ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2025) 2025:10

A Collection of New and Flexible Modifications of the Hardy-Hilbert Integral Inequality

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Abstract
The Hardy-Hilbert integral inequality is a classic result in mathematics. It has been widely studied, leading to important developments in functional analysis, including operator theory. In this article, we advance the field by establishing new modifications of this inequality based on parametric power and power-logarithmic functions. The constants in the factor are optimized while the inequality remains sufficiently tractable for further mathematical manipulation. Several existing results are then extended with more versatile bounds. The theoretical framework includes special functions, such as the gamma function, the Lerch transcendent function, and the two-parameter Mittag-Leffler function. Some mixed approaches based on our key results are also examined. The detailed proofs are given and special cases are emphasized. Overall, this article can be summarized by its title: it provides a comprehensive collection of new and flexible modifications of the Hardy-Hilbert integral inequality with potential applications in mathematical analysis and related areas.

Keywords: Integral inequalities; optimal constants; gamma function; Lerch transcendent function; two-parameter Mittag-Leffler function.

1 Introduction

Some integral inequalities are considered classic results in mathematics because of their fundamental role in analysis. One such inequality is the Hardy-Hilbert integral inequality, introduced by Hardy over a thousand years ago in [15]. It gives an upper bound on the weighted double integral of the product of two non-negative functions. The bound is given as the product of a certain constant, called the constant in the factor, and the unweighted integral norms of the two main functions. The Hardy-Hilbert integral inequality is stated formally below.

Hardy-Hilbert integral inequality. Let $p > 1$, $q = p/(p-1)$ be the Hölder conjugate of p, i.e., such that $1/p+1/q = 1$, and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals involved in the upper bound converge. These integrals are the main components of the unweighted L_p and L_q norms of f and g, respectively. The constant in the factor, i.e., $\pi/\sin(\pi/p)$, has the property of being ratio-trigonometric in nature and dependent on p. Furthermore, it is optimal in the following sense: if we consider a smaller constant, we can find functions f and g for which the inequality no

Received: 2025-04-15

longer holds. This optimality is a crucial property in the theory of integral inequalities. It is often established using extremal functions or by analyzing cases where the inequality becomes an equality.

When $p = 2$, the Hardy-Hilbert integral inequality is reduced to the classical Hilbert integral inequality. It thus has a simpler form, as expressed formally below.
Hilbert integral inequality. Let $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \leq \pi\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy},$$

provided that the two integrals involved in the upper bound converge. These integrals are thus the main components of the unweighted integral L_2 norms of f and g, respectively. The constant in the factor reduces to π, preserving its optimality in this context. The Hilbert integral inequality is therefore more fundamental than the Hardy-Hilbert integral inequality, but better suited to certain scenarios involving L_2 norms. This is especially relevant for error analysis with various L_2 error benchmarks, approximation theory and related applications.

These two classical integral inequalities have been studied extensively over time. Research efforts have extended their applicability to various weighted, fractional and dynamic settings, explored different norm conditions and developed multidimensional versions. The literature on this topic is very large. Notable "theory oriented" contributions can be found in [1, 2, 4–11, 13, 16–34, 36, 37], with additional results and historical perspectives provided in the book [35].

To highlight the significance of this article, we first introduce two key modifications of the Hardy-Hilbert integral inequality established in [12]. They are expressed in turn below.
First key modification of the Hardy-Hilbert integral inequality in [12, Theorem 2.2]. Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}f(x)g(y)dxdy \leq \frac{1}{\alpha}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals involved in the upper bound converge. The constant in the factor, i.e., $1/\alpha$, is optimal.
Second key modification of the Hardy-Hilbert integral inequality in [12, Theorem 3.1]. Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}f(x)g(y)dxdy \leq \frac{1}{\alpha}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

provided that the two integrals involved in the upper bound converge. The constant in the factor, i.e., $1/\alpha$, is optimal. This second modification can be seen as a product version of the first. We say "product" because the sum or difference of the variables is transformed into the sum or difference of a constant and the product of the variables, i.e., $x - y$ becomes $1 - xy$ and $x + y$ becomes $1 + xy$.

Compared with the existing results, these modifications present three remarkable features. The first is the originality of the integrand in the double integral term, which depends on an adjustable parameter α, with the simple condition $\alpha > 0$. The second feature is the moderate complexity of this integrand. In particular, focusing on the first main result, we can write

$$\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}} = \frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^{\alpha}$$

and the triangle inequality ensures that $|x-y|/(x+y) \in (0, 1)$ for $x > 0$ and $y > 0$. The power form combined with this interval property makes the integrand suitable for use in power series expansions and various integral transformations with respect to α. The final but important feature is the simplicity of the constant in the factor, i.e., $1/\alpha$, which can be manipulated using the same analytical tools. This flexibility is exploited in [12] through various approaches, leading to other new modifications of the Hardy-Hilbert integral inequality. In addition, numerous illustrative examples are given to support the theory.

In this article, we extend the main results in [12] by incorporating a one-parameter power-logarithmic term in the integrands of the main double integral terms, thus increasing their flexibility and applicability. More precisely, our first contributions are centered on bounding the following double integral term:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy,$$

where β is a new parameter that governs the introduced power-logarithmic term. This leads to more general constants in the factor defined by the gamma function. Their optimality is rigorously established. Inspired by techniques of [12], we derive other new modifications of the Hardy-Hilbert integral inequality, which are innovative by the functional originality of the integrands and also by the versatility of the associated constants in the factor. In total, twenty eights theorems are formulated. Some of them involve the primitive of the main functions, in accordance with the classical Hardy integral inequality (see [16]). The theoretical framework integrates special functions, including the gamma function, the Lerch transcendent function (see [3]), sometimes reduced to the Riemann zeta function, and the two-parameter Mittag-Leffler function (see [14]). Several examples illustrate the results, and all proofs are given in full detail. Overall, this article offers a collection of new modifications of the Hardy-Hilbert integral inequality. It contributes to mathematical analysis and its applications in related fields.

The rest of the article is organized as follows: Section 2 introduces the first modification of the Hardy-Hilbert integral inequality, along with related results and examples. Section 3 examines natural product modifications of these inequalities. The detailed proofs are given in Section 4. Finally, Section 5 concludes the article with remarks and future research directions.

2 First contributions

Our first contributions to the topic of the Hardy-Hilbert integral inequality are presented in this section.

2.1 Main theorem

The theorem below proposes a new modification of the Hardy-Hilbert integral inequality that has the originality of depending on two adjustable parameters and a power-logarithmic term.

Theorem 2.1 *Let $p>1$, $q=p/(p-1)$, $\alpha>0$, $\beta>0$ and $f,g:[0,+\infty)\mapsto[0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function defined by

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx,$$

with $a>0$, provided that the two integrals involved in the upper bound converge. Furthermore, the constant in the factor, i.e., $\Gamma(\beta)/\alpha^{\beta}$, is optimal.

The proof, which is deferred in Section 4 (as are all the proofs), relies on a suitable decomposition of the integrand, the Hölder integral inequality and an auxiliary proposition of a new integral result.

If we set $\beta = 1$, then Theorem 2.1 gives

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}} f(x)g(y)dxdy \le \frac{1}{\alpha} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

which corresponds to [12, Theorem 2.2]. Therefore, Theorem 2.1 can be seen as a power-logarithmic generalization of this result. As far as we know, it is new in the literature. The constant in the factor is also remarkable, since the gamma function plays a crucial role and does not depend on the parameter p.

Two other special cases of interest are presented below.

- If we set $\alpha = 1$, then we have

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{1}{(x+y)^2} \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\le \Gamma(\beta) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

  The constant in the factor is thus reduced to $\Gamma(\beta)$, and it is optimal in this context.

- If we set $\beta = m + 1/2$, where $m \in \mathbb{N}$, then $\Gamma(\beta) = \Gamma(m + 1/2) = [(2m)!/(m!4^m)]\sqrt{\pi}$, and we have

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}} \log^{m-1/2}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\le \frac{(2m)!}{\alpha^{m+1/2} m! 4^m} \sqrt{\pi} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{1}$$

  The constant in the factor is still optimal, and it is interesting to see how $\sqrt{\pi}$ naturally emerges in its definition.

Another important aspect of Theorem 2.1 is its degree of applicability. This is due to the fact that the parameters α and β modulate different components of the integrand, while the constant in the factor has a simple form. This makes the inequality a valuable intermediate result that can facilitate the derivation of new modifications of the Hardy-Hilbert integral inequality. This idea is explored in the subsection below from different theoretical perspectives.

As a more secondary remark on Theorem 2.1, using the decompositions $1 = (x+y)^{\alpha-1}/(x+y)^{\alpha-1}$ and $x^2 - y^2 = (x-y)(x+y)$ with $x > 0$ and $y > 0$, the main inequality can be written as follows:

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x^2-y^2|^{\alpha-1}}{(x+y)^{2\alpha}} \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\le \frac{1}{\alpha^\beta} \Gamma(\beta) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The result below proposes an alternative formulation of Theorem 2.1, more related to the L_p norm and using only one function.

Theorem 2.2 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$ and $f : [0, +\infty) \mapsto [0, +\infty)$ be a function. Then the inequality in Theorem 2.1 is equivalent to*

$$\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}} \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)dx\right]^p dy \le \frac{1}{\alpha^{\beta p}} \Gamma^p(\beta) \int_0^{+\infty} f^p(x)dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

The proof uses a suitable decomposition of the integrand and the Hölder integral inequality.

2.2 Primitive versions

The theorem below can be seen as a primitive version of Theorem 2.1. We emphasize the use of the unweighted L_p and L_q norms of f and g, respectively, not their primitives.

Theorem 2.3 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0, \beta > 0$, $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions and $F, G : [0, +\infty) \mapsto [0, +\infty)$ be defined their respective primitives given by*

$$F(x) = \int_0^x f(r)dr, \quad G(y) = \int_0^y g(r)dr,$$

provided that they exist. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)G(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 2.1, which serves as an intermediate result, and the Hardy integral inequality. It is important to note that the two constants in the factor of these results are combined in order to have the sharpest possible constant for the new inequality. The obtained upper bound still depends on the L_p and L_q norms of f and g, and the constant in the factor now depends on p.

On the same mathematical basis, a primitive version of Theorem 2.2 is formulated below.

Theorem 2.4 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $f : [0, +\infty) \mapsto [0, +\infty)$ be a function and $F : [0, +\infty) \mapsto [0, +\infty)$ be its primitive given by*

$$F(x) = \int_0^x f(r)dr,$$

provided that it exists. Then the following inequality holds:

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)dx\right]^p dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

The proof uses Theorem 2.2 and the Hardy integral inequality.

2.3 New modifications

In the result below, a sophisticated two-parameter modification of Theorem 2.1 is presented.

Theorem 2.5 *Let $p > 1$, $q = p/(p-1)$, $\beta > 1$, $\delta > 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 2.1 and uses an integration method with respect to the parameter α. Note that

$$\lim_{\beta \to 1} \frac{1}{\beta - 1}(1 - \delta^{1-\beta})\Gamma(\beta) = \log(\delta),$$

which corresponds to the constant in the factor obtained in [12, Proposition 2.9].

As an example, if we set $\delta = 2$, noticing that $x + y - |x - y| = 2\min(x, y)$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{\min(x,y)}{(x+y)^3}\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\ \leq \frac{1}{2(\beta-1)}(1-2^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

As a secondary remark, note that the main inequality of the theorem can be written as follows:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\ \leq \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \\ + \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy.$$

A three-parameter modification of Theorem 2.1 is described below. The constant in the factor has the property of being dependent on a special function: the Lerch transcendent function.

Theorem 2.6 *Let $p > 1$, $q = p/(p-1)$, $\beta > 1$, $\epsilon > 0$, $\gamma \in (0, 1]$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon-1}(x+y)^{\epsilon-1}}{[(x+y)^\epsilon - \gamma|x-y|^\epsilon]^2}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\ \leq \frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma, \beta-1, 1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function defined by

$$\Phi(a, b, c) = \sum_{i=0}^{+\infty}\frac{a^i}{(i+c)^b},$$

with $a \in (-1, 1)$, $b \in \mathbb{R}$ and $c > 0$, or $|a| = 1$, $b > 1$ and $c > 0$, provided that the two integrals involved in the upper bound converge.

The proof uses Theorem 2.1 and power series expansion method. The flexibility in the choice of the parameter α is exploited.

Note that

$$\Phi(\gamma, \beta-1, 1) = \sum_{i=0}^{+\infty}\frac{\gamma^i}{(i+1)^{\beta-1}}.$$

This also corresponds to the polylogarithm function at γ and $\beta - 1$ divided by γ and, for $\gamma = 1$, to the Riemann zeta function at $\beta - 1$ (see [3]).

