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New Variants of the Steffensen Integral Inequality

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Abstract
In this paper, we present some new variants of the Steffensen integral inequality. Original assumptions are made, including a monotonicity condition of a transformation of the functions involved. Some of our variants have the property of depending on three adaptable functions. The theory is illustrated by several examples.

Keywords: Steffensen inequality, integral inequalities, monotonicity.

1 Introduction

Integral inequalities have always played a central role in mathematics. They aim to give tractable bounds on integrals that are difficult to determine. They form the basis of key results in functional analysis, probability theory, statistical theory, differential equations, optimization and numerical analysis. Classical examples include the Cauchy-Schwarz integral inequality, the Jensen integral inequality, the Hölder integral inequality, the Hilbert integral inequality, the Hardy integral inequality and the Steffensen integral inequality, each of which contributes to different goals. For more details on this classic topic, see [1], [2], [3], [4] and [5]. Let us focus here on the Steffensen integral inequality introduced by Johan Frederik Steffensen in [6]. It is presented in the proposition below.

Proposition 1.1 (Steffensen integral inequality, [6]). *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and $f, g : (a, b) \mapsto (0, +\infty)$ be two integrable functions, such that*

- *f is non-increasing,*
- *for any $t \in (a, b)$, we have*

$$0 \leq g(t) \leq 1.$$

Let us set

$$\lambda = \int_a^b g(t)dt.$$

Then we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g(t)dt.$$

This inequality has the ability to autocomplete itself with the following lower bound:

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt.$$

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In recent decades, the Steffensen integral inequality has been extended, generalized and modified in several ways. Significant progress on this topic can be found in [7], [8], [9], [10], [11] and [12]. See also the detailed monograph in [13].

This paper proposes a new approach to this inequality. It consists mainly in modifying the monotonicity assumption; instead of imposing it only on f, we make a monotonicity assumption on a special transformation of f and g. This, together with a different choice of λ, i.e.,

$$\lambda = \int_a^b \sqrt{g(t)}dt,$$

leads to a new integral inequality which can be seen as a variant of the Steffensen integral inequality. Examples and auxiliary results are given. Then, under different assumptions, a more general variant is established. It is characterized by the addition of a new adaptable function. Further results for integral inequalities are also derived. The theory is illustrated by numerous examples with specific functional configurations and numerical studies.

The rest of the paper consists of the following sections: Section 2 discusses our main integral inequality. Section 3 is devoted to its generalization. A conclusion is given in Section 4.

2 A variant of the Steffensen integral inequality

2.1 Main result

Our main variant of the Steffensen integral inequality is described in the proposition below.

Proposition 2.1 *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and $f, g : (a, b) \mapsto (0, +\infty)$ be two functions such that*

- $\sqrt{g}$ *is integrable,*
- *for any $t \in (a, b)$, we have $0 \leq g(t) \leq 1$,*
- $[1 + \sqrt{g}]f$ *is non-increasing.*

Let us set

$$\lambda = \int_a^b \sqrt{g(t)}dt.$$

Then we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b f(t)\sqrt{g(t)}dt.$$

Proof of Proposition 2.1. Note that, since $0 \leq g(t) \leq 1$ for any $t \in (a, b)$, we have $0 \leq \lambda = \int_a^b \sqrt{g(t)}dt \leq \int_a^b dt = b - a$. Using this and the Chasles integral relation, we can write

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} f(t)dt - \int_a^{a+\lambda} f(t)g(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} [1 - g(t)]f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} \left[1 - \sqrt{g(t)}\right]\left[1 + \sqrt{g(t)}\right] f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt. \qquad (1)
\end{aligned}$$

This last decomposition, and in particular the factorization carried out, is the key point of our approach. Using $0 \leq g(t) \leq 1$ for any $t \in (a, b)$, the fact that $[1 + \sqrt{g}]f$ is non-increasing, the definition of λ and the Chasles integral relation, we have

$$\begin{aligned}
&\int_a^{a+\lambda} \left[1 - \sqrt{g(t)}\right]\left[1 + \sqrt{g(t)}\right] f(t)dt \\
&\geq \left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) \int_a^{a+\lambda} \left[1 - \sqrt{g(t)}\right] dt \\
&= \left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) \left[\lambda - \int_a^{a+\lambda} \sqrt{g(t)}dt\right] \\
&= \left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) \left[\int_a^{b} \sqrt{g(t)}dt - \int_a^{a+\lambda} \sqrt{g(t)}dt\right] \\
&= \left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) \int_{a+\lambda}^{b} \sqrt{g(t)}dt. \qquad (2)
\end{aligned}$$

