \\
&\leq \int_a^b \left||\hat{F}(x) - \hat{f}(x)| - |F(x) - f(x)|\right| dx \\
&\leq \int_a^b \left|\hat{F}(x) - \hat{f}(x) - (F(x) - f(x))\right| dx \\
&\leq \int_a^b \left[|\hat{F}(x) - F(x)| + |\hat{f}(x) - f(x)|\right] dx \\
&\leq \int_a^b |\hat{F}(x) - F(x)|dx + \int_a^b |\hat{f}(x) - f(x)|dx.
\end{aligned} \tag{7}$$

It follows from Equations (6) and (7) and the linearity of the expectation operator that

$$\left|\mathbb{E}\left(\hat{D}\right) - D\right| \leq \mathbb{E}\left[\int_a^b |\hat{F}(x) - F(x)|dx\right] + \mathbb{E}\left[\int_a^b |\hat{f}(x) - f(x)|dx\right].$$

Therefore, under the considered convergence assumptions, we have

$$\begin{aligned}
0 &\leq \lim_{n\to+\infty} \left|\mathbb{E}\left(\hat{D}\right) - D\right| \\
&\leq \lim_{n\to+\infty} \mathbb{E}\left[\int_a^b |\hat{F}(x) - F(x)|dx\right] + \lim_{n\to+\infty} \mathbb{E}\left[\int_a^b |\hat{f}(x) - f(x)|dx\right] = 0,
\end{aligned}$$

implying that $\lim_{n\to+\infty}\left|\mathbb{E}\left(\hat{D}\right)-D\right|=0$, so $\lim_{n\to+\infty}\mathbb{E}\left(\hat{D}\right)=D$. This demonstrates the asymptotically unbiased nature of $\hat{D}$. □

Note that the convergence assumptions made about $\hat{f}(x)$ and $\hat{F}(x)$ are quite standard for most existing estimation methods. We refer again to [21]. Therefore, if n is large enough, $\hat{D}$ can be an efficient estimate of D. We can think of using the criterion discussed in Subsection 2.2 to discriminate the underlying distributions based on multiple data sets.

In this article, we will not develop this estimation aspect any further, but will concentrate on the full explanation of D.

3 Examples

In this section, we determine the exact expression of D for selected unit distributions, namely the power distribution, the transmuted uniform distribution over $(0, 1)$, the $(0, 1)$-truncated exponential distribution, the special exponential distribution, the $(0, 1)$-truncated sine distribution, the simple $(0, 1)$-truncated Lomax distribution, the $(0, 1)$-truncated tangent distribution, and the special inverse exponential distribution. We also comment on the results obtained and apply some previously established propositions.

3.1 Power distribution

One of the simplest unit distributions is the power distribution. It provides a manageable model for measures with values in $(0, 1)$, where smaller events are more likely to occur. See [1] and, for more modern developments, [6].

Given this distribution, the result below exhibits the exact expression of D.

Proposition 3.1. *We consider the power distribution with parameter $\alpha > 0$, defined by the CDF $F(x) = x^\alpha$ for $x \in (0, 1)$, $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, and the PDF $f(x) = \alpha x^{\alpha-1}$ for $x \in (0, 1)$, and $f(x) = 0$ for $x \notin (0, 1)$. Then the following expressions hold for D:*

- *For any $\alpha \geq 1$, we have*
$$D = \frac{\alpha}{\alpha + 1}.$$
- *For any $\alpha \in (0, 1)$, we have*
$$D = \frac{2\alpha^\alpha - \alpha}{\alpha + 1}.$$

Proof. First, we have

$$D = \int_0^1 |F(x) - f(x)|dx = \int_0^1 |x^\alpha - \alpha x^{\alpha-1}|dx = \int_0^1 x^{\alpha-1}|x - \alpha|dx.$$

Based on this formula, let us prove each item in turn.

- For any $\alpha \geq 1$ and any $x \in (0, 1)$, we have $\alpha - x \geq 1 - x \geq 0$, implying that
$$D = \int_0^1 x^{\alpha-1}(\alpha - x)dx = \alpha \int_0^1 x^{\alpha-1}dx - \int_0^1 x^\alpha dx = 1 - \frac{1}{\alpha + 1} = \frac{\alpha}{\alpha + 1}.$$
  An equivalent argument is possible: since $f(x)$ is increasing on $(0, 1)$, Proposition 2.3 also directly gives $D = 1 - E$, with $E = \int_0^1 F(x)dx = 1/(\alpha + 1)$.

- For any $\alpha \in (0, 1)$, we have

$$\begin{aligned} D &= \int_0^\alpha x^{\alpha-1}(\alpha - x)dx + \int_\alpha^1 x^{\alpha-1}(x-\alpha)dx \\ &= \alpha^\alpha - \frac{\alpha^{\alpha+1}}{\alpha+1} + \frac{1}{\alpha+1} - \frac{\alpha^{\alpha+1}}{\alpha+1} - 1 + \alpha^\alpha \\ &= 2\alpha^\alpha\left(1 - \frac{\alpha}{\alpha+1}\right) - \frac{\alpha}{\alpha+1} = \frac{2\alpha^\alpha - \alpha}{\alpha+1}. \end{aligned}$$

This ends the proof. □

Choosing $\alpha = 1$, the power distribution corresponds to the uniform distribution over $(0, 1)$, and we get

$$D = \frac{1}{2}.$$

Since $D < 0.8$, the uniform distribution over $(0, 1)$ obviously corresponds to State I of the established criterion.

To complete Proposition 3.1, Figure 3 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$ for some selected values of α.

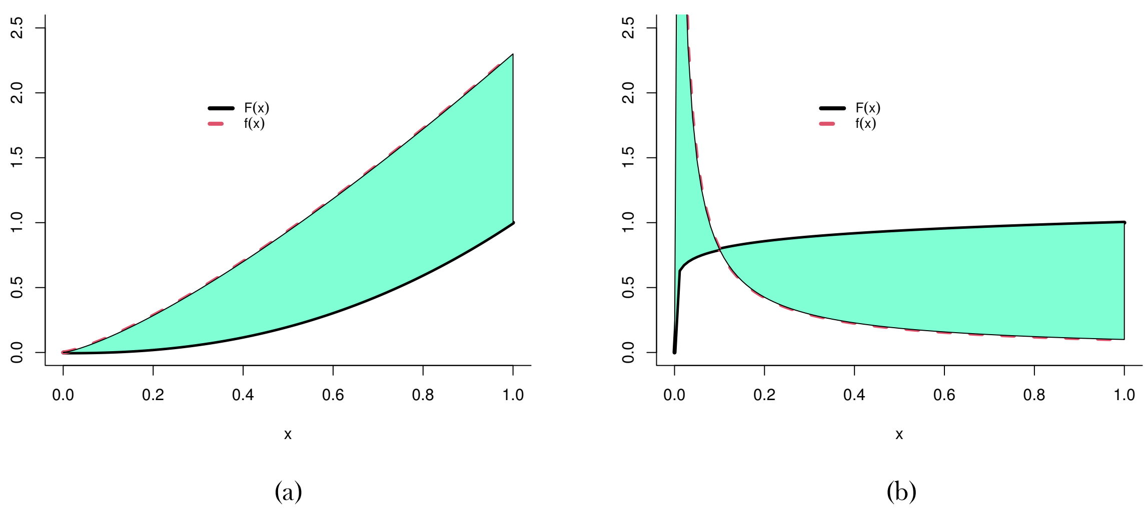


Figure 3: Plots of the area calculated by D for the power distribution with parameter α for (a) $\alpha = 2.3$ and (b) $\alpha = 0.1$

Note that the y-axis is truncated at $y = 2.5$ for comparison purposes; in the case of $\alpha = 0.1$, a coloured area around the neighborhood of $x = 0$ is not visible.

