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Exponential Stability of Solutions of Lorenz Equations via a Differential Inequality

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Abstract
This paper deals with the exponential convergence of the solutions of nonlinear perturbed differential inequalities to a small ball centred at the origin. The behaviour of the Lorenz system is also investigated, and several sufficient conditions are provided for exponential stability toward a small neighbourhood of the origin.

Keywords: Exponential Stability; Lorenz Equations; Differential Inequality.

1 Introduction

It is well known that differential inequalities play important roles in the study of stability, boundedness, oscillation and stability properties to the solutions in particular for integro-differential equations. Lyapunov's direct method, (see [20]-[25]), states that if a positive-definite function (now called a Lyapunov function) of the state coordinates of a dynamical system can be constructed for which its time rate of change following small perturbations from the system equilibrium is always negative or zero, then the system equilibrium state is stable. In other words, Lyapunov's method is based on the construction of a Lyapunov function that serves as a generalized norm of the solution of a dynamical system ([1]-[10], ([13]-[18])). Many applications are treated in literature in particular for the exponential convergence of the solutions for Lorenz system ([12], [19]). The goal of this paper is to present some new conditions for practical stability of differential inequalities in presence of perturbation. Moreover, we provide an application to Lorenz equations to prove the validity of this approach.

2 Definitions and tools

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. A positive definite function $\mathbb{R}^+ \to \mathbb{R}^+$ is one that is zero at the origin and positive otherwise. We define the closed ball $B_r := \Big\{x \in \mathbb{R}^n : \|x\| \leq r\Big\}$.

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Consider the time varying system described by the following:

$$\dot{x} \quad = \quad f(t,x) + g(t,x) \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $\mathbb{R}^+ \times \mathbb{R}^n$. We consider also the associated nominal system

$$\dot{x} \quad = \quad f(t,x) \tag{2}$$

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution of (1) at time t_0 starting from the point x_0.
We recall now some standard concepts from stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [20, 24]. Let introduce some basic definitions that we shall need in the sequel for the system (2) (see [3], [4], [6]).

Definition 1. *(uniform stability of B_r)*

i *B_r is uniformly stable if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$,*

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall\, t \geq t_0; \tag{3}$$

ii *B_r is globally uniformly stable if it is uniformly stable and the solutions of system (2) are globally uniformly bounded.*

Definition 2. *(uniform attractivity of B_r) B_r is globally uniformly attractive if for all $\epsilon > r$ and $c > 0$, there exists $T(\epsilon,c) > 0$ such that for all $t_0 \geq 0$,*

$$displaystyle\|x(t)\| < \epsilon \quad \forall\, t \geq t_0 + T(\epsilon,c), \quad \|x_0\| < c. \tag{4}$$

The next definition concerns the practical global uniform exponential stability.

Definition 3. *B_r is globally uniformly exponentially stable if there exist γ, k, positive constants, such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$,*

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t-t_0)) + r. \tag{5}$$

System (2) is globally practically uniformly exponentially stable if there exists $r > 0$ such that B_r is globally uniformly exponentially stable.

In the sequel, we will consider the practical stability of a class of perturbed system of the form (1), in the case of differential inequality. We will show that, under some sufficient conditions, the solutions of a differential inequality in presence of perturbation converge to a small ball centred at the origin.

Let consider the following differential inequality:

$$\dot{y}(t) \leq s(t)y(t) + K(t) \tag{6}$$

where the functions $s(.)$ and $K(.)$ are continuous.

Theorem 1. *Suppose that,*

- *there exists $\delta > 0$, such that*

$$\limsup_{l\to+\infty} \limsup_{k\to+\infty} \frac{1}{l} \int_k^{k+l} s(t)dt < -\delta, \tag{7}$$

- *for any fixed* $T < \infty$,

$$\sup_{t\geq 0;0\leq u\leq T}\int_t^{t+u} s(\lambda)d\lambda < \infty, \tag{8}$$

- *there exists* $1 < p < \infty$ *such that,* $K \in L^p([0, +\infty[)$.
  Then,

$$y(t) \leq ke^{-\frac{\delta}{2}t}|y(0)| + r, \tag{9}$$

  where k *and* r *are positive constants.*

Proof. It is easy to see that

$$y(t) \leq \exp(\int_0^t s(\lambda)d\lambda)y(0) + \int_0^t K(u)\exp(\int_u^t s(\lambda)d\lambda)du. \tag{10}$$