In particular, if we set $\gamma = 1$ and $\epsilon = 1$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(x,y)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\ \leq 4\Gamma(\beta)\Phi(1, \beta-1, 1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Some special cases of this inequality are detailed below.

- If we set $\beta = 3$, then $\Gamma(\beta) = \Gamma(3) = 2$ and $\Phi(1, \beta - 1, 1) = \Phi(1, 2, 1) = \pi^2/6$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(x,y)}\log^2\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq \frac{4\pi^2}{3}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

- If we set $\beta = 5$, then $\Gamma(\beta) = \Gamma(4) = 6$ and $\Phi(1, \beta - 1, 1) = \Phi(1, 4, 1) = \pi^4/90$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(x,y)}\log^4\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq \frac{4\pi^4}{15}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Another modification of Theorem 2.1 of the same kind but with three parameters is examined below. The constant in the factor also depends on the Lerch transcendent function.

Theorem 2.7 *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$, $\kappa > 0$, $\eta \in (0, 1]$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^{\kappa}(x+y-\eta|x-y|)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq \Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 2.1 and the power series expansion method, including a geometric series expansion formula.

In particular, if we set $\eta = 1$, using $x + y - |x - y| = 2\min(x, y)$, then we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^{\kappa}\min(x,y)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq 2\Gamma(\beta)\Phi(1,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with

$$\Phi(1,\beta,\kappa) = \sum_{i=0}^{+\infty}\frac{1}{(i+\kappa)^{\beta}}.$$

A two-parameter logarithmic modification of Theorem 2.1 is presented below.

Theorem 2.8 *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$, $\rho > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^{\rho}}{(x+y)^{\rho}-|x-y|^{\rho}}\right]\times \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq \frac{1}{\rho^{\beta}}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 2.1 and the power series expansion method, including the logarithmic series expansion formula.

Note that

$$\Phi(1, \beta+1, 1) = \sum_{i=0}^{+\infty} \frac{1}{(i+1)^{\beta+1}}.$$

This corresponds to the Riemann zeta function at $\beta + 1$.

In particular, if we set $\rho = 1$, using $x + y - |x - y| = 2\min(x, y)$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p} y^{1/q} \frac{1}{(x+y)|x-y|} \log\left[\frac{x+y}{2\min(x,y)}\right] \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\Phi(1, \beta+1, 1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Some special cases of this inequality are detailed below.

- If we set $\beta = 1$, then $\Gamma(\beta) = \Gamma(1) = 1$ and $\Phi(1, \beta+1, 1) = \Phi(1, 2, 1) = \pi^2/6$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p} y^{1/q} \frac{1}{(x+y)|x-y|} \log\left[\frac{x+y}{2\min(x,y)}\right] \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

- If we set $\beta = 3$, then $\Gamma(\beta) = \Gamma(3) = 2$ and $\Phi(1, \beta+1, 1) = \Phi(1, 4, 1) = \pi^4/90$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p} y^{1/q} \frac{1}{(x+y)|x-y|} \log\left[\frac{x+y}{2\min(x,y)}\right] \log^{\beta-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\leq \frac{\pi^4}{45}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The modification of Theorem 2.1 below innovates by including the two-parameter Mittag-Leffler function in its definition, and a relatively simple constant in the factor.

Theorem 2.9 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 1$, $\theta > 0$, $\mu > 0$, $\nu \in (0, \alpha^\theta)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}} \Psi\left[\nu \log^\theta\left(\frac{x+y}{|x-y|}\right), \theta, \mu\right] \log^{\mu-1}\left(\frac{x+y}{|x-y|}\right) f(x)g(y)dxdy$$
$$\leq \frac{\alpha^{\theta-\mu}}{\alpha^\theta - \nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Ψ is the two-parameter Mittag-Leffler function defined by

$$\Psi(a, b, c) = \sum_{i=0}^{+\infty} \frac{a^i}{\Gamma(bi + c)},$$

with $a \in \mathbb{R}$, $b > 0$ and $c > 0$, or $|a| < 1$, $b = 0$ and $c > 0$, provided that the two integrals involved in the upper bound converge.

The proof uses Theorem 2.1 and the power series expansion method.

Considering the standard expressions of the two-parameter Mittag-Leffler function, some special cases of this theorem are presented below. More details about these expressions can be found in [14].

- If we set $\theta = 1$ and $\mu = 1$, then $\Psi(x, \theta, \mu) = \Psi(x, 1, 1) = e^x$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-\nu-1}}{(x+y)^{\alpha-\nu+1}}\log^{\mu-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq \frac{1}{\alpha-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This corresponds to Theorem 2.1 with the parameters "$\alpha = \alpha - \nu$" and $\beta = 1$.

- If we set $\theta = 2$ and $\mu = 1$, then $\Psi(x, \theta, \mu) = \Psi(x, 2, 1) = \cosh[\sqrt{x}]$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left[\left(\frac{x+y}{|x-y|}\right)^{\sqrt{\nu}}+\left(\frac{|x-y|}{x+y}\right)^{\sqrt{\nu}}\right]\times f(x)g(y)dxdy \leq \frac{2\alpha}{\alpha^2-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \quad (2)$$

- If we set $\theta = 2$ and $\mu = 2$, then $\Psi(x, \theta, \mu) = \Psi(x, 2, 2) = \sinh[\sqrt{x}]/\sqrt{x}$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left[\left(\frac{x+y}{|x-y|}\right)^{\sqrt{\nu}}-\left(\frac{|x-y|}{x+y}\right)^{\sqrt{\nu}}\right]\times f(x)g(y)dxdy \leq \frac{2\sqrt{\nu}}{\alpha^2-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \quad (3)$$

If we sum the inequalities in Equations (2) and (3), we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-\sqrt{\nu}-1}}{(x+y)^{\alpha-\sqrt{\nu}+1}}f(x)g(y)dxdy \leq \frac{1}{\alpha-\sqrt{\nu}}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This corresponds to Theorem 2.1 with the parameters "$\alpha = \alpha - \sqrt{\nu}$" and $\beta = 1$.

Thanks to the versatility of the two-parameter Mittag-Leffler function, many other special cases of Theorem 2.9 can be determined in a similar way.

2.4 Functional extensions

This subsection is devoted to functional extensions of the main theorem, using two adaptable functions denoted h and k. An advantage of such extensions is that the inequalities are established for possibly different supports, beyond the strict $[0, +\infty)^2$.

The result below is a functional extension of Theorem 2.1. Note the presence of weighted L_p and L_q norms of f and g, respectively. The weights in question depend on the derivatives of h and k.

Theorem 2.10 *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ *with* $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ *with* $d > c$, $f : [a, b] \mapsto [0, +\infty)$ *and* $g : [c, d] \mapsto [0, +\infty)$ *be two functions, and* $h : [a, b] \mapsto [0, +\infty)$ *and* $k : [c, d] \mapsto [0, +\infty)$ *be two differentiable non-decreasing functions such that* $\lim_{t\to a} h(t) = 0$,

$\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ *and* $\lim_{t\to d} k(t) = +\infty$. *Then the following inequality holds:*

$$\int_c^d \int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q},$$

where Γ *is the gamma function, provided that the two integrals involved in the upper bound converge.*

The proof is based on Theorem 2.1 and well-configured changes of variables.

Some special cases of this theorem are developed below.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{\iota/p}y^{\upsilon/q}\frac{|x^\iota-y^\upsilon|^{\alpha-1}}{[x^\iota+y^\upsilon]^{\alpha+1}}\log^{\beta-1}\left(\frac{x^\iota+y^\upsilon}{|x^\iota-y^\upsilon|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta\iota^{1-1/p}\upsilon^{1-1/q}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)\frac{1}{y^{(\upsilon-1)(q-1)}}dy\right]^{1/q}.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty}\int_1^{+\infty} \log^{1/p}(x)\log^{1/q}(y)\frac{|\log(x/y)|^{\alpha-1}}{[\log(xy)]^{\alpha+1}}\log^{\beta-1}\left[\frac{\log(xy)}{|\log(x/y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_1^{+\infty} f^p(x)x^{p-1}dx\right]^{1/p}\left[\int_1^{+\infty} g^q(y)y^{q-1}dy\right]^{1/q}.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : [1, +\infty) \mapsto \mathbb{R}$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{x/p+y/q}\frac{|e^x-e^y|^{\alpha-1}}{(e^x+e^y)^{\alpha+1}}\log^{\beta-1}\left(\frac{e^x+e^y}{|e^x-e^y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx\right]^{1/p}\left[\int_{-\infty}^{+\infty} g^q(y)e^{-(q-1)y}dy\right]^{1/q}.$$

Thus, different domains of integration can be considered, as well as different weighted L_p and L_q norms of f and g, respectively.

In the same spirit, the theorem below is a functional extension of Theorem 2.2.

Theorem 2.11 *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ *with* $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ *with* $d > c$, $f : [a, b] \mapsto [0, +\infty)$ *be a function, and* $h : [a, b] \mapsto [0, +\infty)$ *and* $k : [c, d] \mapsto [0, +\infty)$ *be two differentiable non-decreasing functions such that* $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ *and* $\lim_{t\to d} k(t) = +\infty$. *Then the following inequality holds:*

$$\int_c^d\left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx,$$

where Γ *is the gamma function, provided that the integral involved in the upper bound converges.*

The proof is based on Theorem 2.1 and suitable changes of variables.

Some special cases of this theorem are developed below.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty} \left\{\int_0^{+\infty} x^{\iota/p} y^{\upsilon/q} \frac{|x^\iota - y^\upsilon|^{\alpha-1}}{[x^\iota + y^\upsilon]^{\alpha+1}} \log^{\beta-1}\left(\frac{x^\iota + y^\upsilon}{|x^\iota - y^\upsilon|}\right) f(x)dx\right\}^p y^{\upsilon-1}dy$$
$$\leq \frac{1}{\upsilon\alpha^{\beta p}\iota^{p-1}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty} \left\{\int_1^{+\infty} \log^{1/p}(x)\log^{1/q}(y) \frac{|\log(x/y)|^{\alpha-1}}{[\log(xy)]^{\alpha+1}} \log^{\beta-1}\left[\frac{\log(xy)}{|\log(x/y)|}\right] f(x)dx\right\}^p \frac{1}{y}dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_1^{+\infty} f^p(x)x^{p-1}dx.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : [1, +\infty) \mapsto \mathbb{R}$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} e^{x/p+y/q} \frac{|e^x - e^y|^{\alpha-1}}{(e^x + e^y)^{\alpha+1}} \log^{\beta-1}\left(\frac{e^x + e^y}{|e^x - e^y|}\right) f(x)dx\right\}^p e^y dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx.$$

A functional extension of Theorem 2.3 is formulated below.

Theorem 2.12 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ with $d > c$, $f : [a, b] \mapsto [0, +\infty)$ and $g : [c, d] \mapsto [0, +\infty)$ be two functions, $F : [a, b] \mapsto [0, +\infty)$ and $G : [c, d] \mapsto [0, +\infty)$ be their respective primitives given by*

$$F(x) = \int_a^x f(r)dr, \quad G(y) = \int_c^y g(r)dr,$$

provided that they exist, and $h : [a, b] \mapsto [0, +\infty)$ and $k : [c, d] \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ and $\lim_{t\to d} k(t) = +\infty$. Then the following inequality holds:

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|h(x) - k(y)|^{\alpha-1}}{[h(x) + k(y)]^{\alpha+1}} \log^{\beta-1}\left[\frac{h(x) + k(y)}{|h(x) - k(y)|}\right] F(x)G(y)h'(x)k'(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q}.$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

Some special cases of this theorem are developed below.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{\iota(1/p-1)} y^{\upsilon(1/q-1)} \frac{|x^\iota - y^\upsilon|^{\alpha-1}}{[x^\iota + y^\upsilon]^{\alpha+1}} \log^{\beta-1}\left(\frac{x^\iota + y^\upsilon}{|x^\iota - y^\upsilon|}\right) F(x)G(y) x^{\iota-1} y^{\upsilon-1} dxdy$$
$$\leq \frac{p^2}{\alpha^\beta \iota^{2-1/p} \upsilon^{2-1/q}(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx\right]^{1/p} \times$$
$$\left[\int_0^{+\infty} g^q(y)\frac{1}{y^{(\upsilon-1)(q-1)}}dy\right]^{1/q}.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty}\int_1^{+\infty} \log^{1/p-1}(x)\log^{1/q-1}(y) \frac{|\log(x/y)|^{\alpha-1}}{[\log(xy)]^{\alpha+1}} \log^{\beta-1}\left[\frac{\log(xy)}{|\log(x/y)|}\right] F(x)G(y)\frac{1}{xy}dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_1^{+\infty} f^p(x)x^{p-1}dx\right]^{1/p}\left[\int_1^{+\infty} g^q(y)y^{q-1}dy\right]^{1/q}.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : \mathbb{R} \mapsto [0, +\infty)$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{x(1/p-1)+y(1/q-1)} \frac{|e^x - e^y|^{\alpha-1}}{(e^x + e^y)^{\alpha+1}} \log^{\beta-1}\left(\frac{e^x + e^y}{|e^x - e^y|}\right) F(x)G(y)e^{x+y}dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx\right]^{1/p}\left[\int_{-\infty}^{+\infty} g^q(y)e^{-(q-1)y}dy\right]^{1/q}.$$

We conclude this section with a functional extension of Theorem 2.4, followed by some special cases of interest.