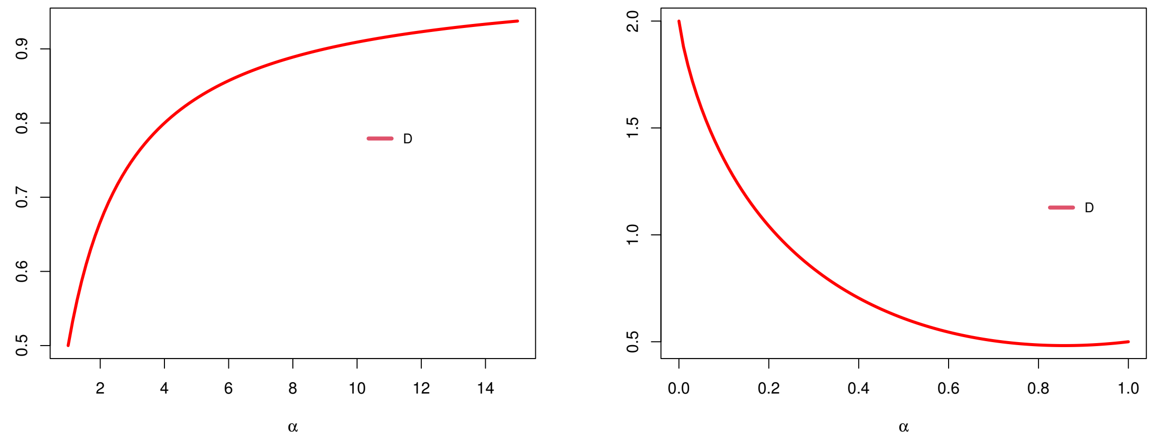
In fact, for $\alpha = 2.3$, we have $D \approx 0.6969697$, corresponding to State I (because $D < 0.8$), and for $\alpha = 0.1$, we have $D \approx 1.353324$, corresponding to State II (because $D \geq 0.8$).

For a more global view of the possible values for D, Figure 4 shows its curve with respect to α.

As can be seen from this figure, the power distribution reaches all the states of the considered criterion; smooth or moderate, and significant or abrupt variations are possible, depending on α.

Some inequalities based on the established propositions are described below. Based on Proposition 2.1, for any $\alpha \in (0, 1)$, we have

$$D = \frac{2\alpha^\alpha - \alpha}{\alpha+1} \geq e^{-1}.$$



(a) (b)

Figure 4: Curves of D for the power distribution with parameter α for (a) $\alpha \in (1, 15)$ and (b) $\alpha \in (0, 1)$

(For any $\alpha \geq 1$, we obviously have $\alpha/(1+\alpha) \geq 1/2 \geq e^{-1}$). Because of Proposition 2.4, for any $\alpha \in (1/2, 1)$, we have

$$\frac{1}{2\alpha+1} + \frac{\alpha^2}{2(\alpha-1)+1} = \int_0^1 [F(x)]^2 dx + \int_0^1 [f(x)]^2 dx \geq 1 + D^2$$
$$= 1 + \left(\frac{2\alpha^\alpha - \alpha}{\alpha+1}\right)^2.$$

These inequalities are only of mathematical interest and have potential for use in a variety of analysis situations.

3.2 Transmuted uniform distribution over $(0, 1)$

A simple compromise distribution between the uniform distribution and the square-root uniform distribution is the transmuted uniform distribution. See [22].

In the proposition below, we determine the exact expression of D for this distribution over $(0, 1)$.

Proposition 3.2. *We consider the transmuted uniform distribution over $(0, 1)$ with parameter $\beta \in [-1, 1]$, defined by the CDF $F(x) = (1+\beta)x - \beta x^2$ for $x \in (0, 1)$, $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, and the PDF $f(x) = 1 + \beta - 2\beta x$ for $x \in (0, 1)$, and $f(x) = 0$ for $x \notin (0, 1)$. Then the following expressions hold for D:*

- *For any $\beta \in [-1, 0]$, we have*

$$D = \frac{1}{6}(3 - \beta).$$

- *For any $\beta \in (0, 1]$, we have*

$$D = \frac{2}{3}\beta \left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right]^3 - (1 + 3\beta)\left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right]^2$$
$$+ 2(1+\beta)\left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right] + \frac{1}{6}(\beta - 3).$$

Proof. First, we note that

$$D = \int_0^1 |F(x) - f(x)|dx = \int_0^1 |(1+\beta)x - \beta x^2 - (1+\beta - 2\beta x)|dx$$
$$= \int_0^1 |\beta x^2 - (1+3\beta)x + 1 + \beta|dx.$$

Let us prove each item in turn.

- For any $\beta \in [-1, 0]$ and any $x \in (0, 1)$, we have $f'(x) = -2\beta \geq 0$, meaning that $f(x)$ is increasing. It follows from Proposition 2.3 that $D = 1 - E$, with

  $$E = \int_0^1 F(x)dx = \int_0^1 [(1+\beta)x - \beta x^2]dx = (1+\beta)\int_0^1 xdx - \beta \int_0^1 x^2 dx$$
  $$= (1+\beta)\frac{1}{2} - \beta\frac{1}{3} = \frac{1}{6}(\beta + 3).$$

  We thus obtain

  $$D = \frac{1}{6}(3 - \beta).$$

- For any $\beta \in (0, 1]$ and any $x \in (0, 1)$, we have $\beta x^2 - (1+3\beta)x + 1 + \beta = 0$ if and only if $x_* = [1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}]/(2\beta)$ or $x = x_{**} = [1 + 3\beta + \sqrt{1 + 2\beta + 5\beta^2}]/(2\beta)$. Since $1 + 2\beta + 5\beta^2 > 1 + 2\beta + \beta^2 = (1+\beta)^2$, we have

  $$x_* < \frac{1 + 3\beta - (1+\beta)}{2\beta} = 1,$$

  and since $1 + 2\beta + 5\beta^2 < 1 + 6\beta + 9\beta^2 = (1+3\beta)^2$, we have

  $$x_* > \frac{1 + 3\beta - (1+3\beta)}{2\beta} = 0.$$

  Therefore, we have $x_* \in (0, 1)$. On the other hand, we have

  $$x_{**} = \frac{1 + 3\beta + \sqrt{1 + 2\beta + 5\beta^2}}{2\beta} \geq \frac{3\beta}{2\beta} = 1.5 > 1,$$

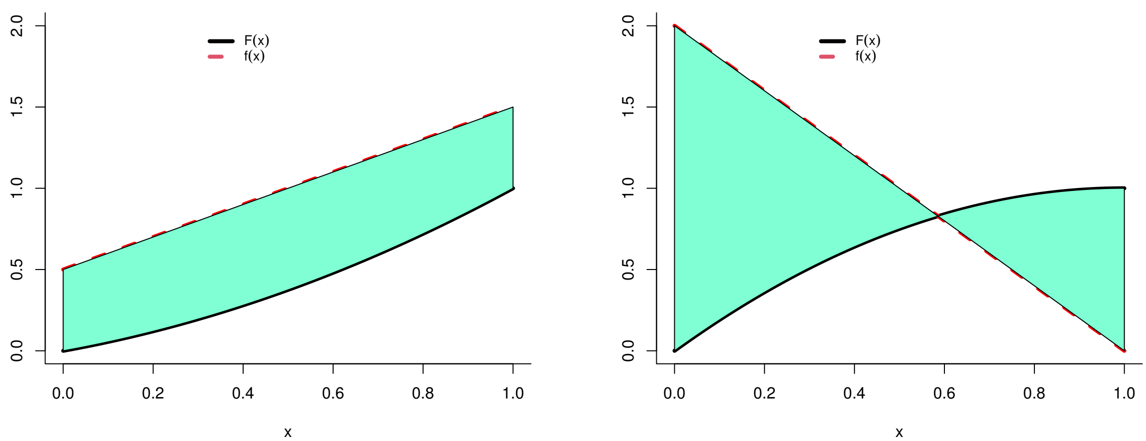
  implying that $x_{**} \notin (0, 1)$. Therefore, we have

  $$D = \int_0^{x_*} [\beta x^2 - (1+3\beta)x + 1 + \beta]dx + \int_{x_*}^1 [-\beta x^2 + (1+3\beta)x - (1+\beta)]dx$$
  $$= \left[\beta\frac{x^3}{3} - (1+3\beta)\frac{x^2}{2} + (1+\beta)x\right]_0^{x_*} + \left[-\beta\frac{x^3}{3} + (1+3\beta)\frac{x^2}{2} - (1+\beta)x\right]_{x_*}^1$$
  $$= 2\left[\beta\frac{x_*^3}{3} - (1+3\beta)\frac{x_*^2}{2} + (1+\beta)x_*\right] + \frac{1}{6}(\beta - 3)$$
  $$= \frac{2}{3}\beta\left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right]^3 - (1+3\beta)\left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right]^2$$
  $$+ 2(1+\beta)\left[\frac{1 + 3\beta - \sqrt{1 + 2\beta + 5\beta^2}}{2\beta}\right] + \frac{1}{6}(\beta - 3).$$

The desired formula is obtained. □

Choosing $\beta = 0$, the transmuted uniform distribution over $(0, 1)$ corresponds to the uniform distribution over $(0, 1)$, and we find again $D = 3/6 = 1/2$. Furthermore, by choosing $\beta = -1$, it corresponds to the square-root uniform distribution over $(0, 1)$, and we get $D = 4/6 = 2/3$.