It follows from Equations (1) and (2), the Chasles integral relation and the fact that $[1+\sqrt{g}]f$ is non-increasing that

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^{b} f(t)g(t)dt \\
&\geq \left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) \int_{a+\lambda}^{b} \sqrt{g(t)}dt - \int_{a+\lambda}^{b} f(t)g(t)dt \\
&= \int_{a+\lambda}^{b} \left\{\left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) - f(t)\sqrt{g(t)}\right\} \sqrt{g(t)}dt \\
&= \int_{a+\lambda}^{b} \left\{\left[1 + \sqrt{g(a+\lambda)}\right] f(a+\lambda) - \left[1 + \sqrt{g(t)}\right] f(t)\right\} \sqrt{g(t)}dt + \int_{a+\lambda}^{b} f(t)\sqrt{g(t)}dt \\
&\geq \int_{a+\lambda}^{b} f(t)\sqrt{g(t)}dt.
\end{aligned}$$

We thus concludes that

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^{b} f(t)g(t)dt + \int_{a+\lambda}^{b} f(t)\sqrt{g(t)}dt,$$

which is the desired inequality. □

Remark 2.2 *In Proposition 2.1, if we assume that $[1 + \sqrt{g}]f$ is non-decreasing instead of non-increasing, we can show that the final inequality is reversed, i.e.,*

$$\int_a^{a+\lambda} f(t)dt \leq \int_a^{b} f(t)g(t)dt + \int_{a+\lambda}^{b} f(t)\sqrt{g(t)}dt.$$

The main difference between the Proposition 2.1 and the classical Steffensen integral inequality is the non-increasing assumption, which applies to $[1+\sqrt{g}]f$, not just to f. Also, the parameter λ is redefined, and a sum of two integral terms is the lower bound. This makes the result in Proposition 2.1 a novel integral inequality to the best of our knowledge.

As an immediate remark, in the exact framework of Proposition 2.1, we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^{b} f(t)g(t)dt + \int_{a+\lambda}^{b} f(t)\sqrt{g(t)}dt \geq \int_a^{b} f(t)g(t)dt,$$

which is the form of the Steffensen integral inequality, but with a different definition of λ and, more importantly, different assumptions on f and g. More refined results related to Proposition 2.1 are developed in Subsection 2.3.

2.2 Some examples

Let us now illustrate Proposition 2.1 with two examples which consider specific functions and lead to calculating various integrals.

- Let us consider $a = 0$, $b = 1$, $g(t) = 1 - t$, and

$$f(t) = \frac{1}{\sqrt{t}\left[1 + \sqrt{1-t}\right]}.$$

It is clear that $0 \leq g(t) = 1 - t \leq 1$ for any $t \in (0, 1)$, and so g is integrable on $(0, 1)$. Furthermore, f is non-monotonic because we have

$$\lim_{t \to 0} f(t) = +\infty, \quad f\left(\frac{1}{2}\right) = 2[\sqrt{2} - 1] \approx 0.828427, \quad f(1) = 1,$$

so that $+\infty > 0.828427 < 1$. Therefore, the standard Steffensen integral inequality can not be applied. However, keeping in mind the framework of Proposition 2.1, for any $t \in (0, 1)$, we have

$$[1 + \sqrt{g(t)}]f(t) = \left[1 + \sqrt{1-t}\right] \frac{1}{\sqrt{t}\left[1 + \sqrt{1-t}\right]} = \frac{1}{\sqrt{t}},$$

which is obviously non-increasing. The monotonicity assumption of Proposition 2.1 is thus satisfied. In order to apply this proposition, we calculate

$$\lambda = \int_a^b \sqrt{g(t)}dt = \int_0^1 \sqrt{1-t}dt = \frac{2}{3},$$

$$\int_a^{a+\lambda} f(t)dt = \int_0^{2/3} \frac{1}{\sqrt{t}\left[1 + \sqrt{1-t}\right]}dt = \sqrt{2} - \sqrt{6} + 2\arcsin\left[\sqrt{\frac{2}{3}}\right] \approx 0.8754,$$