Theorem 2.13 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ with $d > c$, $f : [a, b] \mapsto [0, +\infty)$, $F : [a, b] \mapsto [0, +\infty)$ be its primitive given by*

$$F(x) = \int_a^x f(r)dr,$$

provided that it exists, and $h : [a, b] \mapsto [0, +\infty)$ and $k : [c, d] \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ and $\lim_{t\to d} k(t) = +\infty$. Then the following inequality holds:

$$\int_c^d \left\{\int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|h(x) - k(y)|^{\alpha-1}}{[h(x) + k(y)]^{\alpha+1}} \log^{\beta-1}\left[\frac{h(x) + k(y)}{|h(x) - k(y)|}\right] F(x)h'(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p \Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{[h'(x)]^{p-1}}dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

Some special cases of this theorem are developed below.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty}\left\{\int_0^{+\infty} x^{\iota(1/p-1)} y^{\upsilon/q} \frac{|x^\iota - y^\upsilon|^{\alpha-1}}{[x^\iota + y^\upsilon]^{\alpha+1}} \log^{\beta-1}\left(\frac{x^\iota + y^\upsilon}{|x^\iota - y^\upsilon|}\right) F(x)x^{\iota-1}dx\right\}^p y^{\upsilon-1}dy$$
$$\leq \frac{1}{\upsilon\alpha^{\beta p}\iota^{2p-1}}\left(\frac{p}{p-1}\right)^p \Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty} \left\{ \int_1^{+\infty} \log^{(1/p-1)}(x) \log^{1/q}(y) \frac{|\log(x/y)|^{\alpha-1}}{[\log(xy)]^{\alpha+1}} \log^{\beta-1}\left[\frac{\log(xy)}{|\log(x/y)|}\right] F(x)\frac{1}{x}dx \right\}^p \frac{1}{y}dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\left(\frac{p}{p-1}\right)^p \int_1^{+\infty} f^p(x)x^{p-1}dx.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : [1, +\infty) \mapsto \mathbb{R}$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} e^{x(1/p-1)+y/q} \frac{|e^x - e^y|^{\alpha-1}}{(e^x + e^y)^{\alpha+1}} \log^{\beta-1}\left(\frac{e^x + e^y}{|e^x - e^y|}\right) F(x)e^x dx \right\}^p e^y dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p \Gamma^p(\beta) \int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx.$$

To the best of our knowledge, the Hardy-Hilbert-type integral inequalities presented in this section are new. They extend the theoretical framework of a classical topic. In the rest of the article, we complement these contributions with further modifications of the Hardy-Hilbert integral inequality of potential interest.

3 Second contributions

This section presents our second set of contributions to the subject. They are mainly product modifications of the inequalities presented in the previous section. We say "product" because, in the main integrands, the sum or difference of the variables is transformed into the sum or difference of a constant and the product of the variables, i.e., $x - y$ becomes $1 - xy$ and $x + y$ becomes $1 + xy$. The theory needs to be adjusted accordingly.

3.1 Main result

The result below proposes a new modification of the Hardy-Hilbert integral inequality, which can be described as a product version of Theorem 2.1.

Theorem 3.1 *Let $p > 1$, $q = p/(p - 1)$, $\alpha > 0$, $\beta > 0$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p}y^{1/q} \frac{|1 - xy|^{\alpha-1}}{(1 + xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1 + xy}{|1 - xy|}\right) f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge. Furthermore, the constant in the factor, i.e., $\Gamma(\beta)/\alpha^\beta$, is optimal.

The proof uses the main lines of Theorem 2.1, with a notable change of variables. This change takes into account the product nature of the variables. If we set $\beta = 1$, then Theorem 3.1 gives

$$\int_0^{+\infty} \int_0^{+\infty} x^{1/p}y^{1/q} \frac{|1 - xy|^{\alpha-1}}{(1 + xy)^{\alpha+1}} f(x)g(y)dxdy \leq \frac{1}{\alpha}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The constant in the factor is thus reduced to $1/\alpha$, which remains optimal. This modification of the Hardy-Hilbert integral inequality corresponds to [12, Theorem 3.1]. Therefore, Theorem 3.1 can be seen as a power-logarithmic generalization of this result. As for Theorem 2.1, the constant in the factor is also characterized by the gamma function. In addition, there is no dependence with the parameter p.

Two other special cases of interest are described below.

- If we set $\alpha = 1$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The constant in the factor is thus reduced to $\Gamma(\beta)$, and it is optimal in this context.

- If we set $\beta = 1/2$, then $\Gamma(\beta) = \Gamma(1/2) = \sqrt{\pi}$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\frac{1}{\sqrt{\log\left[(1+xy)/|1-xy|\right]}}f(x)g(y)dxdy$$
$$\leq \sqrt{\frac{\pi}{\alpha}}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Similarly to Equation (1), we can generalize this inequality easily by considering $\beta = m+1/2$ with $m \in \mathbb{N}$.

In each case, we emphasize the simplicity of the constant in the factor, which is clearly an advantage for further mathematical manipulations. This will be supported later via integral and power series manipulations with respect to the parameters α or β.

The theorem below proposes an alternative formulation of Theorem 3.1. It is also a product version of Theorem 2.2.

Theorem 3.2 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$ and $f : [0,+\infty) \mapsto [0,+\infty)$ be a function. Then the inequality in Theorem 3.1 is equivalent to*

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)dx\right]^p dy \leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

The proof adapts that of Theorem 2.2.

As an example, if we set $p = 2$ and $\beta = 1/2$, then $\Gamma(\beta) = \Gamma(1/2) = \sqrt{\pi}$, and we have

$$\int_0^{+\infty}\left[\int_0^{+\infty}\sqrt{xy}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\frac{1}{\sqrt{\log\left[(1+xy)/|1-xy|\right]}}f(x)dx\right]^2 dy \leq \frac{\pi}{\alpha}\int_0^{+\infty} f^2(x)dx.$$

Such a result could be useful in operator theory. The subsection below deals with primitive versions of Theorems 3.1 and 3.2.

3.2 Primitive versions

A product version of Theorem 2.3, as well as a primitive version of Theorem 3.1, is given below.

Theorem 3.3 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0, \beta > 0$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions and $F, G : [0,+\infty) \mapsto [0,+\infty)$ be defined their respective primitives given by*

$$F(x) = \int_0^x f(r)dr, \quad G(y) = \int_0^y g(r)dr,$$

provided that they exist. Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 3.1 and the Hardy integral inequality for the treatment of integrals of power-weighted primitives. It is not claimed that the obtained constant in the factor is optimal, but since it is based on two optimal constants in two different contexts, it can be considered as sharp. The obtained upper bound still depends on the L_p and L_q norms of f and g, and the constant in the factor now depends on p.

On the same mathematical basis, a product version of Theorem 2.4, as well as a primitive modification of Theorem 3.2, is formulated in the statement below.

Theorem 3.4 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $f : [0, +\infty) \mapsto [0, +\infty)$ be a function and $F : [0, +\infty) \mapsto [0, +\infty)$ be its primitive given by*

$$F(x) = \int_0^x f(r)dr,$$

provided that it exists. Then the following inequality holds:

$$\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)dx\right]^p dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p \Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

New modifications of Theorem 3.1 are examined in the subsection below.

3.3 New modifications

New modifications of Theorem 3.1 are developed in this section, starting with the result below. It can be seen as a product version of Theorem 2.5.

Theorem 3.5 *Let $p > 1$, $q = p/(p-1)$, $\beta > 1$, $\delta > 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 3.1 and uses an integration method with respect to the parameter α. Applying $\beta \to 1$, this result reduces to [12, Proposition 3.8].

As a special case, if we set $\delta = 2$, noticing that $1 + xy - |1 - xy| = 2\min(1, xy)$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{\min(1,xy)}{(1+xy)^3}\log^{\beta-2}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{2(\beta-1)}(1-2^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

A three-parameter modification of Theorem 3.1 is described below. It can also be seen as a product version of Theorem 2.6.

Theorem 3.6 *Let $p > 1$, $q = p/(p-1)$, $\beta > 1$, $\epsilon > 0$, $\gamma \in (0, 1]$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon-1}(1+xy)^{\epsilon-1}}{[(1+xy)^{\epsilon}-\gamma|1-xy|^{\epsilon}]^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\epsilon^{\beta}}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function, provided that the two integrals involved in the upper bound converge.

The proof uses Theorem 3.1 and the power series expansion method.

In particular, if we set $\gamma = 1$ and $\epsilon = 1$, noticing that $1 + xy - |1 - xy| = 2\min(1, xy)$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(1,xy)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq 4\Gamma(\beta)\Phi(1,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Some special cases of this inequality are detailed below, inspired by those of Theorem 2.6.

- If we set $\beta = 3$, then $\Gamma(\beta) = \Gamma(3) = 2$ and $\Phi(\gamma, \beta - 1, 1) = \Phi(1, 2, 1) = \pi^2/6$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(1,xy)}\log^{2}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{4\pi^2}{3}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

- If we set $\beta = 5$, then $\Gamma(\beta) = \Gamma(4) = 6$ and $\Phi(\gamma, \beta - 1, 1) = \Phi(1, 4, 1) = \pi^4/90$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{\min^2(1,xy)}\log^{4}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{4\pi^4}{15}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Another modification of Theorem 3.1 with three parameters and a power-log transformation is examined below. It is also a product version of Theorem 2.7. The constant in the factor is defined as the product of the Lerch transcendent function and the gamma function, which remains quite original for the topic.

Theorem 3.7 *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$, $\kappa > 0$, $\eta \in (0, 1]$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa}(1+xy-\eta|1-xy|)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function, provided that the two integrals involved in the upper bound converge.

The proof follows the main lines than that of Theorem 2.7. More precisely, it relies on Theorem 2.1 and the power series expansion method.

Two special cases are detailed below, inspired by those of Theorem 2.7.

- If we set $\eta = 1$, using $1 + xy - |1 - xy| = 2\min(1, xy)$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa}\min(1,xy)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq 2\Gamma(\beta)\Phi(1,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with

$$\Phi(1,\beta,\kappa) = \sum_{i=0}^{+\infty}\frac{1}{(i+\kappa)^{\beta}}.$$

- If we set $\eta = 1/2$, using the same arguments, then we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa}(1+xy+2\min(1,xy))}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{2}\Gamma(\beta)\Phi\left(\frac{1}{2},\beta,\kappa\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with

$$\Phi\left(\frac{1}{2},\beta,\kappa\right) = \sum_{i=0}^{+\infty}\frac{1}{2^i(i+\kappa)^{\beta}}.$$

A two-parameter modification of Theorem 3.1, as well as a product version of Theorem 2.8, is presented below.

Theorem 3.8 *Let $p > 1$, $q = p/(p-1)$, $\beta > 0$, $\rho > 0$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\rho}}{(1+xy)^{\rho}-|1-xy|^{\rho}}\right]\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\rho^{\beta}}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Φ is the Lerch transcendent function, provided that the two integrals involved in the upper bound converge.

The proof uses Theorem 3.1 and the logarithmic series expansion.

In particular, if we set $\rho = 1$, using $1 + xy - |1 - xy| = 2\min(1, xy)$, then we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1+xy}{2\min(1,xy)}\right]\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with

$$\Phi(1,\beta+1,1) = \sum_{i=0}^{+\infty}\frac{1}{(i+1)^{\beta+1}}.$$

On this basis, some special cases are detailed below, inspired by those of Theorem 2.8.