By (7), we can find a sufficiently large number $T < \infty$ such that for all $u \geq T$ and $t-u \geq T$,

$$\int_u^t s(\lambda)d\lambda \leq -\frac{\delta}{2}(t-u) \tag{11}$$

For any $t \geq 2T$,

$$\begin{aligned}
y(t) &\leq \exp(\int_0^t s(\lambda)d\lambda)\exp(-\frac{\delta}{2}(t-T))|y(0)| + \int_0^T |K(u)|\exp(\int_u^T s(\lambda)d\lambda)du\, e^{-\frac{\delta}{2}(t-T)}\\
&+ \int_T^{t-T}|K(u)|\exp(\int_u^t s(\lambda)d\lambda)du + \int_{t-T}^t |K(u)|\exp(\int_u^t s(\lambda)d\lambda)du\\
&\leq e^{M_1}\exp(-\frac{\delta}{2}(t-T))|y(0)| + e^{M_2}\int_0^T|K(u)|du\exp(-\frac{\delta}{2}(t-T))\\
&+ \int_T^{t-T}|K(u)|\exp(-\frac{\delta}{2}(t-u))du + \int_{t-T}^t|K(u)|\exp(\int_u^t s(\lambda)d\lambda)du\\
&\leq e^{M_1}\exp(-\frac{\delta}{2}(t-T))|y(0)| + e^{M_2}\int_0^T|K(u)|du\exp(-\frac{\delta}{2}(t-T))\\
&+ (\int_T^{t-T}|K(u)|^p du)^{\frac{1}{p}}\int_T^{t-T}\exp(-\frac{\delta}{2q}(t-u))du + e^{M_3}\int_{t-T}^t|K(u)|du\\
&\leq e^{M_1}\exp(-\frac{\delta}{2}(t-T))|y(0)| + e^{M_2}T^{\frac{1}{q}}\|K\|_p\exp(-\frac{\delta}{2}(t-T))\\
&+ \frac{2\|K\|_p q}{\delta}[\exp(-\frac{\delta}{2q}T) - \exp(-\frac{\delta}{2q}(t-T))] + e^{M_3}T^{\frac{1}{q}}\|K\|_p\\
&\leq e^{M_1}\exp(\frac{\delta}{2}T)\exp(-\frac{\delta}{2}t)|y(0)| + (e^{M_2}T^{\frac{1}{q}} + 2\frac{q}{\delta} + e^{M_3}T^{\frac{1}{q}})\|K\|_p.
\end{aligned}$$

Then,

$$y(t) \leq ke^{-\frac{\delta}{2}t}|y(0)| + r, \tag{12}$$

where, $k = e^{M_1}\exp(\frac{\delta}{2}T)$ and $r = (e^{M_2}T^{\frac{1}{q}} + 2\frac{q}{\delta} + e^{M_3}T^{\frac{1}{q}})\|K\|_p$ with respect inequality (9).

□

Consider the differential inequality (6).

Theorem 2. *Suppose that,*

- *there exists* $\delta > 0$, $l > 0$ *and* $T \geq 0$ *such that*

$$\frac{1}{l}\int_k^{k+l} s(t)dt \leq -\delta, \quad \forall k \geq T. \tag{13}$$

- *for any fixed* $T < \infty$

$$\sup_{t\geq 0; 0\leq u\leq T} \int_t^{t+u} s(\lambda)d\lambda < \infty, \tag{14}$$

- *there exists* $1 < p < \infty$ *such that* $K \in L^p([0, +\infty[)$.
Then,

$$y(t) \leq ke^{-\delta(t-t_0)}|y(t_0)| + r, \tag{15}$$

where k *and* r *are positive constants.*

Proof. We have,

$$y(t) \leq \exp(\int_{t_0}^t s(\lambda)d\lambda)y(t_0) + \int_{t_0}^t K(u)\exp(\int_u^t s(\lambda)d\lambda)du. \tag{16}$$