$$\int_a^b f(t)g(t)dt = \int_0^1 \frac{1}{\sqrt{t}\left[1 + \sqrt{1-t}\right]}(1-t)dt = \frac{3\pi}{2} - 4 \approx 0.7124$$

and

$$\int_{a+\lambda}^b f(t)\sqrt{g(t)}dt = \int_{2/3}^1 \frac{1}{\sqrt{t}\left[1 + \sqrt{1-t}\right]}\sqrt{1-t}dt$$
$$= 4 - 5\sqrt{\frac{2}{3}} + \sqrt{2} - 2\arctan\left[\frac{1}{\sqrt{2}}\right] \approx 0.1008.$$

We clearly have

$$0.8754 \geq 0.7124 + 0.1008 = 0.8132,$$

which illustrates the inequality in Proposition 2.1.

- As another example, let us consider $a = 0$, $b = 1$, $g(t) = \sin(\pi t)$, and

$$f(t) = \frac{1}{(1+t)\left[1 + \sqrt{\sin(\pi t)}\right]}.$$

It is clear that $0 \leq g(t) = \sin(\pi t) \leq 1$ for any $t \in (0, 1)$, and so g is integrable on $(0, 1)$. Furthermore, f is non-monotonic because we have

$$\lim_{t\to 0} f(t) = 1, \quad f\left(\frac{1}{2}\right) = \frac{1}{3}, \quad f(1) = \frac{1}{2},$$

so that $1 > 1/3 < 1/2$. Therefore, the standard Steffensen inequality is not directly applicable. However, for any $t \in (0, 1)$, we have

$$[1 + \sqrt{g(t)}]f(t) = \left[1 + \sqrt{\sin(\pi t)}\right] \frac{1}{(1+t)\left[1 + \sqrt{\sin(\pi t)}\right]} = \frac{1}{1+t},$$

which is obviously non-increasing. This is an assumption of Proposition 2.1. In order to apply this proposition, we calculate

$$\lambda = \int_a^b \sqrt{g(t)}dt = \int_0^1 \sqrt{\sin(\pi t)}dt = \frac{2\sqrt{2}}{\pi^{3/2}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2 \approx 0.76276,$$

where $\Gamma(x)$ is the standard gamma function defined by $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$,

$$\int_a^{a+\lambda} f(t)dt \approx \int_0^{0.76276} \frac{1}{(1+t)\left[1 + \sqrt{\sin(\pi t)}\right]}dt \approx 0.322034,$$

$$\int_a^b f(t)g(t)dt = \int_0^1 \frac{1}{(1+t)\left[1 + \sqrt{\sin(\pi t)}\right]}\sin(\pi t)dt \approx 0.233646$$

and

$$\int_{a+\lambda}^b f(t)\sqrt{g(t)}dt \approx \int_{0.76276}^1 \frac{1}{(1+t)\left[1 + \sqrt{\sin(\pi t)}\right]}\sqrt{\sin(\pi t)}dt$$
$$\approx 0.0445287.$$

We clearly have

$$0.322034 \geq 0.233646 + 0.0445287 = 0.2781747,$$

which illustrates the inequality in Proposition 2.1.

Other examples of various kinds can be presented similarly.

2.3 Complementary results

We now establish some additional results which, in a sense, complete Proposition 2.1.

The proposition below is about an inequality involving the integrals of f and fg over (a, b).

Proposition 2.3 *In the exact framework of Proposition 2.1, we have*

$$\int_a^b f(t)dt \geq \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b [1 + \sqrt{g(t)}]f(t)dt.$$

If a and b are finite, we can possibly lower bound the last term as follows:

$$\int_{a+\lambda}^b [1 + f(t)]\sqrt{g(t)}dt \geq (b - a - \lambda)[1 + \sqrt{g(b)}]f(b).$$

Proof of Proposition 2.3. Thanks to Proposition 2.1, we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b f(t)\sqrt{g(t)}dt.$$

If we add the integral term $\int_{a+\lambda}^b f(t)dt$ on both sides, by the Chasles integral relation, we obtain

$$\begin{aligned}\int_a^b f(t)dt &= \int_a^{a+\lambda} f(t)dt + \int_{a+\lambda}^b f(t)dt \\ &\geq \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b f(t)\sqrt{g(t)}dt + \int_{a+\lambda}^b f(t)dt \\ &= \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b [1+\sqrt{g(t)}]f(t)dt.\end{aligned}$$