Figure 5 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$ for selected values of β.



(a) (b)

Figure 5: Plots of the area calculated by D for the transmuted uniform distribution over $(0, 1)$ with parameter β for (a) $\beta = -0.5$ and (b) $\beta = 1$

For $\beta = -0.5$, we have $D \approx 0.5833333$, and for $\beta = 1$, we have $D \approx 0.771237$. These values are close, but as we can see, the monotonicity of the PDFs is completely different. The constant however is that only smooth or moderate variations are observed.

Figure 6 presents the curve of D with respect to β.

Clearly, we have D approximately in $(0.5, 0.8)$, so the power distribution is mainly under State I "smooth or moderate variations".

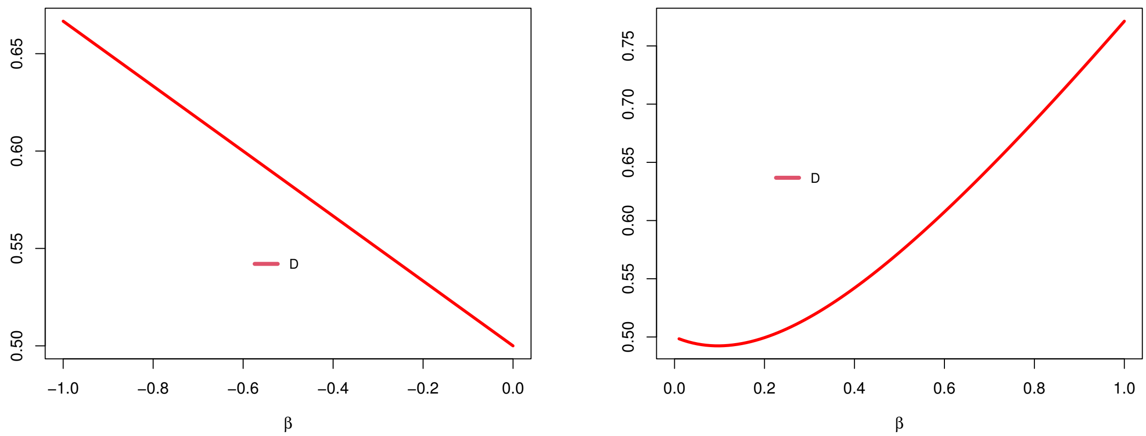
3.3 $(0, 1)$-truncated exponential distribution

The $(0, 1)$-truncated exponential distribution can be described as an adaptation of the standard exponential distribution to the unit interval. It is useful for modeling measures with values in $(0, 1)$ that have an exponentially decaying probability. See [23] for more details.

The result below determines the exact expression of D in the setting of this distribution.

Proposition 3.3. *We consider the* $(0, 1)$*-truncated exponential distribution with parameter* $\lambda > 0$*, defined by the CDF* $F(x) = (1 - e^{-\lambda x})/(1 - e^{-\lambda})$ *for* $x \in (0, 1)$*,* $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$*, and the PDF* $f(x) = \lambda e^{-\lambda x}/(1 - e^{-\lambda})$ *for* $x \in (0, 1)$*, and* $f(x) = 0$ *for* $x \notin (0, 1)$*. Then we have*

$$D = -\frac{2}{\lambda(1 - e^{-\lambda})} \log(1 + \lambda) - \frac{1}{\lambda} + \frac{2 + e^{-\lambda}}{1 - e^{-\lambda}}.$$



(a) (b)

Figure 6: Curves of D for the transmuted uniform distribution over $(0, 1)$ with parameter β for (a) $\beta \in [-1, 0]$ and (b) $\beta \in (0, 1]$

Proof. We can decompose D as

$$D = \int_0^1 |F(x) - f(x)|dx = \int_0^1 \left| \frac{1}{1-e^{-\lambda}}(1 - e^{-\lambda x}) - \frac{1}{1-e^{-\lambda}} \lambda e^{-\lambda x} \right| dx$$
$$= \frac{1}{1-e^{-\lambda}} \int_0^1 |1 - (1+\lambda)e^{-\lambda x}|dx.$$

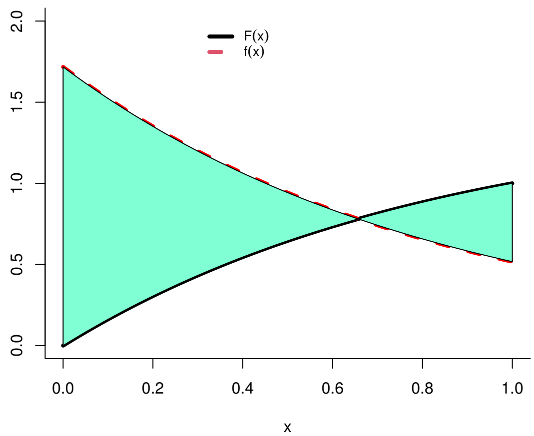
We have $1 - (1+\lambda)e^{-\lambda x} = 0$ if and only if $x = x_* = \log(1+\lambda)/\lambda$, and we have $x_* \in (0, 1)$ because of the following standard logarithmic inequality: $\log(1+\lambda) < \lambda$. Hence, we have

$$D = \frac{1}{1-e^{-\lambda}} \left\{ \int_0^{x_*} [(1+\lambda)e^{-\lambda x} - 1]dx + \int_{x_*}^1 [1 - (1+\lambda)e^{-\lambda x}]dx \right\}$$
$$= \frac{1}{1-e^{-\lambda}} \left\{ (1+\lambda) \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^{x_*} - x_* + 1 - x_* - (1+\lambda) \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_{x_*}^1 \right\}$$
$$= \frac{1}{1-e^{-\lambda}} \left[(1+\lambda)\frac{1}{\lambda}(1 - e^{-\lambda x_*}) + 1 - 2x_* - (1+\lambda)\frac{1}{\lambda}(e^{-\lambda x_*} - e^{-\lambda}) \right]$$
$$= \frac{1}{1-e^{-\lambda}} \left[(1+\lambda)\frac{1}{\lambda}\left(1 - \frac{1}{1+\lambda}\right) + 1 - \frac{2}{\lambda}\log(1+\lambda) - (1+\lambda)\frac{1}{\lambda}\left(\frac{1}{1+\lambda} - e^{-\lambda}\right) \right]$$
$$= \frac{1}{1-e^{-\lambda}} \left\{ 2\left[1 - \frac{1}{\lambda}\log(1+\lambda)\right] - \frac{1}{\lambda}(1 - e^{-\lambda}) + e^{-\lambda} \right\}$$
$$= -\frac{2}{\lambda(1-e^{-\lambda})}\log(1+\lambda) - \frac{1}{\lambda} + \frac{2+e^{-\lambda}}{1-e^{-\lambda}}.$$

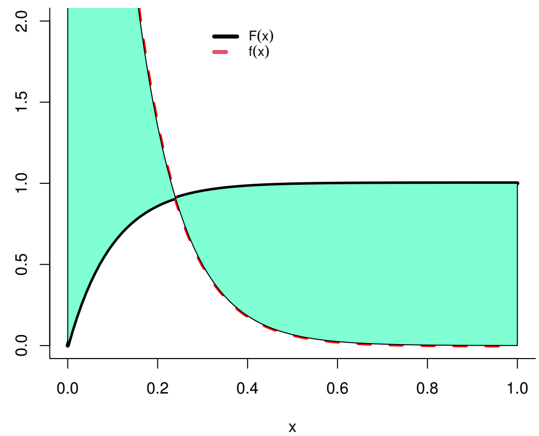
This ends the proof. □

The appearance of the logarithmic term in D is surprising at first sight, but is due to the management of the absolute values in its definition.