Case 1: when $T \geq t_0$, $\int_{t_0}^t s(\lambda)d\lambda = \int_{t_0}^T s(\lambda)d\lambda + \int_T^t s(\lambda)d\lambda$.
Let N be the integer for which

$$(N-1)l \leq t - T \leq Nl$$

$$\begin{aligned}
\int_{t_0}^t s(\lambda)d\lambda &= \int_{t_0}^T s(\lambda)d\lambda + \int_T^t s(\lambda)d\lambda \\
&= \int_{t_0}^T s(\lambda)d\lambda + \sum_{i=0}^{N-2} \int_{T+il}^{T+(i+1)l} s(\lambda)d\lambda + \int_{T+(N-1)l}^t s(\lambda)d\lambda \\
&\leq \int_{t_0}^T s(\lambda)d\lambda - (N-1)\delta l + \int_{T+(N-1)l}^t s(\lambda)d\lambda \\
&\leq M_1 - (N-1)\delta l + M_2 \\
&\leq -\delta(t-t_0) + M.
\end{aligned}$$

Case 2: when $T \leq t_0$,
Let N be the integer for which

$$(N-1)l \leq t - t_0 \leq Nl$$

$$\begin{aligned}
\int_{t_0}^t s(\lambda)d\lambda &= \sum_{i=0}^{N-2} \int_{t_0+il}^{t_0+(i+1)l} s(\lambda)d\lambda + \int_{t_0+(N-1)l}^t s(\lambda)d\lambda \\
&\leq -(N-1)\delta l + M_1^{`} \\
&\leq -\delta(t-t_0) + M^{`}.
\end{aligned}$$

Then, there exists a constant α such that,

$$\int_{t_0}^t s(\lambda)d\lambda \leq -\delta(t-t_0) + \alpha \ , \ \forall\, t_0 \geq 0, \quad \forall t \geq t_0. \tag{17}$$

Then, there exists a constant k such that

$$e^{\int_{t_0}^t s(\lambda)d\lambda} \leq ke^{-\delta(t-t_0)}, \quad \forall t \geq t_0. \tag{18}$$

Hence,

$$\begin{aligned} y(t) &\leq k e^{-\delta(t-t_0)}|y(t_0)| + k\int_{t_0}^{t} |K(u)|e^{-\delta(t-u)}du \\ &\leq k e^{-\delta(t-t_0)}|y(t_0)| + k(\int_{t_0}^{t} e^{-\delta q(t-u)}du)^{\frac{1}{q}}\|K\|_p \\ &\leq k e^{-\delta(t-t_0)}|y(t_0)| + \frac{\|K\|_p}{(\delta q)^{\frac{1}{q}}}. \end{aligned}$$

Then,

$$y(t) \leq k e^{-\delta(t-t_0)}|y(t_0)| + r, \tag{19}$$

where $r = \dfrac{\|K\|_p}{(\delta q)^{\frac{1}{q}}}$ with respect inequality (15). □

3 Stability of Lorenz equations

The Lorenz equations is one of the most famous models of nonlinear dynamics, which is a nonlinear system that evolves in $\mathbb{R}^3$ whose equations are given by:

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases} \tag{20}$$

where the parameters a, b and c are assumed positive real numbers. For considered assumption on parameters, if $c < 1$ then the system (20) has a unique equilibrium point $S_0(0,0,0)$ and if $c > 1$ then the system (20) has a three equilibrium point $S_0(0,0,0)$ and $q_{\pm} = (\pm\sqrt{b(c-1)}, \pm\sqrt{b(c-1)}, c-1)$.

The Lorenz system has played a fundamental role in the area of nonlinear science and chaotic dynamics. Therefore, the study of the stability and attractivity of the origin as an equilibrium point of the Lorenz system is theoretically significant, and also practically important. When $a = 10,\ b = 8/3,\ c = 28$, the system is chaotic, with the attractor as shown in Fig 1. Consequently, the authors in [11] proposed the following Lorenz family:

$$\begin{cases} \dot{x} = a_\alpha(y-x) \\ \dot{y} = c_\alpha x - xz - d_\alpha y \\ \dot{z} = xy - b_\alpha z \end{cases} \tag{21}$$