The first result is proved. For the last term, since $[1+\sqrt{g}]f$ is non-increasing, it is immediate that

$$\int_{a+\lambda}^b [1+\sqrt{g(t)}]f(t)dt \geq [1+\sqrt{g(b)}]f(b)\int_{a+\lambda}^b dt = (b-a-\lambda)[1+\sqrt{g(b)}]f(b).$$

The second result is established, ending the proof. □

Remark 2.4 *In Proposition 2.3, if we assume that $[1+\sqrt{g}]f$ is non-decreasing instead of non-increasing, we can show that the main inequalities are reversed, i.e.,*

$$\int_a^b f(t)dt \leq \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b [1+\sqrt{g(t)}]f(t)dt$$

and, if a and b are finite,

$$\int_{a+\lambda}^b [1+f(t)]\sqrt{g(t)}dt \leq (b-a-\lambda)[1+\sqrt{g(b)}]f(b).$$

In the result below, with a new definition of λ and under different assumptions, a variant of Proposition 2.1 is proposed.

Proposition 2.5 *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a<b$, and $f,g:(a,b)\mapsto(0,+\infty)$ be two functions such that*

- *$\sqrt{1-g}$ is integrable,*
- *for any $t\in(a,b)$, we have $0\leq g(t)\leq 1$,*
- *$[1+\sqrt{1-g}]f$ is non-increasing.*

Let us set

$$\lambda = \int_a^b \sqrt{1-g(t)}dt.$$

Then we have

$$\int_a^b f(t)g(t)dt \geq \int_{a+\lambda}^b f(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt.$$

Proof of Proposition 2.5. The proof is based on Proposition 2.1 under a particular configuration. Let us define $g_\star:(a,b)\mapsto(0,+\infty)$ by $g_\star = 1-g$. Then the assumptions of Proposition 2.1 are satisfied for f and $g_\star$ instead of g. Noticing that

$$\lambda = \int_a^b \sqrt{1-g(t)}dt = \int_a^b \sqrt{g_\star(t)}dt,$$

the main integral inequality can be rewritten as

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt &\geq \int_a^b f(t)g_\star(t)dt + \int_{a+\lambda}^b f(t)\sqrt{g_\star(t)}dt \\ &= \int_a^b f(t)[1-g(t)]dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt \\ &= \int_a^b f(t)dt - \int_a^b f(t)g(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt.\end{aligned}$$

Therefore, according to the Chasles integral relation, we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &\geq \int_a^b f(t)dt - \int_a^{a+\lambda} f(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt \\ &= \int_{a+\lambda}^b f(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt.\end{aligned}$$

This concludes the proof. □

Remark 2.6 *In Proposition 2.5, if we assume that $[1+\sqrt{1-g}]f$ is non-decreasing instead of non-increasing, we can show that the final inequality is reversed, i.e.,*

$$\int_a^b f(t)g(t)dt \leq \int_{a+\lambda}^b f(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt.$$

In the exact framework of Proposition 2.5, we also have

$$\int_a^b f(t)g(t)dt \geq \int_{a+\lambda}^b f(t)dt + \int_{a+\lambda}^b f(t)\sqrt{1-g(t)}dt \geq \int_{a+\lambda}^b f(t)dt,$$

which can be seen as a reversed variant of the Steffensen integral inequality.

The proposition below considers the definition of the parameter λ in the standard Steffensen integral inequality and offers an alternative inequality. The proof actually adapts the result in Proposition 2.1.