Figure 7 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$ for selected values of λ.



(a)



(b)

Figure 7: Plots of the area calculated by D for the $(0, 1)$-truncated exponential distribution with parameter λ for (a) $\lambda = 1.2$ and (b) $\lambda = 10$

In fact, for $\lambda = 1.2$, we have $D \approx 0.5792174$, corresponding to State I, and for $\lambda = 10$, we have $D \approx 1.420537$, corresponding to State II.

Figure 8 presents the curve of D with respect to λ.

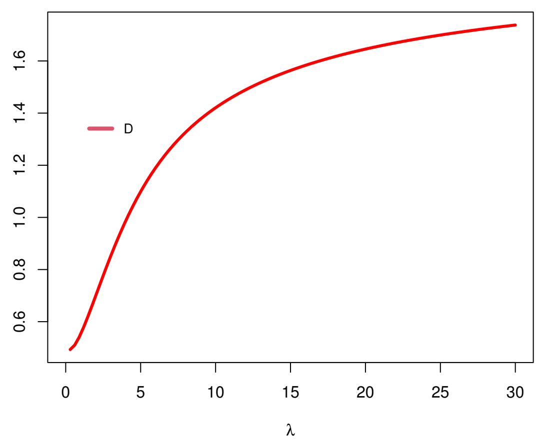


Figure 8: Curve of D for the $(0, 1)$-truncated exponential distribution with parameter λ for $\lambda \in (0, 30)$

We notice that D has its values in $(0.5, 1.7)$, so the $(0, 1)$-truncated exponential distribution has all the possible states.

To complete this part, we note that the PDF of the $(0, 1)$-truncated exponential distribution is also valid for $\lambda < 0$. This "modified" $(0, 1)$-truncated exponential distribution is considered in the next result.

Proposition 3.4. *We consider the* $(0, 1)$*-truncated exponential distribution as presented in Proposition 3.3, but with parameter* $\lambda < 0$*. Then we have*

$$D = 1 + \frac{1}{\lambda} - \frac{1}{1 - e^{-\lambda}}.$$

Proof. Let us remark that, for any $x \in (0, 1)$ and any $\lambda < 0$, we have

$$f'(x) = \frac{1}{e^{-\lambda} - 1}\lambda^2 e^{-\lambda x} \geq 0,$$

meaning that $f(x)$ is increasing. Proposition 2.3 gives $D = 1 - E$, with

$$E = \int_0^1 F(x)dx = \frac{1}{1 - e^{-\lambda}} \int_0^1 (1 - e^{-\lambda x})dx = \frac{1}{1 - e^{-\lambda}} \left[x + \frac{1}{\lambda} e^{-\lambda x}\right]_0^1$$
$$= \frac{1}{1 - e^{-\lambda}} \left[1 - \frac{1}{\lambda}(1 - e^{-\lambda})\right] = \frac{1}{1 - e^{-\lambda}} - \frac{1}{\lambda}.$$

Therefore, we obtain

$$D = 1 + \frac{1}{\lambda} - \frac{1}{1 - e^{-\lambda}}.$$

The proof ends. □

Figure 9 presents the curve of D with respect to λ.

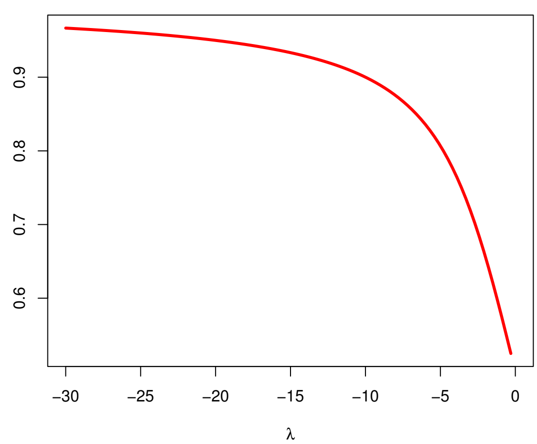


Figure 9: Curve of D for the "modified" $(0, 1)$-truncated exponential distribution with parameter λ for $\lambda \in (-30, 0)$

We observe that D has its values in $(0.5, 1)$, so the "modified" $(0, 1)$-truncated exponential distribution has all the possible states.

3.4 Special exponential distribution

The special exponential distribution can be thought of as an exponentially weighted version of the uniform distribution over $(0, 1)$. The weighting function that affects the CDF of the uniform distribution over $(0, 1)$ is defined by $w(x) = e^{\lambda(1-x)}$, where $\lambda \leq 1$, with possible negative values.

Given this distribution, the result below determines the exact expression of D.

Proposition 3.5. *We consider the special exponential distribution with parameter* $\lambda \leq 1$*, defined by the CDF* $F(x) = xe^{\lambda(1-x)}$ *for* $x \in (0,1)$*,* $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$*, and the PDF* $f(x) = (1-\lambda x)e^{\lambda(1-x)}$ *for* $x \in (0,1)$*, and* $f(x) = 0$ *for* $x \notin (0,1)$*. Then the following expressions hold for D:*

- *For any* $\lambda < 0$*, we have*
$$D = 1 + \frac{1}{\lambda} - \frac{1}{\lambda^2}(e^{\lambda} - 1).$$
- *For any* $\lambda \in (0,1]$*, we have*
$$D = \frac{1}{\lambda}(e^{\lambda} - \lambda) + \frac{1}{\lambda^2}(\lambda+1)\left[2e^{\lambda^2/(\lambda+1)} - e^{\lambda} - 1\right].$$

Proof. We have

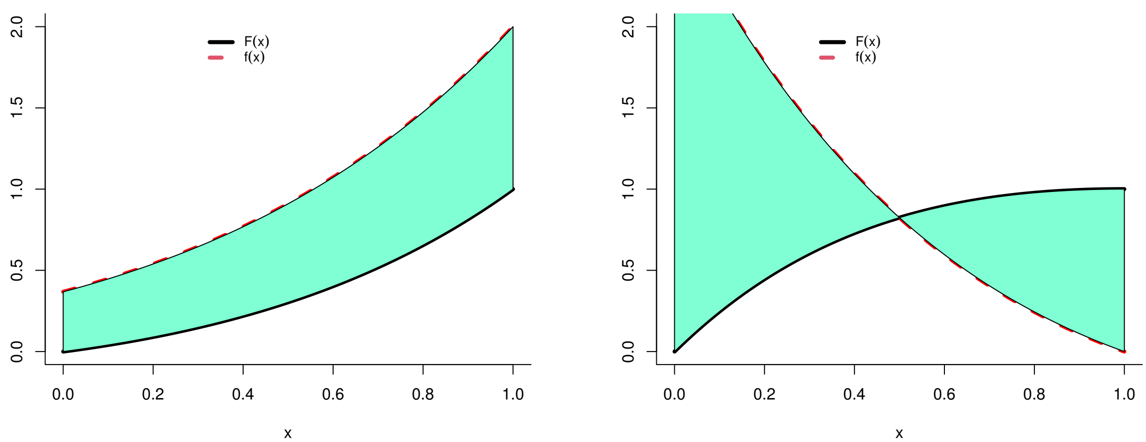
$$D = \int_0^1 |F(x) - f(x)|dx = \int_0^1 |xe^{\lambda(1-x)} - (1-\lambda x)e^{\lambda(1-x)}|dx$$
$$= \int_0^1 e^{\lambda(1-x)}|1-(\lambda+1)x|dx.$$

Let us prove each item in turn.