- If we set $\beta = 1$, then $\Gamma(\beta) = \Gamma(1) = 1$ and $\Phi(1, \beta+1, 1) = \Phi(1, 2, 1) = \pi^2/6$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1+xy}{2\min(1,xy)}\right]\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{\pi^2}{6}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

- If we set $\beta = 3$, then $\Gamma(\beta) = \Gamma(3) = 2$ and $\Phi(1, \beta+1, 1) = \Phi(1, 4, 1) = \pi^4/90$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1+xy}{2\min(1,xy)}\right]\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{\pi^4}{45}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The modification of Theorem 3.1 below has the originality of using the two-parameter Mittag-Leffler function in the integrand. It is also a product version of Theorem 2.9.

Theorem 3.9 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 1$, $\theta > 0$, $\mu > 0$, $\nu \in (0, \alpha^\theta)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\Psi\left[\nu\log^\theta\left(\frac{1+xy}{|1-xy|}\right), \theta, \mu\right]\log^{\mu-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{\alpha^{\theta-\mu}}{\alpha^\theta - \nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function and Ψ is the two-parameter Mittag-Leffler function, provided that the two integrals involved in the upper bound converge.

The proof uses Theorem 3.1 and power series expansions, including a geometric series formula.
Some special cases of this theorem are presented below.

- If we set $\theta = 1$ and $\mu = 1$, then $\Psi(x, \theta, \mu) = \Psi(x, 1, 1) = e^x$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-\nu-1}}{(1+xy)^{\alpha-\nu+1}}\log^{\mu-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

  This corresponds to Theorem 3.1 with "$\alpha = \alpha - \nu$" and $\beta = 1$.

- If we set $\theta = 2$ and $\mu = 1$, then $\Psi(x, \theta, \mu) = \Psi(x, 2, 1) = \cosh[\sqrt{x}]$, and we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left[\left(\frac{1+xy}{|1-xy|}\right)^{\sqrt{\nu}} + \left(\frac{|1-xy|}{1+xy}\right)^{\sqrt{\nu}}\right]\times$$
$$f(x)g(y)dxdy$$
$$\leq \frac{2\alpha}{\alpha^2-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{4}$$

- If we set $\theta = 2$ and $\mu = 2$, then $\Psi(x, \theta, \mu) = \Psi(x, 2, 2) = \sinh[\sqrt{x}]/\sqrt{x}$, and we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left[\left(\frac{1+xy}{|1-xy|}\right)^{\sqrt{\nu}} - \left(\frac{|1-xy|}{1+xy}\right)^{\sqrt{\nu}}\right]\times$$
$$f(x)g(y)dxdy$$
$$\leq \frac{2\sqrt{\nu}}{\alpha^2-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{5}$$

If we sum the inequalities in Equations (4) and (5), we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-\sqrt{\nu}-1}}{(1+xy)^{\alpha-\sqrt{\nu}+1}}f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha-\sqrt{\nu}}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

This corresponds to Theorem 3.1 with "$\alpha=\alpha-\sqrt{\nu}$" and $\beta=1$.

These are examples among many that can be derived from the versatile expressions of the two-parameter Mittag-Leffler function, demonstrating the flexibility of the theorem.

The rest of the section is devoted to various functional extensions of some of the above results, as well mixed approaches.

3.4 Functional extensions

The theorem below is a functional extension of Theorem 3.1, using two intermediate functions, h and k. It can be seen as a product version of Theorem 2.10. We emphasize the presence of weighted L_p and L_q norms of f and g, respectively. The weights in question depend on the derivatives of h and k.

Theorem 3.10 *Let $p>1$, $q=p/(p-1)$, $\alpha>0$, $\beta>0$, $a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}\cup\{+\infty\}$ with $b>a$, $c\in\mathbb{R}\cup\{-\infty\}$, $d\in\mathbb{R}\cup\{+\infty\}$ with $d>c$, $f:[a,b]\mapsto[0,+\infty)$ and $g:[c,d]\mapsto[0,+\infty)$ be two functions, and $h:[a,b]\mapsto[0,+\infty)$ and $k:[c,d]\mapsto[0,+\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a}h(t)=0$, $\lim_{t\to b}h(t)=+\infty$, $\lim_{t\to c}k(t)=0$ and $\lim_{t\to d}k(t)=+\infty$. Then the following inequality holds:*

$$\int_c^d\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof is based on Theorem 3.1 and well-configured changes of variables.

Some special cases of this theorem are developed below, inspired by those of Theorem 2.10.

- If we set $a=0$, "$b=+\infty$", $c=0$, "$d=+\infty$", $h:[0,+\infty)\mapsto[0,+\infty)$ defined by $h(x)=x^\iota$ with $\iota>0$ and $k:[0,+\infty)\mapsto[0,+\infty)$ defined by $k(y)=y^\upsilon$ with $\upsilon>0$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{\iota/p}y^{\upsilon/q}\frac{|1-x^\iota y^\upsilon|^{\alpha-1}}{[1+x^\iota y^\upsilon]^{\alpha+1}}\log^{\beta-1}\left(\frac{1+x^\iota y^\upsilon}{|1-x^\iota y^\upsilon|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta\iota^{1-1/p}\upsilon^{1-1/q}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)\frac{1}{y^{(\upsilon-1)(q-1)}}dy\right]^{1/q}.$$

- If we set $a=1$, "$b=+\infty$", $c=1$, "$d=+\infty$", $h:[1,+\infty)\mapsto[0,+\infty)$ defined by $h(x)=\log(x)$ and $k:[1,+\infty)\mapsto[0,+\infty)$ defined by $k(y)=\log(y)$, then we have

$$\int_1^{+\infty}\int_1^{+\infty}\log^{1/p}(x)\log^{1/q}(y)\frac{|1-\log(x)\log(y)|^{\alpha-1}}{[1+\log(x)\log(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+\log(x)\log(y)}{|1-\log(x)\log(y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_1^{+\infty} f^p(x)x^{p-1}dx\right]^{1/p}\left[\int_1^{+\infty} g^q(y)y^{q-1}dy\right]^{1/q}.$$

- If we set "$a=-\infty$", "$b=+\infty$", "$c=-\infty$", "$d=+\infty$", $h:\mathbb{R}\mapsto[0,+\infty)$ defined by $h(x)=e^x$ and $k:\mathbb{R}\mapsto[0,+\infty)$ defined by $k(y)=e^y$, then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{x/p+y/q}\frac{|1-e^{x+y}|^{\alpha-1}}{(1+e^{x+y})^{\alpha+1}}\log^{\beta-1}\left(\frac{1+e^{x+y}}{|1-e^{x+y}|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx\right]^{1/p}\left[\int_{-\infty}^{+\infty} g^q(y)e^{-(q-1)y}dy\right]^{1/q}.$$

A complementary result to Theorem 3.10 is proposed below. It is also a product version of Theorem 3.2.

Theorem 3.11 *Let $p>1$, $q=p/(p-1)$, $\alpha>0$, $\beta>0$, $a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}\cup\{+\infty\}$ with $b>a$, $c\in\mathbb{R}\cup\{-\infty\}$, $d\in\mathbb{R}\cup\{+\infty\}$ with $d>c$, $f:[a,b]\mapsto[0,+\infty)$ be a function, and $h:[a,b]\mapsto[0,+\infty)$ and $k:[c,d]\mapsto[0,+\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a}h(t)=0$, $\lim_{t\to b}h(t)=+\infty$, $\lim_{t\to c}k(t)=0$ and $\lim_{t\to d}k(t)=+\infty$. Then the following inequality holds:*

$$\int_c^d\left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

Some special cases of this theorem are developed below, inspired by those of Theorem 2.11.

- If we set $a=0$, "$b=+\infty$", $c=0$, "$d=+\infty$", $h:[0,+\infty)\mapsto[0,+\infty)$ defined by $h(x)=x^{\iota}$ with $\iota>0$ and $k:[0,+\infty)\mapsto[0,+\infty)$ defined by $k(y)=y^{\upsilon}$ with $\upsilon>0$, then we have

$$\int_0^{+\infty}\left\{\int_0^{+\infty} x^{\iota/p}y^{\upsilon/q}\frac{|1-x^{\iota}y^{\upsilon}|^{\alpha-1}}{[1+x^{\iota}y^{\upsilon}]^{\alpha+1}}\log^{\beta-1}\left(\frac{1+x^{\iota}y^{\upsilon}}{|1-x^{\iota}y^{\upsilon}|}\right)f(x)dx\right\}^p y^{\upsilon-1}dy$$
$$\leq \frac{1}{\upsilon\alpha^{\beta p}\iota^{p-1}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx.$$

- If we set $a=1$, "$b=+\infty$", $c=1$, "$d=+\infty$", $h:[1,+\infty)\mapsto[0,+\infty)$ defined by $h(x)=\log(x)$ and $k:[1,+\infty)\mapsto[0,+\infty)$ defined by $k(y)=\log(y)$, then we have

$$\int_1^{+\infty}\left\{\int_1^{+\infty}\log^{1/p}(x)\log^{1/q}(y)\frac{|1-\log(x)\log(y)|^{\alpha-1}}{[1+\log(x)\log(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+\log(x)\log(y)}{|1-\log(x)\log(y)|}\right]f(x)dx\right\}^p\frac{1}{y}dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_1^{+\infty} f^p(x)x^{p-1}dx.$$

- If we set "$a=-\infty$", "$b=+\infty$", "$c=-\infty$", "$d=+\infty$", $h:\mathbb{R}\mapsto[0,+\infty)$ defined by $h(x)=e^x$ and $k:\mathbb{R}\mapsto[0,+\infty)$ defined by $k(y)=e^y$, then we have

$$\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} e^{x/p+y/q}\frac{|1-e^{x+y}|^{\alpha-1}}{(1+e^{x+y})^{\alpha+1}}\log^{\beta-1}\left(\frac{1+e^{x+y}}{|1-e^{x+y}|}\right)f(x)dx\right\}^p e^y dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx.$$

A product version of Theorem 3.3 is given below, still using the primitives of the main functions.

Theorem 3.12 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ with $d > c$, $f : [a, b] \mapsto [0, +\infty)$ and $g : [c, d] \mapsto [0, +\infty)$ be two functions, $F : [a, b] \mapsto [0, +\infty)$ and $G : [c, d] \mapsto [0, +\infty)$ be their respective primitives given by*

$$F(x) = \int_a^x f(r)dr, \quad G(y) = \int_c^y g(r)dr,$$

provided that they exist, and $h : [a, b] \mapsto [0, +\infty)$ and $k : [c, d] \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ and $\lim_{t\to d} k(t) = +\infty$. Then the following inequality holds:

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}} \log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right] F(x)G(y)h'(x)k'(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

Some special cases of this theorem are developed below, inspired by those of Theorem 2.12.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{\iota(1/p-1)}y^{\upsilon(1/q-1)}\frac{|1-x^\iota y^\upsilon|^{\alpha-1}}{[1+x^\iota y^\upsilon]^{\alpha+1}}\log^{\beta-1}\left(\frac{1+x^\iota y^\upsilon}{|1-x^\iota y^\upsilon|}\right)F(x)G(y)x^{\iota-1}y^{\upsilon-1}dxdy$$
$$\leq \frac{p^2}{\alpha^\beta \iota^{2-1/p}\upsilon^{2-1/q}(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx\right]^{1/p}\times$$
$$\left[\int_0^{+\infty} g^q(y)\frac{1}{y^{(\upsilon-1)(q-1)}}dy\right]^{1/q}.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty}\int_1^{+\infty} \log^{1/p-1}(x)\log^{1/q-1}(y)\frac{|1-\log(x)\log(y)|^{\alpha-1}}{[1+\log(x)\log(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+\log(x)\log(y)}{|1-\log(x)\log(y)|}\right]\times$$
$$F(x)G(y)\frac{1}{xy}dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_1^{+\infty} f^p(x)x^{p-1}dx\right]^{1/p}\left[\int_1^{+\infty} g^q(y)y^{q-1}dy\right]^{1/q}.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : \mathbb{R} \mapsto [0, +\infty)$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{x(1/p-1)+y(1/q-1)}\frac{|1-e^{x+y}|^{\alpha-1}}{(1+e^{x+y})^{\alpha+1}}\log^{\beta-1}\left(\frac{1+e^{x+y}}{|1-e^{x+y}|}\right)F(x)G(y)e^{x+y}dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx\right]^{1/p}\left[\int_{-\infty}^{+\infty} g^q(y)e^{-(q-1)y}dy\right]^{1/q}.$$

The theorem below is a functional extension of Theorem 3.4, and also a product version of Theorem 2.13.