Proposition 2.7 *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a<b$, and $f,g : (a,b) \mapsto (0,+\infty)$ be two functions such that*

- *g is integrable,*
- *for any $t \in (a,b)$, we have $0 \leq g(t) \leq 1$,*
- *$(1+g)f$ is non-increasing.*

Let us set

$$\lambda = \int_a^b g(t)dt.$$

Then we have

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g^2(t)dt + \int_{a+\lambda}^b f(t)g(t)dt.$$

Proof of Proposition 2.7. The proof is based on Proposition 2.1 under a particular configuration. Let us define $g_\dagger : (a,b) \mapsto (0,+\infty)$ by $g_\dagger = g^2$. Then the assumptions of Proposition 2.1 are satisfied for f and $g_\dagger$ instead of g. Noticing that

$$\lambda = \int_a^b g(t)dt = \int_a^b \sqrt{g_\dagger(t)}dt,$$

the main integral inequality can be rewritten as

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g_\dagger(t)dt + \int_{a+\lambda}^b f(t)\sqrt{g_\dagger(t)}dt,$$

so that

$$\int_a^{a+\lambda} f(t)dt \geq \int_a^b f(t)g^2(t)dt + \int_{a+\lambda}^b f(t)g(t)dt.$$

This completes the proof. □

Remark 2.8 *In Proposition 2.7, if we assume that $[1+g]f$ is non-decreasing instead of non-increasing, we can show that the final inequality is reversed, i.e.,*

$$\int_a^{a+\lambda} f(t)dt \leq \int_a^b f(t)g^2(t)dt + \int_{a+\lambda}^b f(t)g(t)dt.$$

Proposition 2.7 thus shows that, under integrability and monotonicity assumptions involving some interaction between f and g, by considering $\lambda = \int_a^b g(t)dt$, we can obtain a new lower bound for $\int_a^{a+\lambda} f(t)dt$. This provides an alternative inequality to the Steffensen integral inequality.

3 Generalization

The generalization of some of the above results is discussed in this section.

3.1 A general result

The proposition below can be seen as a general version of Proposition 2.1, with the addition of an intermediate function h.

Proposition 3.1 *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and $f, g, h : (a,b) \mapsto (0,+\infty)$ be three functions such that*

- *$\sqrt{g}$ is integrable,*
- *by setting*

  $$\lambda = \int_a^b \sqrt{g(t)}dt,$$

  for any $t \in (a,b)$, we have

  $$0 \leq \lambda^2 g(t) \leq h(t),$$

- *$[\sqrt{h} + \lambda\sqrt{g}]f$ is non-increasing.*

Let us consider the real number $\theta \in (0, b-a)$ such that

$$\int_a^{a+\theta} \sqrt{h(t)}dt = \lambda^2.$$

Then we have

$$\int_a^{a+\theta} f(t)h(t)dt \geq \lambda\left[\lambda \int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)}f(t)dt\right].$$

Proof of Proposition 3.1. Since $\theta \in (0, b-a)$, the Chasles integral relation gives

$$\begin{aligned}
&\int_a^{a+\theta} f(t)h(t)dt - \lambda^2 \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\theta} f(t)h(t)dt - \lambda^2 \int_a^{a+\theta} f(t)g(t)dt - \lambda^2 \int_{a+\theta}^b f(t)g(t)dt \\
&= \int_a^{a+\theta} [h(t) - \lambda^2 g(t)] f(t)dt - \lambda^2 \int_{a+\theta}^b f(t)g(t)dt \\
&= \int_a^{a+\theta} \left[\sqrt{h(t)} - \lambda\sqrt{g(t)}\right]\left[\sqrt{h(t)} + \lambda\sqrt{g(t)}\right] f(t)dt - \lambda^2 \int_{a+\theta}^b f(t)g(t)dt. \qquad (3)
\end{aligned}$$

Using $0 \le \lambda^2 g(t) \le h(t)$ for any $t \in (a, b)$, which implies that $\sqrt{h(t)} - \lambda\sqrt{g(t)} \ge 0$, the fact that $[\sqrt{h} + \lambda\sqrt{g}]f$ is non-increasing, the Chasles integral relation and the definitions of λ and θ, we have

$$\begin{aligned}
&\int_a^{a+\theta} \left[\sqrt{h(t)} - \lambda\sqrt{g(t)}\right]\left[\sqrt{h(t)} + \lambda\sqrt{g(t)}\right] f(t)dt \\
&\ge \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \int_a^{a+\theta} \left[\sqrt{h(t)} - \lambda\sqrt{g(t)}\right] dt \\
&= \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \left[\int_a^{a+\theta} \sqrt{h(t)}dt - \lambda \int_a^{a+\theta} \sqrt{g(t)}dt\right] \\
&= \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \times \\
&\left[\int_a^{a+\theta} \sqrt{h(t)}dt - \lambda \int_a^b \sqrt{g(t)}dt + \lambda \int_{a+\theta}^b \sqrt{g(t)}dt\right] \\
&= \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \left\{\left[\int_a^{a+\theta} \sqrt{h(t)}dt - \lambda^2\right] + \lambda \int_{a+\theta}^b \sqrt{g(t)}dt\right\} \\
&= \lambda \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \int_{a+\theta}^b \sqrt{g(t)}dt. \qquad (4)
\end{aligned}$$