- For any $\lambda < 0$ and any $x \in (0,1)$, we have
$$f'(x) = -\lambda(2-\lambda x)e^{\lambda(1-x)} \geq 0,$$
meaning that $f(x)$ is increasing. It follows from Proposition 2.3 that $D = 1 - E$, with
$$E = \int_0^1 F(x)dx = \int_0^1 xe^{\lambda(1-x)}dx = e^{\lambda}\left\{\left[-x\frac{1}{\lambda}e^{-\lambda x}\right]_0^1 + \int_0^1 \frac{1}{\lambda}e^{-\lambda x}dx\right\}$$
$$= e^{\lambda}\left\{-\frac{1}{\lambda}e^{-\lambda} + \frac{1}{\lambda^2}(1-e^{-\lambda})\right\} = -\frac{1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda}-1).$$
This gives us
$$D = 1 + \frac{1}{\lambda} - \frac{1}{\lambda^2}(e^{\lambda} - 1).$$
- For any $\lambda \in (0,1]$ and any $x \in (0,1)$, we have $1-(\lambda+1)x = 0$ if and only if $x_* = 1/(\lambda+1)$ and we have $x_* \in (0,1)$. Based on this, we get
$$D = \int_0^{x_*} e^{\lambda(1-x)}[1-(\lambda+1)x]dx + \int_{x_*}^1 e^{\lambda(1-x)}[(\lambda+1)x-1]dx$$
$$= \left[-\frac{1}{\lambda}e^{\lambda(1-x)}[1-(\lambda+1)x]\right]_0^{x_*} - \frac{1}{\lambda}(\lambda+1)\int_0^{x_*} e^{\lambda(1-x)}dx$$
$$+ \left[-\frac{1}{\lambda}e^{\lambda(1-x)}[(\lambda+1)x-1]\right]_{x_*}^1 + \frac{1}{\lambda}(\lambda+1)\int_{x_*}^1 e^{\lambda(1-x)}dx$$
$$= -\frac{1}{\lambda}e^{\lambda(1-x_*)}[1-(\lambda+1)x_*] + \frac{1}{\lambda}e^{\lambda} - \frac{1}{\lambda^2}(\lambda+1)e^{\lambda}(1-e^{-\lambda x_*})$$
$$-1 + \frac{1}{\lambda}e^{\lambda(1-x_*)}[(\lambda+1)x_*-1] + \frac{1}{\lambda^2}(\lambda+1)e^{\lambda}(e^{-\lambda x_*} - e^{-\lambda})$$
$$= \frac{1}{\lambda}e^{\lambda} + \frac{1}{\lambda^2}(\lambda+1)\left[e^{\lambda^2/(\lambda+1)} - e^{\lambda}\right] - 1 + \frac{1}{\lambda^2}(\lambda+1)\left[e^{\lambda^2/(\lambda+1)} - 1\right]$$
$$= \frac{1}{\lambda}(e^{\lambda} - \lambda) + \frac{1}{\lambda^2}(\lambda+1)\left[2e^{\lambda^2/(\lambda+1)} - e^{\lambda} - 1\right].$$

The desired formula is established. □

Figure 10 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$.

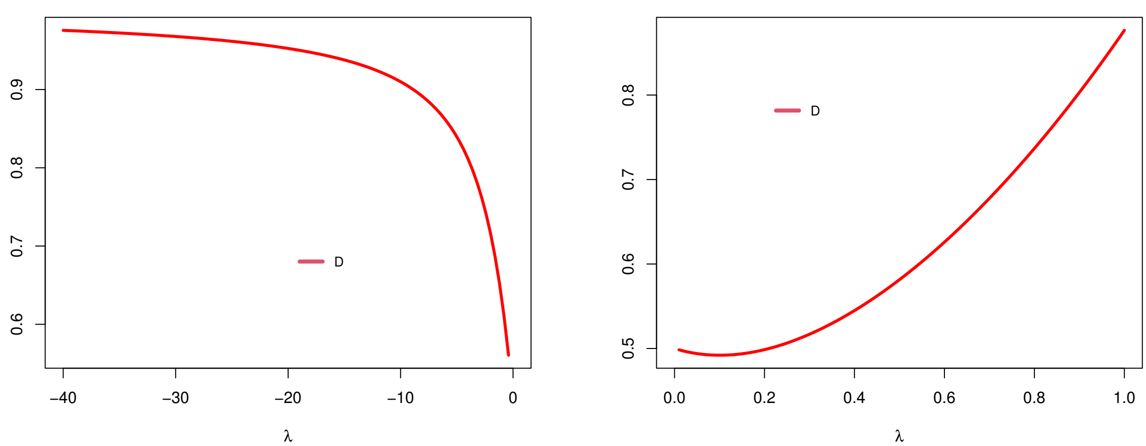


(a) (b)

Figure 10: Plots of the area calculated by D for the special exponential distribution with parameter λ for (a) $\lambda = -1$ and (b) $\lambda = 1$

For $\lambda = -1$, we have $D \approx 0.6321206$, corresponding to State I, and for $\lambda = 1$, we have $D \approx 0.8766033$, corresponding to State II.

Figure 11 presents the curve of D with respect to λ.



(a) (b)

Figure 11: Curves of D for the special exponential distribution with parameter λ for (a) $\lambda \in (-40, 0)$ and (b) $\lambda \in (0, 1)$

For any $\lambda \leq 1$, we observe that the values of D are approximately in $(0.45, 1)$, meaning that the special exponential distribution has all the possible states.

3.5 $(0, 1)$-truncated sine distribution

Trigonometric distributions have undergone considerable development in recent decades. Their flexibility has made them valuable models for various data analysis purposes. The most popular are those defined around the sine function, such as the $(0, 1)$-truncated sine distribution. See [24] and [25].

The proposition below gives the exact expression of D for this distribution.

Proposition 3.6. *We consider the* $(0, 1)$*-truncated sine distribution with parameter* $\lambda \in (0, \pi/2]$*, defined by the CDF* $F(x) = \sin(\lambda x)/\sin(\lambda)$ *for* $x \in (0, 1)$*,* $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$*, and the PDF* $f(x) = \lambda\cos(\lambda x)/\sin(\lambda)$ *for* $x \in (0, 1)$*, and* $f(x) = 0$ *for* $x \notin (0, 1)$*. Then we have*

$$D = \frac{1}{\sin(\lambda)}\left\{2\frac{\sqrt{\lambda^2+1}}{\lambda} - \frac{1}{\lambda}[1+\cos(\lambda)] - \sin(\lambda)\right\}.$$

Proof. We can write

$$\begin{aligned} D &= \int_0^1 |F(x) - f(x)|dx = \int_0^1 \left|\frac{1}{\sin(\lambda)}\sin(\lambda x) - \frac{1}{\sin(\lambda)}\lambda\cos(\lambda x)\right|dx \\ &= \frac{1}{\sin(\lambda)}\int_0^1 |\sin(\lambda x) - \lambda\cos(\lambda x)|dx. \end{aligned}$$

We have $\sin(\lambda x) - \lambda\cos(\lambda x) = 0$ if and only if $x = x_* = \arctan(\lambda)/\lambda$, and we have $x_* \in (0, 1)$ because of the following standard arctangent inequality: $\arctan(\lambda) < \lambda$. Therefore, we get

$$\begin{aligned} D &= \frac{1}{\sin(\lambda)}\left\{\int_0^{x_*}[\lambda\cos(\lambda x) - \sin(\lambda x)]dx + \int_{x_*}^1[\sin(\lambda x) - \lambda\cos(\lambda x)]dx\right\} \\ &= \frac{1}{\sin(\lambda)}\left\{\left[\sin(\lambda x) + \frac{1}{\lambda}\cos(\lambda x)\right]_0^{x_*} + \left[-\frac{1}{\lambda}\cos(\lambda x) - \sin(\lambda x)\right]_{x_*}^1\right\} \\ &= \frac{1}{\sin(\lambda)}\left\{2\left[\sin(\lambda x_*) + \frac{1}{\lambda}\cos(\lambda x_*)\right] - \frac{1}{\lambda}[1+\cos(\lambda)] - \sin(\lambda)\right\}. \end{aligned}$$

Using $\sin[\arctan(x)] = x/\sqrt{x^2+1}$ and $\cos[\arctan(x)] = 1/\sqrt{x^2+1}$, we obtain

$$\begin{aligned} D &= \frac{1}{\sin(\lambda)}\left\{2\left[\frac{\lambda}{\sqrt{\lambda^2+1}} + \frac{1}{\lambda\sqrt{\lambda^2+1}}\right] - \frac{1}{\lambda}[1+\cos(\lambda)] - \sin(\lambda)\right\} \\ &= \frac{1}{\sin(\lambda)}\left\{2\frac{\sqrt{\lambda^2+1}}{\lambda} - \frac{1}{\lambda}[1+\cos(\lambda)] - \sin(\lambda)\right\}. \end{aligned}$$

This ends the proof. □

Figure 12 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$.