Theorem 3.13 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$, $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R} \cup \{+\infty\}$ with $d > c$, $f : [a, b] \mapsto [0, +\infty)$, $F : [a, b] \mapsto [0, +\infty)$ be its primitive given by*

$$F(x) = \int_a^x f(r)dr,$$

provided that it exists, and $h : [a, b] \mapsto [0, +\infty)$ and $k : [c, d] \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that $\lim_{t\to a} h(t) = 0$, $\lim_{t\to b} h(t) = +\infty$, $\lim_{t\to c} k(t) = 0$ and $\lim_{t\to d} k(t) = +\infty$. Then the following inequality holds:

$$\int_c^d \left\{ \int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}} \log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right] F(x)h'(x)dx \right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}} \left(\frac{p}{p-1}\right)^p \Gamma^p(\beta) \int_0^{+\infty} f^p(x)\frac{1}{[h'(x)]^{p-1}}dx,$$

where Γ is the gamma function, provided that the integral involved in the upper bound converges.

Some special cases of this theorem are developed below, inspired by those of Theorem 2.13.

- If we set $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $h : [0, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = x^\iota$ with $\iota > 0$ and $k : [0, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = y^\upsilon$ with $\upsilon > 0$, then we have

$$\int_0^{+\infty} \left\{ \int_0^{+\infty} x^{\iota(1/p-1)}y^{\upsilon/q}\frac{|1-x^\iota y^\upsilon|^{\alpha-1}}{[1+x^\iota y^\upsilon]^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x^\iota y^\upsilon}{|1-x^\iota y^\upsilon|}\right) F(x)x^{\iota-1}dx \right\}^p y^{\upsilon-1}dy$$
$$\leq \frac{1}{\upsilon\alpha^{\beta p}\iota^{2p-1}} \left(\frac{p}{p-1}\right)^p \Gamma^p(\beta) \int_0^{+\infty} f^p(x)\frac{1}{x^{(\iota-1)(p-1)}}dx.$$

- If we set $a = 1$, "$b = +\infty$", $c = 1$, "$d = +\infty$", $h : [1, +\infty) \mapsto [0, +\infty)$ defined by $h(x) = \log(x)$ and $k : [1, +\infty) \mapsto [0, +\infty)$ defined by $k(y) = \log(y)$, then we have

$$\int_1^{+\infty} \left\{ \int_1^{+\infty} \log^{(1/p-1)}(x)\log^{1/q}(y)\frac{|1-\log(x)\log(y)|^{\alpha-1}}{[1+\log(x)\log(y)]^{\alpha+1}} \log^{\beta-1}\left[\frac{1+\log(x)\log(y)}{|1-\log(x)\log(y)|}\right] \times \right.$$
$$\left. F(x)\frac{1}{x}dx \right\}^p \frac{1}{y}dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta) \left(\frac{p}{p-1}\right)^p \int_1^{+\infty} f^p(x)x^{p-1}dx.$$

- If we set "$a = -\infty$", "$b = +\infty$", "$c = -\infty$", "$d = +\infty$", $h : \mathbb{R} \mapsto [0, +\infty)$ defined by $h(x) = e^x$ and $k : \mathbb{R} \mapsto [0, +\infty)$ defined by $k(y) = e^y$, then we have

$$\int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} e^{x(1/p-1)+y/q}\frac{|1-e^{x+y}|^{\alpha-1}}{(1+e^{x+y})^{\alpha+1}} \log^{\beta-1}\left(\frac{1+e^{x+y}}{|1-e^{x+y}|}\right) F(x)e^x dx \right\}^p e^y dy$$
$$\leq \frac{1}{\alpha^{\beta p}} \left(\frac{p}{p-1}\right)^p \Gamma^p(\beta) \int_{-\infty}^{+\infty} f^p(x)e^{-(p-1)x}dx.$$

Some mixed approaches of our main integral results are presented in the subsection below.

3.5 Some mixed approaches

A mixed approach of Theorems 2.1 and 3.1 is proposed below.

Theorem 3.14 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\theta > 0$, $\lambda \in [0,1]$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions. Then the following inequality holds:*

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof relies on the Hölder integral inequality and Theorems 2.1 and 3.1. The key to this mixed approach is characterized by the presence of λ. If $\lambda = 1$, the result reduces to Theorem 2.1, and if $\lambda = 0$, it reduces to Theorem 3.1; the intermediate values of λ lead to new integral inequalities.

A mixed approach using the primitives of the main functions, based on Theorems 2.3 and 3.3, is given below.

Theorem 3.15 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\theta > 0$, $\lambda \in [0,1]$ and $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two functions and $F, G : [0,+\infty) \mapsto [0,+\infty)$ be defined their respective primitives given by*

$$F(x) = \int_0^x f(r)dr, \quad G(y) = \int_0^y g(r)dr,$$

Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy$$
$$\leq \frac{p^2}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}(p-1)}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where Γ is the gamma function, provided that the two integrals involved in the upper bound converge.

The proof uses the Hölder integral inequality and Theorems 2.3 and 3.3. Again, we emphasise the linking role of the parameter λ. If $\lambda = 1$, the result reduces to Theorem 2.3, and if $\lambda = 0$, it reduces to Theorem 3.3; the intermediate values of λ lead to new integral inequalities.

4 Proofs

The proofs of all results are provided in this section. For the sake of self-containment, reproducibility, and ease of verification, we have included the maximum level of detail.

4.1 A key integral result

The proposition below is needed to prove our main results.

Proposition 4.1 *Let $\alpha > 0$ and $\beta > 0$. Then we have*

$$\int_0^{+\infty}\frac{|1-x|^{\alpha-1}}{(1+x)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+x}{|1-x|}\right)dx = \frac{1}{\alpha^{\beta}}\Gamma(\beta).$$

Proof of Proposition 4.1. Applying the Chasles integral formula at the threshold value $x = 1$, we obtain

$$\int_0^{+\infty} \frac{|1-x|^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{|1-x|}\right) dx = \int_0^{1} \frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{1-x}\right) dx$$
$$+ \int_1^{+\infty} \frac{(x-1)^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{x-1}\right) dx.$$

Using the change of variables $x = 1/s$, the last integral term can be expressed as

$$\int_1^{+\infty} \frac{(x-1)^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{1-x}\right) dx = \int_1^{0} \frac{(1/s-1)^{\alpha-1}}{(1+1/s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+1/s}{1-1/s}\right) \left(-\frac{1}{s^2}ds\right)$$
$$= \int_0^{1} \frac{(1-s)^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{s-1}\right) ds.$$

As a result, we have

$$\int_0^{+\infty} \frac{|1-x|^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{|1-x|}\right) dx = 2\int_0^{1} \frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{1-x}\right) dx. \tag{6}$$

Using the change of variables $t = (1-x)/(1+x)$, we have

$$2\int_0^{1} \frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{1-x}\right) dx = \int_0^{1} \left(\frac{1-x}{1+x}\right)^{\alpha-1} \log^{\beta-1}\left(\frac{1+x}{1-x}\right) \frac{2}{(1+x)^2} dx$$
$$= \int_1^{0} t^{\alpha-1} \log^{\beta-1}\left(\frac{1}{t}\right)(-dt) = \int_0^{1} t^{\alpha-1} \log^{\beta-1}\left(\frac{1}{t}\right) dt. \tag{7}$$

The expression of the logarithmic term suggests the change of variables $t = e^{-u}$, and we obtain

$$\int_0^{1} t^{\alpha-1} \log^{\beta-1}\left(\frac{1}{t}\right) dt = \int_{+\infty}^{0} e^{-u(\alpha-1)} u^{\beta-1}\left(-e^{-u}du\right) = \int_0^{+\infty} u^{\beta-1} e^{-\alpha u} du. \tag{8}$$

To match the expression of the gamma function, the change of variables $v = \alpha u$ gives

$$\int_0^{+\infty} u^{\beta-1} e^{-\alpha u} du = \int_0^{+\infty} \left(\frac{v}{\alpha}\right)^{\beta-1} e^{-v}\left(\frac{1}{\alpha}dv\right) = \frac{1}{\alpha^{\beta}} \int_0^{+\infty} v^{\beta-1} e^{-v} dv = \frac{1}{\alpha^{\beta}}\Gamma(\beta). \tag{9}$$

Combining Equations (6), (7), (8) and (9), we get

$$\int_0^{+\infty} \frac{|1-x|^{\alpha-1}}{(1+x)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+x}{|1-x|}\right) dx = \frac{1}{\alpha^{\beta}}\Gamma(\beta).$$

Proposition 4.1 is proved. □

4.2 Proofs of the first contributions

Proof of Theorem 2.1.

Main integral inequality. By a suitable product decomposition of the integrand via the equality $1/p + 1/q = 1/p + (p-1)/p = 1$ and the Hölder integral inequality, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}\frac{|x-y|^{(\alpha-1)/p}}{(x+y)^{(\alpha+1)/p}}\log^{(\beta-1)/p}\left(\frac{x+y}{|x-y|}\right)f(x)\times$$
$$y^{1/q}\frac{|x-y|^{(\alpha-1)/q}}{(x+y)^{(\alpha+1)/q}}\log^{(\beta-1)/q}\left(\frac{x+y}{|x-y|}\right)g(y)dxdy$$
$$\leq \left[\int_0^{+\infty}\int_0^{+\infty} x\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f^p(x)dxdy\right]^{1/p}\times$$
$$\left[\int_0^{+\infty}\int_0^{+\infty} y\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)g^q(y)dxdy\right]^{1/q}. \tag{10}$$

We need to express each double integral term of this bound. It follows from the Fubini-Tonelli integral theorem (ensuring the exchange of the order of integration), the change of variables $s = y/x$ and Proposition 4.1, that

$$\int_0^{+\infty}\int_0^{+\infty} x\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f^p(x)dxdy$$
$$= \int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-y/x|^{\alpha-1}}{(1+y/x)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+y/x}{|1-y/x|}\right)\frac{1}{x}dy\right]dx$$
$$= \int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)ds\right]dx$$
$$= \int_0^{+\infty} f^p(x)\times\frac{1}{\alpha^\beta}\Gamma(\beta)dx = \frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} f^p(x)dx. \tag{11}$$

Similarly but with the change of variables $t = x/y$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} y\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)g^q(y)dxdy$$
$$= \int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-x/y|^{\alpha-1}}{(1+x/y)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+x/y}{|1-x/y|}\right)\frac{1}{y}dx\right]dy$$
$$= \int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-t|^{\alpha-1}}{(1+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+t}{|1-t|}\right)dt\right]dy$$
$$= \int_0^{+\infty} g^p(y)\times\frac{1}{\alpha^\beta}\Gamma(\beta)dy = \frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} g^p(y)dy. \tag{12}$$

Combining Equations (10), (11) and (12), we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \left[\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Optimality of the constant in the factor. We reason by contradiction by assuming the existence of a better constant than $\Gamma(\beta)/\alpha^{\beta}$. So we consider a constant C such that $C \in (0, \Gamma(\beta)/\alpha^{\beta})$ and, for any $f, g : [0,+\infty) \mapsto [0,+\infty)$,

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \leq C\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{13}$$

provided that the two integrals involved in the upper bound converge. For the candidate functions of the contradiction, we consider $f_\star : [0,+\infty) \mapsto [0,+\infty)$ defined by

$$f_\star(x) = \begin{cases} x^{-(1+1/n)/p} & \text{if } x \in [1,+\infty) \\ 0 & \text{if } x \in [0,1) \end{cases}$$

and $g_\star : [0,+\infty) \mapsto [0,+\infty)$ defined by

$$g_\star(y) = \begin{cases} y^{-(1+1/n)/q} & \text{if } y \in [1,+\infty) \\ 0 & \text{if } y \in [0,1) \end{cases},$$

where $n \in \mathbb{N}\backslash\{0\}$. Note that

$$\int_0^{+\infty} f_\star^p(x)dx = \int_1^{+\infty}(x^{-(1+1/n)/p})^p dx = \left[-nx^{-1/n}\right]_{x=1}^{x\to+\infty} = n$$

and

$$\int_0^{+\infty} g_\star^q(y)dy = \int_1^{+\infty}(y^{-(1+1/n)/q})^q dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty} = n.$$

It follows from the equality $1/p+1/q=1$ and Equation (13) that

$$C = C\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{C\left[\int_0^{+\infty} f_\star^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\star^q(y)dy\right]^{1/q}\right\} \geq \frac{1}{n}\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\star(x)g_\star(y)dxdy. \tag{14}$$