It follows from Equations (3) and (4), the Chasles integral relation and the fact that $[\sqrt{h} + \lambda\sqrt{g}]f$ is non-increasing that

$$\begin{aligned}
&\int_a^{a+\theta} f(t)h(t)dt - \lambda^2 \int_a^b f(t)g(t)dt \\
&\ge \lambda \left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) \int_{a+\theta}^b \sqrt{g(t)}dt - \lambda^2 \int_{a+\theta}^b f(t)g(t)dt \\
&= \lambda \int_{a+\theta}^b \left\{\left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) - \lambda f(t)\sqrt{g(t)}\right\} \sqrt{g(t)}dt \\
&= \lambda \int_{a+\theta}^b \left\{\left[\sqrt{h(a+\theta)} + \lambda\sqrt{g(a+\theta)}\right] f(a+\theta) - \left[\sqrt{h(t)} + \lambda\sqrt{g(t)}\right] f(t)\right\} \sqrt{g(t)}dt \\
&+ \lambda \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)} f(t)dt \\
&\ge \lambda \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)} f(t)dt.
\end{aligned}$$

We thus obtain

$$\int_a^{a+\theta} f(t)h(t)dt \ge \lambda \left[\lambda \int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)} f(t)dt\right],$$

which is the desired result. □

Remark 3.2 *In Proposition 3.1, if we assume that* $[\sqrt{h}+\lambda\sqrt{g}]f$ *is non-decreasing instead of non-increasing, we can show that the final inequality is reversed, i.e.,*

$$\int_a^{a+\theta} f(t)h(t)dt \le \lambda\left[\lambda\int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)}f(t)dt\right].$$

If we take $h(t) = \lambda^2$ in Proposition 3.1, we get $\theta = \lambda$, and the assumptions and result are the same as in Proposition 2.1. Proposition 3.1 is thus a generalization of Proposition 2.1 in this sense.

3.2 Complementary general results

The proposition below presents an inequality involving the integrals of fh and fg over (a, b).

Proposition 3.3 *In the exact framework of Proposition 3.1, we have*

$$\int_a^b f(t)h(t)dt \ge \lambda^2\int_a^b f(t)g(t)dt + \int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt.$$

If $\int_{a+\theta}^b \sqrt{h(t)}dt$ *is finite, we can possibly lower bound the last term as follows:*

$$\int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt \ge [\sqrt{h(b)}+\lambda\sqrt{g(b)}]f(b)\int_{a+\theta}^b \sqrt{h(t)}dt.$$

Proof of Proposition 3.3. Thanks to Proposition 3.1, we get

$$\int_a^{a+\theta} f(t)h(t)dt \ge \lambda\left[\lambda\int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)}f(t)dt\right].$$

If we add the integral term $\int_{a+\theta}^b f(t)h(t)dt$ on both sides, by the Chasles integral relation, we obtain

$$\begin{aligned}\int_a^b f(t)h(t)dt &= \int_a^{a+\theta} f(t)h(t)dt + \int_{a+\theta}^b f(t)h(t)dt\\ &\ge \lambda\left[\lambda\int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g(t)}f(t)dt\right] + \int_{a+\theta}^b f(t)h(t)dt\\ &= \lambda^2\int_a^b f(t)g(t)dt + \int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt.\end{aligned}$$

The first result is proved. For the last term, since $[\sqrt{h}+\sqrt{g}]f$ is non-increasing, we have

$$\int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt \ge [\sqrt{h(b)}+\lambda\sqrt{g(b)}]f(b)\int_{a+\theta}^b \sqrt{h(t)}dt.$$

The second result is obtained, ending the proof. □

Remark 3.4 *In Proposition 3.3, if we assume that* $[\sqrt{h}+\lambda\sqrt{g}]f$ *is non-decreasing instead of non-increasing, we can show that the main inequalities are reversed, i.e.,*

$$\int_a^b f(t)h(t)dt \le \lambda^2\int_a^b f(t)g(t)dt + \int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt$$

and, if $\int_{a+\theta}^b \sqrt{h(t)}dt$ *is finite,*

$$\int_{a+\theta}^b [\sqrt{h(t)}+\lambda\sqrt{g(t)}]f(t)\sqrt{h(t)}dt \le [\sqrt{h(b)}+\lambda\sqrt{g(b)}]f(b)\int_{a+\theta}^b \sqrt{h(t)}dt.$$

If we take $h(t) = \lambda^2$, Proposition 3.3 becomes Proposition 2.3.