where

$$a_\alpha = 25\alpha + 10,\ b_\alpha = \frac{1}{3}(\alpha+8),\ c_\alpha = 28 - 35\alpha,\ d_\alpha = 1 - 29\alpha,$$

with

$$\alpha \in [0, 1/29[.$$

Consider the following Lyapunov function:

$$V_\lambda = \frac{1}{2}[\lambda x^2 + y^2 + (z - \lambda a_\alpha - c_\alpha)^2] \tag{22}$$

this function satisfies the following relation on the derivative with respect to system (21):

$$\begin{aligned} \frac{dV_\lambda}{dt} &= \lambda x\dot{x} + y\dot{y} + \dot{z}(z - \lambda a_\alpha - c_\alpha) \\ &= \lambda x(a_\alpha(y-x)) + y(c_\alpha x - xz - d_\alpha y) + (xy - b_\alpha z)(z - \lambda a_\alpha - c_\alpha) \\ &= \lambda a_\alpha xy - \lambda a_\alpha x^2 + c_\alpha xy - xzy - d_\alpha y^2 - b_\alpha z^2 + \lambda a_\alpha b_\alpha z + c_\alpha b_\alpha z + xyz - \lambda a_\alpha xy - c_\alpha xy \\ &= -\lambda a_\alpha x^2 - d_\alpha y^2 - b_\alpha z^2 + b_\alpha(\lambda a_\alpha + c_\alpha)z. \end{aligned}$$

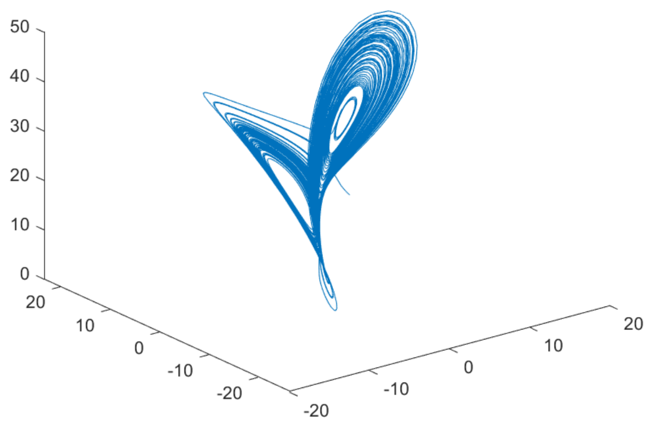

Figure 1: Phase portrait of Lorenz attractor

Then, using the facts that $a_\alpha > 1$ and $0 < d_\alpha \le 1$, we obtain

$$\begin{aligned}\frac{dV_\lambda}{dt} &\le -\lambda x^2 - d_\alpha y^2 - d_\alpha z^2 - (b_\alpha - d_\alpha)z^2 - d_\alpha(\lambda a_\alpha + c_\alpha)^2 \\ &\quad + 2d_\alpha(\lambda a_\alpha + c_\alpha)z + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z + d_\alpha(\lambda a_\alpha + c_\alpha)^2 \\ &= -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z + d_\alpha(\lambda a_\alpha + c_\alpha)^2 - (b_\alpha - d_\alpha)z^2 \\ &= -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + g(z) \end{aligned} \tag{23}$$

with

$$g(z) = -(b_\alpha - d_\alpha)z^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z.$$

Then, setting

$$g'(z) = -2(b_\alpha - d_\alpha)z + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)$$

zero, yields

$$z_0 = \frac{(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)}{2(b_\alpha - d_\alpha)}. \tag{24}$$

Since $b_\alpha > 2 > d_\alpha$, $0 < d_\alpha \le 1$, it follows that $z_0 > 0$ and $g''(z_0) = -2(b_\alpha - d_\alpha) < 0$. Thus,

$$\sup_{z\in\mathbb{R}} g(z) = g(z_0) = \frac{[(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)]^2}{4(b_\alpha - d_\alpha)}. \tag{25}$$