We need to express the double integral term of this bound. Using the definitions of $f_\star$ and $g_\star$, the change of variables $x = sy$, the Fubini-Tonelli integral theorem and the equality $1/p+1/q=1$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\star(x)g_\star(y)dxdy \\
&= \int_1^{+\infty}\int_1^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)x^{-(1+1/n)/p}y^{-(1+1/n)/q}dxdy \\
&= \int_1^{+\infty}\left[\int_1^{+\infty}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)x^{-1/(np)}dx\right]y^{-1/(nq)}dy \\
&= \int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\frac{1}{y^2}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)s^{-1/(np)}y^{-1/(np)}yds\right]y^{-1/(nq)}dy \\
&= \int_1^{+\infty}\left[\int_{1/y}^{+\infty}\frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)s^{-1/(np)}ds\right]y^{-(1+1/n)}dy.
\end{aligned} \tag{15}$$

It follows from the Chasles integral formula at the threshold value $s = 1$, the Fubini-Tonelli integral theorem and the equality $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_1^{+\infty} \left[\int_{1/y}^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\right] y^{-(1+1/n)} dy \\
&= \int_1^{+\infty} \left[\int_{1/y}^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\right] y^{-(1+1/n)} dy \\
&+ \int_1^{+\infty} \left[\int_{1}^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\right] y^{-(1+1/n)} dy \\
&= \int_0^{1} \left[\int_{1/s}^{+\infty} y^{-(1+1/n)} dy\right] \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds \\
&+ \left[\int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\right]\left[\int_1^{+\infty} y^{-(1+1/n)} dy\right] \\
&= \int_0^{1} (ns^{1/n}) \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds \\
&+ n\left[\int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\right] \\
&= n\Bigg[\int_0^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(nq)} ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds\Bigg].
\end{aligned} \tag{16}$$

Combining Equations (14), (15) and (16), we get

$$\begin{aligned}
C &\geq \int_0^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(nq)} ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds.
\end{aligned}$$

Since this inequality is valid for any $n \in \mathbb{N}\backslash\{0\}$, we can take the inferior limit with respect to the integer n, denoted by $\underline{\lim}_{n\to+\infty}$. By virtue of the Fatou integral lemma, which is possible because the integrand is non-negative, $\underline{\lim}_{n\to+\infty} s^{1/(nq)} = 1$ for $s \in (0,1)$, $\underline{\lim}_{n\to+\infty} s^{-1/(np)} = 1$ for $s \in [1,+\infty)$, the Chasles integral formula at the threshold value $s = 1$ and Proposition 4.1, we obtain

$$\begin{aligned}
C &\geq \underline{\lim}_{n\to+\infty} \int_0^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(nq)} ds \\
&+ \underline{\lim}_{n\to+\infty} \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(np)} ds \\
&\geq \int_0^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) \left[\underline{\lim}_{n\to+\infty} s^{1/(nq)}\right] ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) \left[\underline{\lim}_{n\to+\infty} s^{-1/(np)}\right] ds \\
&= \int_0^{1} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds \\
&= \int_0^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds = \frac{1}{\alpha^{\beta}}\Gamma(\beta).
\end{aligned}$$

A contradiction comes since C is assumed to satisfy $C < \Gamma(\beta)/\alpha^{\beta}$. As a result, the constant $\Gamma(\beta)/\alpha^{\beta}$ is optimal. Theorem 2.1 is established. □

Proof of Theorem 2.2.

Proof that Theorem 2.1 implies the presented inequality. We assume that the inequality in Theorem 2.1 holds. It follows from the Fubini-Tonelli integral theorem and a suitable decomposition of the integrand that

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy\\
&=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]\times\\
&\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^{p-1} dy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)\times\\
&\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^{p-1} dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g_{\dagger}(y)dxdy, \qquad (17)
\end{aligned}$$

where

$$g_{\dagger}(y)=\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^{p-1}.$$

Applying Theorem 2.1 to the functions f and $g_{\dagger}$, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g_{\dagger}(y)dxdy\\
&\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_{\dagger}^q(y)dy\right]^{1/q}. \qquad (18)
\end{aligned}$$

Let us now determine the second integral of this bound. Since $q(p-1)=p$, we have

$$\begin{aligned}
&\int_0^{+\infty} g_{\dagger}^q(y)dy=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^{q(p-1)} dy\\
&=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^{p} dy. \qquad (19)
\end{aligned}$$

Combining Equations (17), (18) and (19), we have

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy\\
&\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\times\\
&\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy\right\}^{1/q}.
\end{aligned}$$

We therefore get

$$\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy\right\}^{1-1/q}$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}.$$

Using the relation $1/p+1/q=1$, this is equivalent to

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.$$

The desired inequality is established.

Proof that the presented inequality implies Theorem 2.1. We now assume that the presented inequality holds, and show that it implies Theorem 2.1. It follows from the Fubini-Tonelli integral theorem, the Hölder integral inequality only with respect to the variable y, and the assumed integral inequality that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]g(y)dy$$
$$\leq\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)dx\right]^p dy\right\}^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$\leq\left[\frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$=\frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The desired inequality is established.

The proof of Theorem 2.2 ends. □

Proof of Theorem 2.3. Let us notice that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)G(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\diamond(x)g_\diamond(y)dxdy, \tag{20}$$

where

$$f_\diamond(x)=\frac{1}{x}F(x),\quad g_\diamond(y)=\frac{1}{y}G(y).$$

Applying Theorem 2.1 to the functions $f_\diamond$ and $g_\diamond$, we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\diamond(x)g_\diamond(y)dxdy$$
$$\leq\frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f_\diamond^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\diamond^q(y)dy\right]^{1/q}$$
$$=\frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y^q}G^q(y)dy\right]^{1/q}. \tag{21}$$

It follows from the Hardy integral inequality applied to the function f that

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx. \tag{22}$$

The same applied to the function g, together with the equality $1/p + 1/q = 1$, gives

$$\int_0^{+\infty} \frac{1}{y^q} G^q(y)dy \leq \left(\frac{q}{q-1}\right)^q \int_0^{+\infty} g^q(y)dy = p^q \int_0^{+\infty} g^q(y)dy. \tag{23}$$

Combining Equations (20), (21), (22) and (23), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)G(y)dxdy\\
&\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[p^q\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Theorem 2.3 is demonstrated. □

Proof of Theorem 2.4. Let us notice that

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)dx\right]^p dy\\
&= \int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\diamond(x)dx\right]^p dy,
\end{aligned} \tag{24}$$

where

$$f_\diamond(x) = \frac{1}{x}F(x).$$

Applying Theorem 2.2 to the function $f_\diamond$ and using the Hardy integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f_\diamond(x)dx\right]^p dy\\
&\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f_\diamond^p(x)dx = \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\\
&\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.
\end{aligned} \tag{25}$$

Combining Equations (24) and (25), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)dx\right]^p dy\\
&\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.
\end{aligned}$$

Theorem 2.4 is proved. □

Proof of Theorem 2.5. By virtue of Theorem 2.1, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\\
&\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

with $\alpha > 0$ and $\beta > 0$. Integrating both sides with respect to the parameter α treated as a variable with $\alpha \in (1, \delta)$, and developing only the upper bound by taking into account that $\beta > 1$, we obtain

$$\begin{aligned}
&\int_1^{\delta} \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]d\alpha \\
&\leq \Gamma(\beta)\left[\int_1^{\delta}\frac{1}{\alpha^{\beta}}d\alpha\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \\
&= \Gamma(\beta)\left[\frac{1}{\beta-1}(1-\delta^{1-\beta})\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \\
&= \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (26)
\end{aligned}$$

For the main triple integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the order of integration) and basic integral calculus, we get

$$\begin{aligned}
&\int_1^{\delta} \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]d\alpha \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[\int_1^{\delta}\left(\frac{|x-y|}{x+y}\right)^{\alpha-1}d\alpha\right]\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[-\frac{1}{\log[(x+y)/|x-y|]}\left(\frac{|x-y|}{x+y}\right)^{\alpha-1}\right]_{\alpha=1}^{\alpha=\delta}\times \\
&\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy. \qquad (27)
\end{aligned}$$

Combining Equations (26) and (27), we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)^2}\left[1-\left(\frac{|x-y|}{x+y}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\
&\leq \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Theorem 2.5 is established. □

Proof of Theorem 2.6. Theorem 2.1 gives

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy \\
&\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = \epsilon i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $i\gamma^i$, summing both sides with respect to the integer i and developing only the upper bound using the Lerch transcendent function, we have

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\gamma^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon i-1}}{(x+y)^{\epsilon i+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]\\
&\leq \left[\sum_{i=1}^{+\infty} i\gamma^i \frac{1}{(\epsilon i)^\beta}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\epsilon^\beta}\Gamma(\beta)\left[\sum_{i=1}^{+\infty}\frac{\gamma^i}{i^{\beta-1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\epsilon^\beta}\Gamma(\beta)\gamma\left[\sum_{i=0}^{+\infty}\frac{\gamma^i}{(i+1)^{\beta-1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \gamma\frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (28)
\end{aligned}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and a basic geometric series formula based on $\gamma|x-y|^\epsilon/(x+y)^\epsilon \in (0,1)$ (almost everywhere), we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\gamma^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon i-1}}{(x+y)^{\epsilon i+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{\sum_{i=1}^{+\infty} i\left[\gamma\left(\frac{|x-y|}{x+y}\right)^\epsilon\right]^i\right\}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)\times\\
&f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\gamma\left(\frac{|x-y|}{x+y}\right)^\epsilon\left\{\sum_{i=1}^{+\infty} i\left[\gamma\left(\frac{|x-y|}{x+y}\right)^\epsilon\right]^{i-1}\right\}\times\\
&\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\\
&= \gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left(\frac{|x-y|}{x+y}\right)^\epsilon\times\\
&\frac{1}{[1-\gamma|x-y|^\epsilon/(x+y)^\epsilon]^2}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\\
&= \gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon-1}(x+y)^{\epsilon-1}}{[(x+y)^\epsilon-\gamma|x-y|^\epsilon]^2}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy. \qquad (29)
\end{aligned}$$

Combining Equations (28) and (29), we obtain

$$\begin{aligned}
&\gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon-1}(x+y)^{\epsilon-1}}{[(x+y)^\epsilon-\gamma|x-y|^\epsilon]^2}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\\
&\leq \gamma\frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\epsilon-1}(x+y)^{\epsilon-1}}{[(x+y)^\epsilon-\gamma|x-y|^\epsilon]^2}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\\
&\leq \frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Theorem 2.6 is proved. □

Proof of Theorem 2.7. By virtue of Theorem 2.1, we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = i + \kappa$ with $i \in \mathbb{N}$, which is possible because $\kappa > 0$, multiplying both sides by η^i, summing both sides with respect to the integer i and developing only the upper bound using the Lerch transcendent function, we obtain

$$\sum_{i=0}^{+\infty}\eta^i\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{i+\kappa-1}}{(x+y)^{i+\kappa+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]$$
$$\leq \left[\sum_{i=0}^{+\infty}\eta^i\frac{1}{(i+\kappa)^\beta}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \Gamma(\beta)\left[\sum_{i=0}^{+\infty}\frac{\eta^i}{(i+\kappa)^\beta}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{30}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and a basic geometric series formula based on $\eta|x-y|/(x+y) \in (0,1)$ (almost everywhere), we get

$$\sum_{i=0}^{+\infty}\eta^i\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{i+\kappa-1}}{(x+y)^{i+\kappa+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^{\kappa+1}}\left\{\sum_{i=0}^{+\infty}\left[\eta\left(\frac{|x-y|}{x+y}\right)\right]^i\right\}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^{\kappa+1}}\times\frac{1}{1-\eta|x-y|/(x+y)}\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^\kappa(x+y-\eta|x-y|)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy. \tag{31}$$

Combining Equations (30) and (31), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\kappa-1}}{(x+y)^\kappa(x+y-\eta|x-y|)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 2.7 is demonstrated. □

Proof of Theorem 2.8. Theorem 2.1 gives

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = \rho i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $1/i$, summing both sides with respect to the integer i and developing the upper bound using the Lerch transcendent function, we obtain

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\rho i-1}}{(x+y)^{\rho i+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]$$
$$\leq \left[\sum_{i=1}^{+\infty}\frac{1}{i}\frac{1}{(\rho i)^\beta}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^\beta}\Gamma(\beta)\left[\sum_{i=1}^{+\infty}\frac{1}{i^{\beta+1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^\beta}\Gamma(\beta)\left[\sum_{i=0}^{+\infty}\frac{1}{(i+1)^{\beta+1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^\beta}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{32}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals), and a basic logarithmic series formula based on $|x-y|^\rho/(x+y)^\rho \in (0,1)$ (almost everywhere), we get