In the result below, with a new definition of λ and under different assumptions, a variant of Proposition 3.1 is proposed.

Proposition 3.5 *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, and $f, g, h : (a, b) \mapsto (0, +\infty)$ be three functions such that*

- *$\sqrt{1-g}$ is integrable,*
- *by setting*

$$\lambda = \int_a^b \sqrt{1-g(t)}dt,$$

for any $t \in (a, b)$, we have

$$0 \leq \lambda^2[1-g(t)] \leq h(t),$$

- *$[\sqrt{h} + \lambda\sqrt{1-g}]f$ is non-increasing.*

Let us consider the real number $\theta \in (0, b-a)$ such that

$$\int_a^{a+\theta} \sqrt{h(t)}dt = \lambda^2.$$

Then we have

$$\int_a^{a+\theta} f(t)h(t)dt + \lambda^2 \int_a^b f(t)g(t)dt \geq \lambda \left[\lambda \int_a^b f(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{1-g(t)}f(t)dt\right].$$

Proof of Proposition 3.5. The proof is based on Proposition 3.1 under a particular configuration. Let us define $g_\star : (a, b) \mapsto (0, +\infty)$ by $g_\star = 1 - g$. Then the assumptions of Proposition 3.1 are satisfied for f and $g_\star$ instead of g. Noticing that

$$\lambda = \int_a^b \sqrt{1-g(t)}dt = \int_a^b \sqrt{g_\star(t)}dt,$$

the main integral inequality can be rewritten as

$$\int_a^{a+\theta} f(t)h(t)dt \geq \lambda \left[\lambda \int_a^b f(t)g_\star(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{g_\star(t)}f(t)dt\right],$$

so that

$$\int_a^{a+\theta} f(t)h(t)dt \geq \lambda \left[\lambda \int_a^b f(t)[1-g(t)]dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{1-g(t)}f(t)dt\right]$$

and

$$\int_a^{a+\theta} f(t)h(t)dt \geq \lambda \left[\lambda \int_a^b f(t)dt - \lambda \int_a^b f(t)g(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{1-g(t)}f(t)dt\right].$$

We can transform this inequality as

$$\int_a^{a+\theta} f(t)h(t)dt + \lambda^2 \int_a^b f(t)g(t)dt \geq \lambda \left[\lambda \int_a^b f(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{1-g(t)}f(t)dt\right].$$

This ends the proof. □

Remark 3.6 *In Proposition 3.5, if we assume that* $[\sqrt{h}+\lambda\sqrt{1-g}]f$ *is non-decreasing instead of non-increasing, we can show that the final inequality is reversed, i.e.,*

$$\int_a^{a+\theta} f(t)h(t)dt + \lambda^2 \int_a^b f(t)g(t)dt \leq \lambda \left[\lambda \int_a^b f(t)dt + \int_{a+\theta}^b \sqrt{h(t)}\sqrt{1-g(t)}f(t)dt\right].$$

In a sense, Proposition 3.5 is an adaptation of Proposition 3.1. Other adaptations of this last proposition are possible. For example, following the spirit of what Proposition 2.7 is to Proposition 2.1, we can think of considering the functions $g_\dagger = g^2$ and $h_\dagger = h^2$ instead of g and h, respectively. We do not develop these ideas further.

4 Conclusion

In this paper, we have established new integral inequalities, which can be seen as variants of the famous Steffensen integral inequality. Different assumptions are made on the functions involved, resulting in original lower and upper bounds. Several examples are given. Generalizations involving additional functions are also presented. Applications can be found in mathematical analysis and various branches of applied sciences. A possible future work is the consideration of multivariate integrals and how the current assumptions can be adapted to this scenario.

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