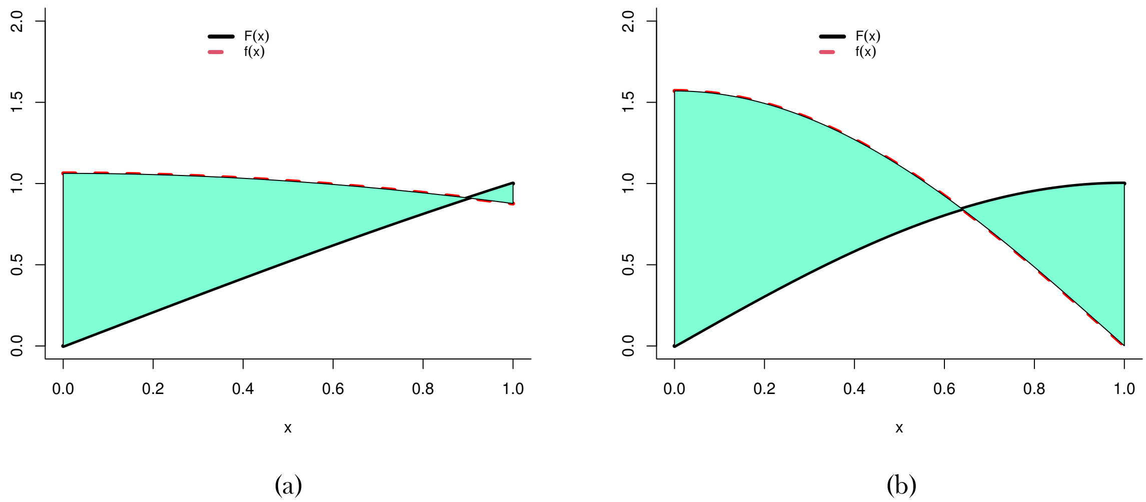


Figure 12: Plots of the area calculated by D for the $(0, 1)$-truncated sine distribution with parameter λ for (a) $\lambda = 0.6$ and (b) $\lambda = 1.57$

For $\lambda = 0.6$, we have $D \approx 0.4966557$, and for $\lambda = 1.57$, we have $D \approx 0.7337918$.

Figure 13 presents the curve of D with respect to λ.

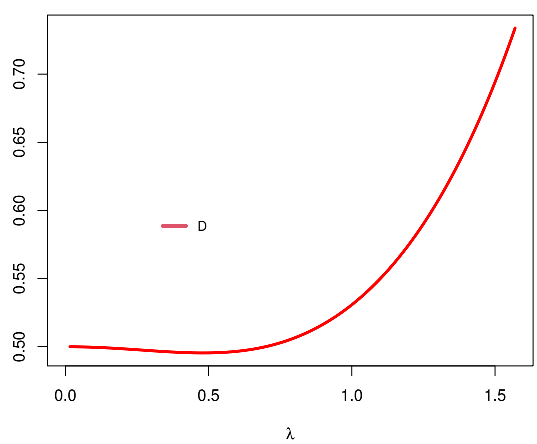


Figure 13: Curve of D for the $(0, 1)$-truncated sine distribution with parameter λ for $\lambda \in (0, 1.57)$

The values of D are approximately in $(0.48, 0.8)$, meaning that the $(0, 1)$-truncated sine distribution is mainly under State I.

3.6 Simple $(0, 1)$-truncated Lomax distribution

The simple $(0, 1)$-truncated Lomax distribution can be considered as a one-parameter alternative to the $(0, 1)$-truncated exponential distribution. It is designed to model measures with values in $(0, 1)$ that have a polynomial rather than an exponential probability of decay. See [26].

The exact expression of D for this distribution is given in the proposition below.

Proposition 3.7. *We consider the simple* $(0,1)$*-truncated Lomax distribution with parameter* $\lambda > 0$*, defined by the CDF* $F(x) = (1+\lambda)x/(x+\lambda)$ *for* $x \in (0,1)$*,* $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$*, and the PDF* $f(x) = \lambda(1+\lambda)/(x+\lambda)^2$ *for* $x \in (0,1)$*, and* $f(x) = 0$ *for* $x \notin (0,1)$*. Then we have*

$$D = (1+\lambda)\Bigg[-\frac{4\lambda}{\lambda+\sqrt{\lambda^2+4\lambda}} + \lambda - \sqrt{\lambda^2+4\lambda} + 2\lambda\log\left(\frac{\lambda+\sqrt{\lambda^2+4\lambda}}{2}\right) - \lambda\log(\lambda) + \frac{\lambda}{1+\lambda} - \lambda\log(1+\lambda) + 2\Bigg].$$

Proof. First, we can decompose D as

$$\begin{aligned} D &= \int_0^1 |F(x)-f(x)|dx = \int_0^1 \left|(1+\lambda)\frac{x}{x+\lambda} - \lambda(1+\lambda)\frac{1}{(x+\lambda)^2}\right| dx \\ &= (1+\lambda)\int_0^1 \frac{|x^2+\lambda x-\lambda|}{(x+\lambda)^2}dx. \end{aligned}$$

We have $x^2+\lambda x-\lambda = 0$ if and only if $x = x_* = [-\lambda+\sqrt{\lambda^2+4\lambda}]/2$ or $x = x_{**} = [-\lambda-\sqrt{\lambda^2+4\lambda}]/2$. Since $\lambda = \sqrt{\lambda^2} < \sqrt{\lambda^2+4\lambda}$ and $\sqrt{\lambda^2+4\lambda} < \sqrt{\lambda^2+4\lambda+4} = \sqrt{(\lambda+2)^2} = \lambda+2$, we have $x_* \in (0,1)$. Moreover, x_{**} is clearly negative. Hence, we establish that

$$\begin{aligned} D &= (1+\lambda)\left\{\int_0^{x_*} \frac{\lambda - x^2-\lambda x}{(x+\lambda)^2}dx + \int_{x_*}^1 \frac{x^2+\lambda x-\lambda}{(x+\lambda)^2}dx\right\} \\ &= (1+\lambda)\left\{\int_0^{x_*}\left[\frac{\lambda}{(x+\lambda)^2} + \frac{\lambda}{x+\lambda} - 1\right]dx + \int_{x_*}^1\left[1 - \frac{\lambda}{(x+\lambda)^2} - \frac{\lambda}{x+\lambda}\right]dx\right\} \\ &= (1+\lambda)\left\{\left[-\frac{\lambda}{x+\lambda} + \lambda\log(x+\lambda) - x\right]_0^{x_*} + \left[\frac{\lambda}{x+\lambda} - \lambda\log(x+\lambda) + x\right]_{x_*}^1\right\} \\ &= (1+\lambda)\left\{2\left[-\frac{\lambda}{x_*+\lambda} + \lambda\log(x_*+\lambda) - x_*\right] - \lambda\log(\lambda) + \frac{\lambda}{1+\lambda} - \lambda\log(1+\lambda) + 2\right\} \\ &= (1+\lambda)\Bigg[-\frac{4\lambda}{\lambda+\sqrt{\lambda^2+4\lambda}} + \lambda - \sqrt{\lambda^2+4\lambda} + 2\lambda\log\left(\frac{\lambda+\sqrt{\lambda^2+4\lambda}}{2}\right) - \lambda\log(\lambda) \\ &\quad + \frac{\lambda}{1+\lambda} - \lambda\log(1+\lambda) + 2\Bigg]. \end{aligned}$$

This concludes the proof. □

Figure 14 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$.