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\rho i-1}}{(x+y)^{\rho i+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{\sum_{i=1}^{+\infty}\frac{1}{i}\left[\left(\frac{|x-y|}{x+y}\right)^\rho\right]^i\right\}\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\left\{-\log\left[1-\left(\frac{|x-y|}{x+y}\right)^\rho\right]\right\}\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{1}{1-|x-y|^\rho/(x+y)^\rho}\right]\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^\rho}{(x+y)^\rho-|x-y|^\rho}\right]\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy. \tag{33}$$

Combining Equations (32) and (33), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(x+y)|x-y|}\log\left[\frac{(x+y)^\rho}{(x+y)^\rho-|x-y|^\rho}\right]\times$$
$$\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\rho^\beta}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 2.8 is proved. □

Proof of Theorem 2.9. By virtue of Theorem 2.1, the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. This can also be written as follows:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left[\frac{1}{\Gamma(\beta)}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Taking $\beta = \theta i + \mu$ with $i \in \mathbb{N}$, which is possible because $\mu > 0$, multiplying both sides by ν^i, summing both sides with respect to the integer i and developing the upper bound using a basic geometric series formula taking into account that $\nu \in (0, \alpha^\theta)$, we have

$$\sum_{i=0}^{+\infty}\nu^i\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left[\frac{1}{\Gamma(\theta i+\mu)}\log^{\theta i+\mu-1}\left(\frac{x+y}{|x-y|}\right)\right]f(x)g(y)dxdy$$
$$\leq \left[\sum_{i=0}^{+\infty}\nu^i\frac{1}{\alpha^{\theta i+\mu}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha^\mu}\left[\sum_{i=0}^{+\infty}\left(\frac{\nu}{\alpha^\theta}\right)^i\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha^\mu}\left(\frac{1}{1-\nu/\alpha^\theta}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{\alpha^{\theta-\mu}}{\alpha^\theta-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (34)$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and the two-parameter Mittag-Leffler function, we have

$$\sum_{i=0}^{+\infty}\nu^i\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left[\frac{1}{\Gamma(\theta i+\mu)}\log^{\theta i+\mu-1}\left(\frac{x+y}{|x-y|}\right)\right]f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\left\{\sum_{i=0}^{+\infty}\frac{1}{\Gamma(\theta i+\mu)}\left[\nu\log^\theta\left(\frac{x+y}{|x-y|}\right)\right]^i\right\}\times$$
$$\log^{\mu-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\Psi\left[\nu\log^{\theta}\left(\frac{x+y}{|x-y|}\right),\theta,\mu\right]\times$$
$$\log^{\mu-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy. \tag{35}$$

Combining Equations (34) and (35), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\Psi\left[\nu\log^{\theta}\left(\frac{x+y}{|x-y|}\right),\theta,\mu\right]\times$$
$$\log^{\mu-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy$$
$$\leq \frac{\alpha^{\theta-\mu}}{\alpha^{\theta}-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 2.9 is established. □

Proof of Theorem 2.10. Using the changes of variables $s=h(x)$ and $t=k(y)$, so that $x=h^{-1}(s)$ and $y=k^{-1}(t)$, and $dx=[1/\{h'[h^{-1}(s)]\}]ds$ and $dy=[1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f[h^{-1}(s)]g[k^{-1}(t)]\frac{1}{h'[h^{-1}(s)]}\frac{1}{k'[k^{-1}(t)]}dsdt$$
$$=\int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f_{\circeq}(s)g_{\circeq}(t)dsdt, \tag{36}$$

where

$$f_{\circeq}(s)=f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]},\quad g_{\circeq}(t)=g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}.$$

Applying Theorem 2.1 to the functions $f_{\circeq}$ and $g_{\circeq}$, we get

$$\int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f_{\circeq}(s)g_{\circeq}(t)dsdt$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f_{\circeq}^p(s)ds\right]^{1/p}\left[\int_0^{+\infty} g_{\circeq}^q(t)dt\right]^{1/q}. \tag{37}$$

Let us now express the integral terms of this bound. Using the change of variables $s=h(u)$, we have

$$\int_0^{+\infty} f_{\circeq}^p(s)ds=\int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds=\int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$=\int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{38}$$

Similarly, using the change of variables $t=k(v)$, we have

$$\int_0^{+\infty} g_{\circeq}^q(t)dt=\int_0^{+\infty}\left\{g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}\right\}^q dt=\int_c^d g^q(v)\frac{1}{[k'(v)]^q}k'(v)dv$$
$$=\int_c^d g^q(v)\frac{1}{[k'(v)]^{q-1}}dv. \tag{39}$$

Combining Equations (36), (37), (38) and (39), we get

$$\int_c^d \int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q}.$$

Theorem 2.10 is proved. □

Proof of Theorem 2.11. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$= \int_0^{+\infty}\left\{\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}ds\right\}^p \times$$
$$k'[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}dt$$
$$= \int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f_{\approx}(s)ds\right]^p dt, \tag{40}$$

where

$$f_{\approx}(s) = f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}.$$

Applying Theorem 2.2 to the function $f_{\approx}$, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)f_{\approx}(s)ds\right]^p dt$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f_{\approx}^p(s)ds. \tag{41}$$

Let us now determine the integral of this bound. Using the change of variables $s = h(u)$, we have

$$\int_0^{+\infty} f_{\approx}^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{42}$$

Combining Equations (40), (41) and (42), we get

$$\int_c^d \left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx.$$

Theorem 2.11 is demonstrated. □

Proof of Theorem 2.12. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]F(x)G(y)h'(x)k'(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F[h^{-1}(s)]G[k^{-1}(t)]\times$$
$$h'[h^{-1}(s)]k'[k^{-1}(t)]\frac{1}{h'[h^{-1}(s)]}\frac{1}{k'[k^{-1}(t)]}dsdt$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F_{\eqsim}(s)G_{\eqsim}(t)dsdt, \tag{43}$$

where

$$F_{\eqsim}(s) = F[h^{-1}(s)], \quad G_{\eqsim}(t) = G[k^{-1}(t)].$$

Using the change of variables $r = h^{-1}(u)$, we have

$$F_{\eqsim}(s) = F[h^{-1}(s)] = \int_a^{h^{-1}(s)} f(r)dr = \int_0^s f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}du = \int_0^s f_{\eqsim}(u)du,$$

where

$$f_{\eqsim}(u) = f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}.$$

Similarly, using the change of variables $r = k^{-1}(v)$, we get

$$G_{\eqsim}(t) = G[k^{-1}(t)] = \int_c^{k^{-1}(t)} g(r)dr = \int_0^t g[k^{-1}(v)]\frac{1}{k'[k^{-1}(v)]}dv = \int_0^t g_{\eqsim}(v)dv,$$

where

$$g_{\eqsim}(v) = g[k^{-1}(v)]\frac{1}{k'[k^{-1}(v)]}.$$

Applying Theorem 2.3 to the functions $f_{\eqsim}$ and $g_{\eqsim}$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F_{\eqsim}(s)G_{\eqsim}(t)dsdt$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f_{\eqsim}^p(s)ds\right]^{1/p}\left[\int_0^{+\infty} g_{\eqsim}^q(t)dt\right]^{1/q}. \tag{44}$$

Let us now express the integral terms of this bound. Using the change of variables $s = h(u)$, we have

$$\int_0^{+\infty} f_{\eqsim}^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{45}$$

Similarly, using the change of variables $t = k(v)$, we have

$$\int_0^{+\infty} g_{\eqsim}^q(t)dt = \int_0^{+\infty}\left\{g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}\right\}^q dt = \int_c^d g^q(v)\frac{1}{[k'(v)]^q}k'(v)dv$$
$$= \int_c^d g^q(v)\frac{1}{[k'(v)]^{q-1}}dv. \tag{46}$$

Combining Equations (43), (44), (45) and (46), we obtain

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]F(x)G(y)h'(x)k'(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q}.$$

Theorem 2.12 is proved. □

Proof of Theorem 2.13. Using the changes of variables $s=h(x)$ and $t=k(y)$, so that $x=h^{-1}(s)$ and $y=k^{-1}(t)$, and $dx=[1/\{h'[h^{-1}(s)]\}]ds$ and $dy=[1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \left\{\int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]F(x)h'(x)dx\right\}^p k'(y)dy$$
$$=\int_0^{+\infty}\left\{\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F[h^{-1}(s)]h'[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}ds\right\}^p\times$$
$$k'[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}dt$$
$$=\int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F_\circ(s)ds\right]^p dt, \tag{47}$$

where

$$F_\circ(s)=F[h^{-1}(s)].$$

Using the change of variables $r=h^{-1}(u)$, we have

$$F_\circ(s)=F[h^{-1}(s)]=\int_a^{h^{-1}(s)} f(r)dr=\int_0^s f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}du=\int_0^s f_\circ(u)du,$$

where

$$f_\circ(u)=f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}.$$

Applying Theorem 2.4 to the function $f_\circ$, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|s-t|^{\alpha-1}}{(s+t)^{\alpha+1}}\log^{\beta-1}\left(\frac{s+t}{|s-t|}\right)F_\circ(s)ds\right]^p dt$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f_\circ^p(s)ds. \tag{48}$$

Let us now determine the integral of this bound. Using the change of variables $s=h(u)$, we have

$$\int_0^{+\infty} f_\circ^p(s)ds=\int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds=\int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$=\int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{49}$$

Combining Equations (47), (48) and (49), we obtain

$$\int_c^d \left\{\int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|h(x)-k(y)|^{\alpha-1}}{[h(x)+k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{h(x)+k(y)}{|h(x)-k(y)|}\right]F(x)h'(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{[h'(x)]^{p-1}}dx.$$

Theorem 2.13 is established. □

4.3 Proofs of the second contributions

Proof of Theorem 3.1.
Main integral inequality. By a suitable product decomposition of the integrand via the equality $1/p + 1/q = 1/p + (p-1)/p = 1$ and the Hölder integral inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}\frac{|1-xy|^{(\alpha-1)/p}}{(1+xy)^{(\alpha+1)/p}}\log^{(\beta-1)/p}\left(\frac{1+xy}{|1-xy|}\right)f(x)\times\\
&y^{1/q}\frac{|1-xy|^{(\alpha-1)/q}}{(1+xy)^{(\alpha+1)/q}}\log^{(\beta-1)/q}\left(\frac{1+xy}{|1-xy|}\right)g(y)dxdy\\
&\le\left[\int_0^{+\infty}\int_0^{+\infty} x\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f^p(x)dxdy\right]^{1/p}\times\\
&\left[\int_0^{+\infty}\int_0^{+\infty} y\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)g^q(y)dxdy\right]^{1/q}. \qquad (50)
\end{aligned}$$

We need to express each double integral term of this bound. It follows from the Fubini-Tonelli integral theorem (ensuring the exchange of the order of integration), the change of variables $s = xy$ and Proposition 4.1, that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f^p(x)dxdy\\
&=\int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)xdy\right]dx\\
&=\int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)ds\right]dx\\
&=\int_0^{+\infty} f^p(x)\times\frac{1}{\alpha^\beta}\Gamma(\beta)dx=\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} f^p(x)dx. \qquad (51)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} y\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)g^q(y)dxdy\\
&=\int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)ydx\right]dy\\
&=\int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)ds\right]dy\\
&=\int_0^{+\infty} g^p(y)\times\frac{1}{\alpha^\beta}\Gamma(\beta)dy=\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} g^p(y)dy. \qquad (52)
\end{aligned}$$

Combining Equations (50), (51) and (52), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&\le\left[\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\frac{1}{\alpha^\beta}\Gamma(\beta)\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&=\frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Optimality of the constant in the factor. We reason by contradiction by assuming the existence of a better constant than $\Gamma(\beta)/\alpha^{\beta}$. So we consider a constant C such that $C \in (0, \Gamma(\beta)/\alpha^{\beta})$ and, for any $f, g : [0, +\infty) \mapsto [0, +\infty)$,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\ &\leq C\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \end{aligned} \tag{53}$$

provided that the two integrals involved in the upper bound converge. For the candidate functions of the contradiction, we consider $f_{>} : [0, +\infty) \mapsto [0, +\infty)$ defined by

$$f_{>}(x) = \begin{cases} 0 & \text{if } x \in \{0\} \cup [1, +\infty) \\ x^{(1/n-1)/p} & \text{if } x \in (0, 1) \end{cases}$$

and $g_{>} : [0, +\infty) \mapsto [0, +\infty)$ defined by

$$g_{>}(y) = \begin{cases} y^{-(1+1/n)/q} & \text{if } y \in [1, +\infty) \\ 0 & \text{if } y \in [0, 1) \end{cases},$$

where $n \in \mathbb{N}\backslash\{0\}$. Note that

$$\int_0^{+\infty} f_{>}^p(x)dx = \int_0^1 (x^{(1/n-1)/p})^p dx = \left[nx^{1/n}\right]_{x=0}^{x=1} = n$$

and

$$\int_0^{+\infty} g_{>}^q(y)dy = \int_1^{+\infty} (y^{-(1+1/n)/q})^q dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty} = n.$$