We obtain

$$\begin{aligned}
\frac{dV_\lambda}{dt} &= -\lambda a_\alpha x^2 - d_\alpha y^2 - b_\alpha z^2 + b_\alpha(\lambda a_\alpha + c_\alpha)z \\
&\le -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + \sup_{z\in\mathbb{R}} g(z) \\
&\le -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + \frac{[(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)]^2}{4(b_\alpha - d_\alpha)} \\
&\le -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)} \\
&\le -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + 2d_\alpha R_\alpha \\
&\le -2d_\alpha V_\lambda + 2d_\alpha R_\alpha. \qquad (26)
\end{aligned}$$

with

$$R_\alpha = \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{8(b_\alpha - d_\alpha)d_\alpha}$$

Thus,

$$\frac{dV_\lambda}{dt} \le 0 \text{ when } V_\lambda \ge R_\lambda.$$

By theorem 1 we get

$$\begin{aligned}
V_\lambda(X(t) &\le V_\lambda(X_0)e^{-2d_\alpha(t-t_0)} + \int_{t_0}^{t} e^{-2d_\alpha(t-\tau})2d_\alpha R_\lambda d_\tau \\
&= V_\lambda(X_0)e^{-2d_\alpha(t-t_0)} + R_\lambda(1 - e^{-2d_\alpha(t-t_0)}). \qquad (27)
\end{aligned}$$

So, if $V_\lambda(X(t)) > R_\lambda,\ t \ge t_0$, we have

$$V_\lambda(X(t)) \le (V_\lambda(X_0) - R_\lambda)e^{-2d_\lambda(t-t_0)} + R_\lambda. \qquad (28)$$

Example

$$\begin{cases} \dot{x} = a_\alpha(y - x) \\ \dot{y} = c_\alpha x - xz + d_\alpha y \\ \dot{z} = xy - b_\alpha z \end{cases} \qquad (29)$$

where $a(\alpha) = 25\alpha + 10,\ b(\alpha) = \frac{1}{3}(\alpha + 8),\ c(\alpha) = 28 - 35\alpha, d(\alpha) = 1 - 29\alpha$.
For $\alpha = \frac{1}{33}$.

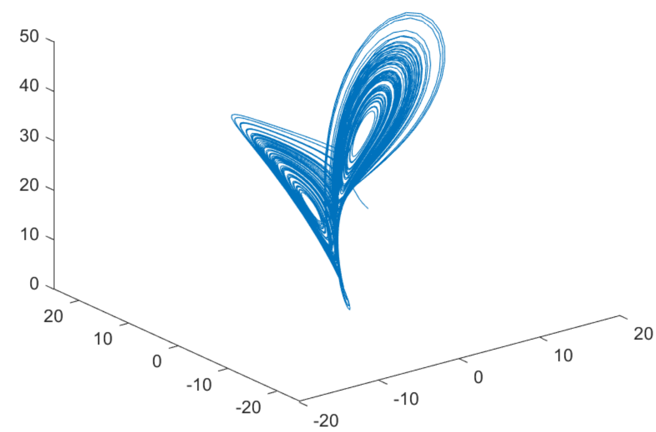

Figure 2: Phase portraits of the Lorenz family for $\alpha = \frac{1}{33}$

For $\alpha = \frac{1}{300}$.

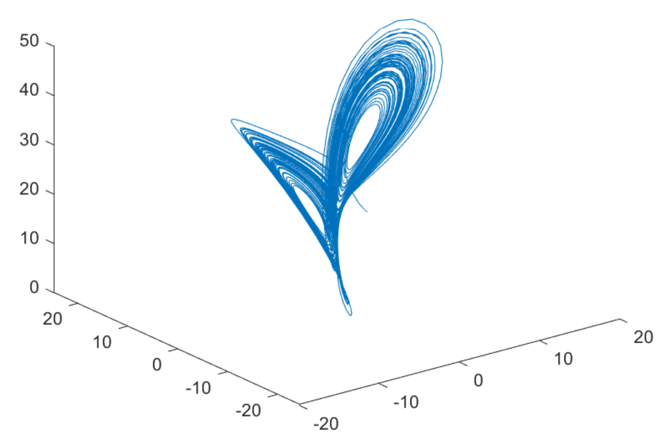

Figure 3: Phase portraits of the Lorenz family for $\alpha = \frac{1}{300}$

4 Conclusion

Exponential practical stability of a class of nonlinear time-varying perturbed systems by the Lyapunov method has been studied. New sufficient conditions for the practical exponential stability of nonlinear perturbed system were given.

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