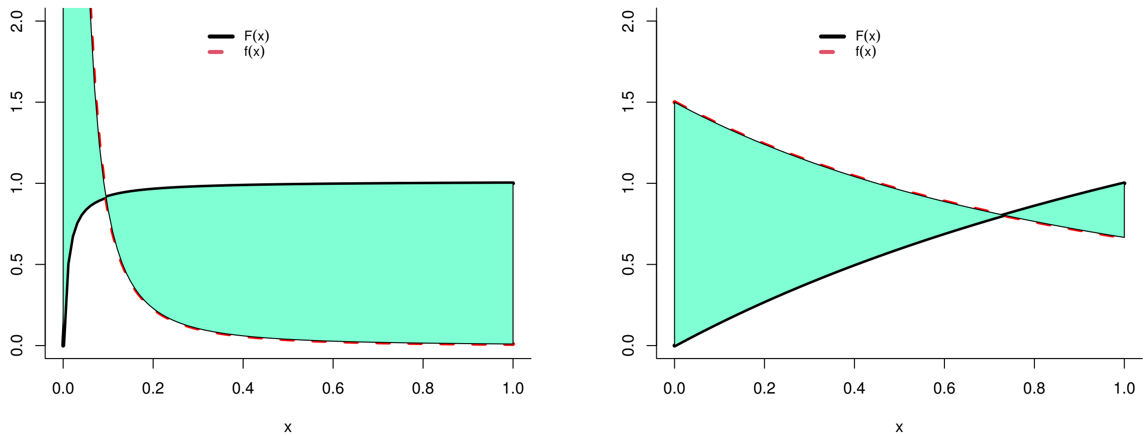


Figure 14: Plots of the area calculated by D for the simple $(0, 1)$-truncated Lomax distribution with parameter λ for (a) $\lambda = 0.01$ and (b) $\lambda = 2$

For $\lambda = 0.01$, we have $D \approx 1.646605$, corresponding to State II, and for $\lambda = 2$, we have $D \approx 0.5254624$, corresponding to State I.

Figure 15 presents the curve of D with respect to λ.

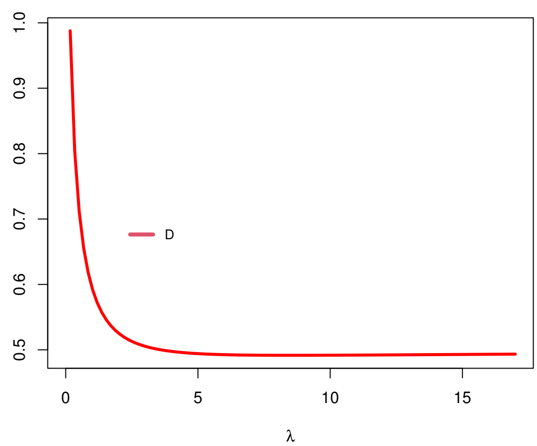


Figure 15: Curve of D for the simple $(0, 1)$-truncated Lomax distribution with parameter λ for $\lambda \in (0, 17)$

The values of D are approximately in $(0.48, 2)$, which means that the $(0, 1)$-truncated Lomax distribution reaches all the states of the considered criterion.

3.7 $(0, 1)$-truncated tangent distribution

Trigonometric distributions include those constructed from the tangent function. The pioneering work on this topic is [27]. From this, we can derive the $(0, 1)$-truncated tangent distribution.

The proposition below examines the exact expression of D for this distribution.

Proposition 3.8. *We consider the* $(0, 1)$*-truncated tangent distribution with parameter* $\lambda \in (0, \pi/4]$*, defined by the CDF* $F(x) = \tan(\lambda x)/\tan(\lambda)$ *for* $x \in (0, 1)$*,* $F(x) = 0$ *for* $x \leq 0$ *and* $F(x) = 1$ *for* $x \geq 1$*, and the PDF* $f(x) = \lambda\{1 + [\tan(\lambda x)]^2\}/\tan(\lambda)$ *for* $x \in (0, 1)$*, and* $f(x) = 0$ *for* $x \notin (0, 1)$*. Then we have*

$$D = 1 + \frac{1}{\lambda \tan(\lambda)} \log[\cos(\lambda)].$$

Proof. For any $\lambda \in (0, \pi/4]$ and any $x \in (0, 1)$, we have

$$f'(x) = \frac{2\lambda^2}{\tan(\lambda)} \tan(\lambda x) \left\{1 + [\tan(\lambda x)]^2\right\} \geq 0.$$

Therefore, $f(x)$ is increasing. Proposition 2.3 gives $D = 1 - E$, with

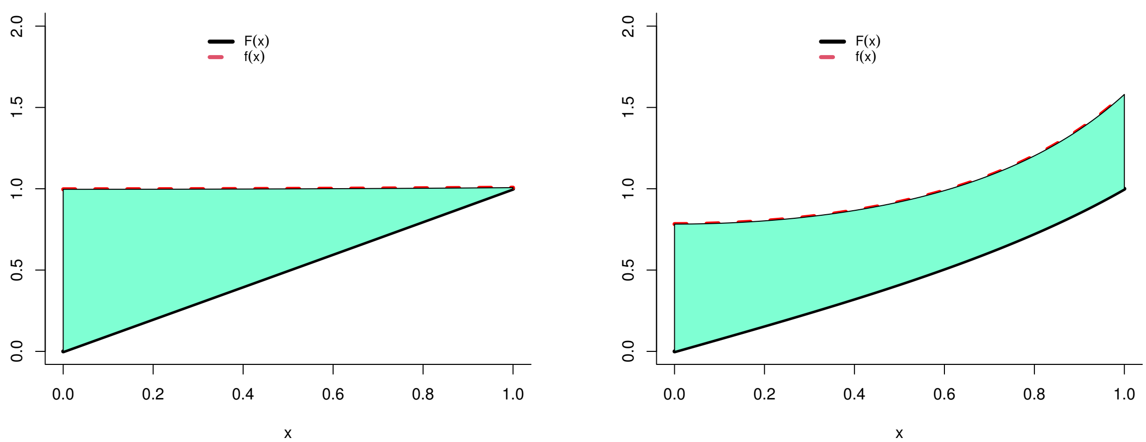
$$E = \int_0^1 F(x)dx = \frac{1}{\tan(\lambda)} \int_0^1 \tan(\lambda x)dx = \frac{1}{\tan(\lambda)} \left[-\frac{1}{\lambda} \log[\cos(\lambda x)]\right]_0^1 = -\frac{1}{\lambda \tan(\lambda)} \log[\cos(\lambda)].$$

Therefore, we obtain

$$D = 1 + \frac{1}{\lambda \tan(\lambda)} \log[\cos(\lambda)].$$

The proof ends. □

Figure 16 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$.



(a) (b)

Figure 16: Plots of the area calculated by D for the $(0, 1)$-truncated tangent distribution with parameter λ for (a) $\lambda = 0.1$ and (b) $\lambda = 0.79$

For $\lambda = 0.1$, we have $D \approx 0.500835$, and for $\lambda = 0.79$, we have $D \approx 0.5595201$.

Figure 17 presents the curve of D with respect to λ.

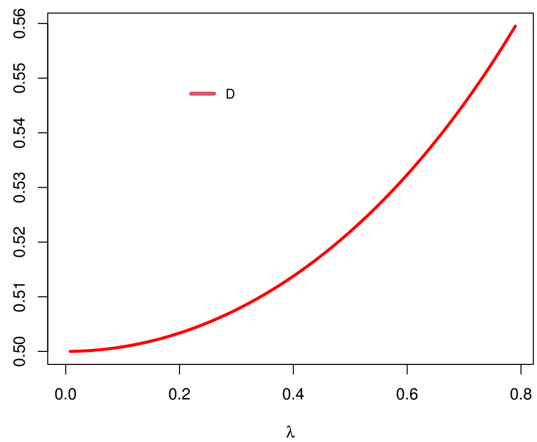


Figure 17: Curve of D for the $(0, 1)$-truncated tangent distribution with parameter λ for $\lambda \in (0, 0.79)$

The values of D are approximately in $(0.5, 0.6)$, meaning that the $(0, 1)$-truncated tangent distribution only reaches State I. The PDF possesses only smooth variations.

3.8 Special inverse exponential distribution

The special inverse exponential distribution can be represented as a special one-parameter case of the unit Gompertz distribution, as defined in [28].

In the result below, we show that, under certain conditions on the parameter, D can be expressed simply as a function of the upper incomplete gamma function.

Proposition 3.9. *We consider the special inverse exponential distribution with parameter $\lambda \geq 2$, defined by the CDF $F(x) = e^{\lambda(1-1/x)}$ for $x \in (0, 1)$, $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, and the PDF $f(x) = (\lambda/x^2)e^{\lambda(1-1/x)}$ for $x \in (0, 1)$, and $f(x) = 0$ for $x \notin (0, 1)$. Then we have*

$$D = \lambda e^{\lambda}\Gamma(0, \lambda),$$

where $\Gamma(a, b)$ denotes the upper incomplete gamma function: $\Gamma(a, b) = \int_b^{+\infty} t^{a-1}e^{-t}dt$ with $a \geq 0$ and $b \geq 0$ such that $a + b > 0$.

Proof. For any $\lambda \geq 2$ and any $x \in (0, 1)$, we have

$$f'(x) = \frac{\lambda}{x^4}e^{\lambda(1-1/x)}(\lambda - 2x) \geq \frac{\lambda}{x^4}e^{\lambda(1-1/x)}(\lambda - 2) \geq 0.$$

Therefore, $f(x)$ is increasing. Proposition 2.3 gives $D = 1 - E$, with

$$E = \int_0^1 F(x)dx = \int_0^1 e^{\lambda(1-1/x)}dx = e^{\lambda}\int_0^1 e^{-\lambda/x}dx.$$

Applying the change of variable $y = \lambda/x$ and doing an integration by parts, we get

$$\int_0^1 e^{-\lambda/x}dx = \lambda\int_{\lambda}^{+\infty}\frac{1}{y^2}e^{-y}dy = \lambda\left\{\left[-\frac{1}{y}e^{-y}\right]_{\lambda}^{+\infty} - \int_{\lambda}^{+\infty}\frac{1}{y}e^{-y}dy\right\}$$
$$= \lambda\left[\frac{1}{\lambda}e^{-\lambda} - \int_{\lambda}^{+\infty}\frac{1}{y}e^{-y}dy\right] = e^{-\lambda} - \lambda\Gamma(0, \lambda).$$

Therefore, we obtain

$$D = 1 - e^{\lambda}\left[e^{-\lambda} - \lambda\Gamma(0,\lambda)\right] = \lambda e^{\lambda}\Gamma(0,\lambda).$$

The desired formula is proved. □

The special inverse exponential distribution is also well defined for $\lambda \in (0, 2)$, but expressing D requires more mathematical effort. A manageable expression for it in this case therefore remains a challenge.

Figure 18 shows a graphical representation of D by a coloured area between the CDF $F(x)$ and the PDF $f(x)$.

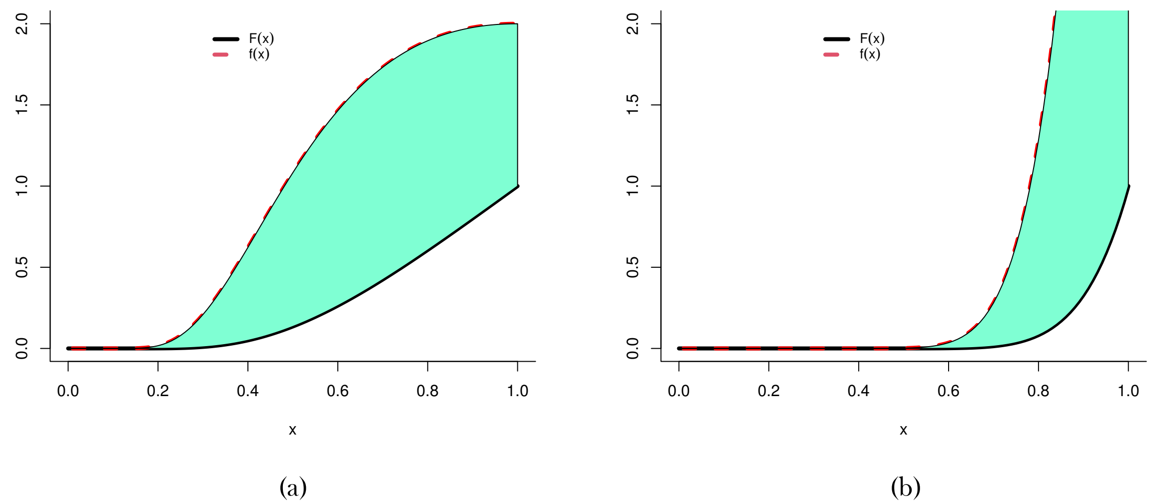


Figure 18: Plots of the area calculated by D for the special inverse exponential distribution with parameter λ for (a) $\lambda = 2$ and (b) $\lambda = 10$

For $\lambda = 2$, we have $D \approx 0.7226572$, corresponding to State I, and for $\lambda = 10$, we have $D \approx 0.9156333$, corresponding to State II.

Figure 19 presents the curve of D with respect to λ.

The values of D are approximately in $(0.7, 0.96)$, meaning that the special inverse exponential distribution reaches all the possible states.

Some inequalities based on the established propositions are described below. Based on Proposition 2.1, for any $\lambda \geq 2$, we have

$$D = \lambda e^{\lambda}\Gamma(0,\lambda) \geq e^{-1}.$$

Applying Proposition 2.4, for any $\lambda \geq 2$, we have

$$1 - 2\lambda e^{2\lambda}\Gamma(0, 2\lambda) + \frac{1}{2} + \frac{1}{4\lambda} + \frac{1}{2}\lambda = \int_0^1 [F(x)]^2 dx + \int_0^1 [f(x)]^2 dx \geq 1 + D^2$$
$$= 1 + \left[\lambda e^{\lambda}\Gamma(0,\lambda)\right]^2,$$

so, by isolating the terms involving the upper incomplete gamma function,

$$\frac{1}{2} + \frac{1}{4\lambda} + \frac{1}{2}\lambda \geq \lambda e^{2\lambda}\left\{\lambda\,[\Gamma(0,\lambda)]^2 + 2\Gamma(0, 2\lambda)\right\}.$$

These inequalities are of mathematical interest only. We do not claim that they are sharp.

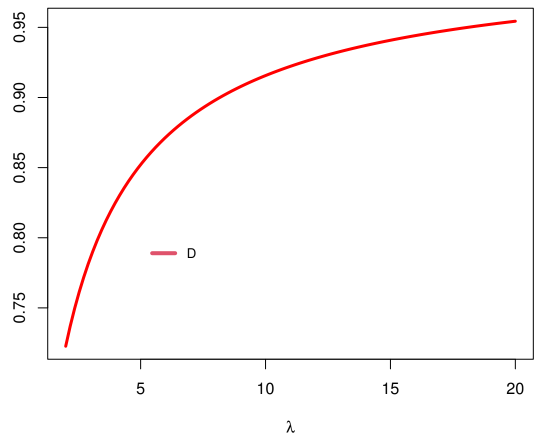


Figure 19: Curve of D for the special inverse exponential distribution with parameter λ for $\lambda \in (2, 20)$

4 Conclusion

In this article, we have proposed a global intra-divergence distribution measure that focuses on distributions with finite support. It provides a numerical indicator that discriminates between distributions with smooth probability density functions and those with abruptly varying probability density functions. We have established several inequalities involving it, discussed a possible discrimination criterion based on some of the bounds obtained in these inequalities, and calculated the exact expressions of it for well-known unit distributions. Namely, the following unit distributions were considered: the power distribution, the transmuted uniform distribution over $(0, 1)$, the $(0, 1)$-truncated exponential distribution, the special exponential distribution, the $(0, 1)$-truncated sine distribution, the simple $(0, 1)$-truncated Lomax distribution, the $(0, 1)$-truncated tangent distribution, and the special inverse exponential distribution. Of course, many other distributions with finite support can be studied in the same way.

A possible refinement of the measure might be to consider

$$D_* = \max\left[\int_a^b |F(x) - f(x)|dx, \int_a^b |S(x) - f(x)|dx\right],$$

where $S(x) = 1 - F(x)$ is the survival function associated with the distribution. In this case we conjecture that $D_* \in [e^{a-b}, 2]$ and that we can have more nuance about the abruptness of all types of distributions, including those with increasing PDFs for which D remains somewhat coarse. We can then construct a new criterion with more nuanced states. In addition, we can also think of developing a global intra-divergence measure for distributions with infinite support. These lines of research remain to be explored. We postpone this work to a future study.

Conflicts of interest: The author declares no conflict of interest.

Funding: The author has received no funds.

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