It follows from the equality $1/p + 1/q = 1$ and Equation (53) that

$$\begin{aligned}C &= C\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{C\left[\int_0^{+\infty} f_{>}^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_{>}^q(y)dy\right]^{1/q}\right\}\\ &\geq \frac{1}{n}\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_{>}(x)g_{>}(y)dxdy. \end{aligned} \tag{54}$$

We need to express the double integral term of this bound. Using the definitions of $f_{>}$ and $g_{>}$, the change of variables $s = xy$, the Fubini-Tonelli integral theorem and the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_{>}(x)g_{>}(y)dxdy\\ &= \int_1^{+\infty}\int_0^1 x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)x^{(1/n-1)/p}y^{-(1+1/n)/q}dxdy\\ &= \int_1^{+\infty}\left[\int_0^1 \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)x^{1/(np)}dx\right]y^{-1/(nq)}dy\\ &= \int_1^{+\infty}\left[\int_0^y \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)s^{1/(np)}y^{-1/(np)}\frac{1}{y}ds\right]y^{-1/(nq)}dy\\ &= \int_1^{+\infty}\left[\int_0^y \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+s}{|1-s|}\right)s^{1/(np)}ds\right]y^{-(1+1/n)}dy. \end{aligned} \tag{55}$$

It follows from the Chasles integral formula at the threshold value $s = 1$, the Fubini-Tonelli integral theorem and the equality $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_1^{+\infty} \left[\int_0^y \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds\right] y^{-(1+1/n)} dy \\
&= \int_1^{+\infty} \left[\int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds\right] y^{-(1+1/n)} dy \\
&+ \int_1^{+\infty} \left[\int_1^y \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds\right] y^{-(1+1/n)} dy \\
&= \left[\int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds\right] \left[\int_1^{+\infty} y^{-(1+1/n)} dy\right] \\
&+ \int_1^{+\infty} \left[\int_s^{+\infty} y^{-(1+1/n)} dy\right] \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds \\
&= n\left[\int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds\right] \\
&+ \int_1^{+\infty} (ns^{-1/n}) \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds \\
&= n\Bigg[\int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(nq)} ds\Bigg]. \qquad (56)
\end{aligned}$$

Combining Equations (54), (55) and (56), we get

$$\begin{aligned}
C &\geq \int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(nq)} ds.
\end{aligned}$$

Since this inequality is valid for any $n \in \mathbb{N}\backslash\{0\}$, we can take the inferior limit with respect to the integer n. By virtue of the Fatou integral lemma, which is possible because the integrand is non-negative, $\underline{\lim}_{n\to+\infty} s^{1/(np)} = 1$ for $s \in (0,1)$, $\underline{\lim}_{n\to+\infty} s^{-1/(nq)} = 1$ for $s \in [1,+\infty)$, the Chasles integral formula at the threshold value $s = 1$ and Proposition 4.1, we obtain

$$\begin{aligned}
C &\geq \underline{\lim}_{n\to+\infty} \int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{1/(np)} ds \\
&+ \underline{\lim}_{n\to+\infty} \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) s^{-1/(nq)} ds \\
&\geq \int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) \left[\underline{\lim}_{n\to+\infty} s^{1/(np)}\right] ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) \left[\underline{\lim}_{n\to+\infty} s^{-1/(nq)}\right] ds \\
&= \int_0^1 \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds \\
&+ \int_1^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds \\
&= \int_0^{+\infty} \frac{|1-s|^{\alpha-1}}{(1+s)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+s}{|1-s|}\right) ds = \frac{1}{\alpha^\beta}\Gamma(\beta).
\end{aligned}$$

A contradiction comes since C is assumed to satisfy $C < \Gamma(\beta)/\alpha^{\beta}$. As a result, the constant in the factor $\Gamma(\beta)/\alpha^{\beta}$ is optimal. Theorem 3.1 is proved. □

Proof of Theorem 3.2.

Proof that Theorem 3.1 implies the presented inequality. We assume that the inequality in Theorem 3.1 holds. It follows from the Fubini-Tonelli integral theorem and a suitable decomposition of the integrand that

$$\begin{aligned}
&\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^p dy \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right] \times \\
&\left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^{p-1} dy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x) \times \\
&\left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^{p-1} dxdy \\
&= \int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x) g_{\ddagger}(y) dxdy,
\end{aligned} \tag{57}$$

where

$$g_{\ddagger}(y) = \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^{p-1}.$$

Applying Theorem 3.1 to the functions f and $g_{\ddagger}$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x) g_{\ddagger}(y) dxdy \\
&\leq \frac{1}{\alpha^{\beta}} \Gamma(\beta) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g_{\ddagger}^q(y)dy\right]^{1/q}.
\end{aligned} \tag{58}$$

Let us now determine the second integral of this bound. Since $q(p-1) = p$, we have

$$\begin{aligned}
&\int_0^{+\infty} g_{\ddagger}^q(y)dy = \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^{q(p-1)} dy \\
&= \int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^p dy.
\end{aligned} \tag{59}$$

Combining Equations (57), (58) and (59), we have

$$\begin{aligned}
&\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^p dy \\
&\leq \frac{1}{\alpha^{\beta}} \Gamma(\beta) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \times \\
&\left\{\int_0^{+\infty} \left[\int_0^{+\infty} x^{1/p} y^{1/q} \frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}} \log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right) f(x)dx\right]^p dy\right\}^{1/q}.
\end{aligned}$$

We therefore get

$$\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)dx\right]^p dy\right\}^{1-1/q}$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}.$$

Using the equality $1/p+1/q=1$, this is equivalent to

$$\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)dx\right]^p dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.$$

The desired inequality is established.

Proof that the presented inequality implies Theorem 3.1. We now assume that the presented inequality holds, and show that it implies Theorem 3.1. It follows from the Fubini-Tonelli integral theorem, the Hölder integral inequality only with respect to the variable y, and the assumed integral inequality that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)dx\right]g(y)dy$$
$$\leq\left\{\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)dx\right]^p dy\right\}^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$\leq\left[\frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$=\frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

The desired inequality is established.

The proof of Theorem 3.2 ends. □

Proof of Theorem 3.3. Let us notice that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_{\diamond}(x)g_{\diamond}(y)dxdy, \tag{60}$$

where

$$f_{\diamond}(x)=\frac{1}{x}F(x),\quad g_{\diamond}(y)=\frac{1}{y}G(y).$$

Applying Theorem 3.1 to the functions $f_\diamond$ and $g_\diamond$, the Hardy integral inequality to the functions f and g, and the equality $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_\diamond(x)g_\diamond(y)dxdy\\
&\le \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f_\diamond^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\diamond^q(y)dy\right]^{1/q}\\
&= \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} \frac{1}{x^p}F^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y^q}G^q(y)dy\right]^{1/q}\\
&\le \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\left(\frac{q}{q-1}\right)^q\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\left(\frac{p}{p-1}\right)^p\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[p^q\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}\tag{61}$$

Combining Equations (60) and (61), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy\\
&\le \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Theorem 3.3 is demonstrated. □

Proof of Theorem 3.4. Let us notice that

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)dx\right]^p dy\\
&= \int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_\diamond(x)dx\right]^p dy,
\end{aligned}\tag{62}$$

where

$$f_\diamond(x) = \frac{1}{x}F(x).$$

Applying Theorem 3.2 to the function $f_\diamond$ and using the Hardy integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f_\diamond(x)dx\right]^p dy\\
&\le \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f_\diamond^p(x)dx = \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\\
&\le \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.
\end{aligned}\tag{63}$$

Combining Equations (62) and (63), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\left[\int_0^{+\infty} x^{1/p-1}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)dx\right]^p dy\\
&\le \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)dx.
\end{aligned}$$

Theorem 3.4 is proved. □

Proof of Theorem 3.5. It follows from Theorem 3.1 that

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Integrating both sides with respect to the parameter α treated as a variable with $\alpha \in (1,\delta)$, and developing only the upper bound by taking into account that $\beta > 1$, we have

$$\int_1^{\delta}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]d\alpha$$
$$\leq \Gamma(\beta)\left[\int_1^{\delta}\frac{1}{\alpha^{\beta}}d\alpha\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \Gamma(\beta)\left[\frac{1}{\beta-1}(1-\delta^{1-\beta})\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{64}$$

For the main triple integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the order of integration) and basic integral calculus, we obtain

$$\int_1^{\delta}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]d\alpha$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[\int_1^{\delta}\left(\frac{|1-xy|}{1+xy}\right)^{\alpha-1}d\alpha\right]\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[-\frac{1}{\log[(1+xy)/|1-xy|]}\left(\frac{|1-xy|}{1+xy}\right)^{\alpha-1}\right]_{\alpha=1}^{\alpha=\delta}\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy. \tag{65}$$

Combining Equations (64) and (65), we get

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)^2}\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\delta-1}\right]\log^{\beta-2}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\beta-1}(1-\delta^{1-\beta})\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 3.5 is established. □

Proof of Theorem 3.6. Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = \epsilon i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $i\gamma^i$, summing both sides with respect to the integer i and developing only the upper bound using the Lerch transcendent function, we have

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\gamma^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon i-1}}{(1+xy)^{\epsilon i+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]\\
&\leq \left[\sum_{i=1}^{+\infty} i\gamma^i \frac{1}{(\epsilon i)^\beta}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\epsilon^\beta}\Gamma(\beta)\left[\sum_{i=1}^{+\infty}\frac{\gamma^i}{i^{\beta-1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \frac{1}{\epsilon^\beta}\Gamma(\beta)\gamma\left[\sum_{i=0}^{+\infty}\frac{\gamma^i}{(i+1)^{\beta-1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\\
&= \gamma\frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \qquad (66)
\end{aligned}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and a basic geometric series formula based on $\gamma|1-xy|^\epsilon/(1+xy)^\epsilon \in (0,1)$ (almost everywhere), we get

$$\begin{aligned}
&\sum_{i=1}^{+\infty} i\gamma^i \left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon i-1}}{(1+xy)^{\epsilon i+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{\sum_{i=1}^{+\infty} i\left[\gamma\left(\frac{|1-xy|}{1+xy}\right)^\epsilon\right]^i\right\}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)\times\\
&f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\gamma\left(\frac{|1-xy|}{1+xy}\right)^\epsilon\left\{\sum_{i=1}^{+\infty} i\left[\gamma\left(\frac{|1-xy|}{1+xy}\right)^\epsilon\right]^{i-1}\right\}\times\\
&\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&= \gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left(\frac{|1-xy|}{1+xy}\right)^\epsilon\times\\
&\frac{1}{[1-\gamma|1-xy|^\epsilon/(1+xy)^\epsilon]^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&= \gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon-1}(1+xy)^{\epsilon-1}}{[(1+xy)^\epsilon-\gamma|1-xy|^\epsilon]^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy. \qquad (67)
\end{aligned}$$

Combining Equations (66) and (67), we obtain

$$\begin{aligned}
&\gamma\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon-1}(1+xy)^{\epsilon-1}}{[(1+xy)^\epsilon-\gamma|1-xy|^\epsilon]^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&\leq \gamma\frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\epsilon-1}(1+xy)^{\epsilon-1}}{[(1+xy)^\epsilon-\gamma|1-xy|^\epsilon]^2}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\\
&\leq \frac{1}{\epsilon^\beta}\Gamma(\beta)\Phi(\gamma,\beta-1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Theorem 3.6 is proved. □

Proof of Theorem 3.7. By virtue of Theorem 3.1, we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = i+\kappa$ with $i \in \mathbb{N}$, which is possible because $\kappa > 0$, multiplying both sides by η^i, summing both sides with respect to the integer i and developing only the upper bound using the Lerch transcendent function, we have

$$\sum_{i=0}^{+\infty}\eta^i\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{i+\kappa-1}}{(1+xy)^{i+\kappa+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]$$
$$\leq \left[\sum_{i=0}^{+\infty}\eta^i\frac{1}{(i+\kappa)^\beta}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$=\Gamma(\beta)\left[\sum_{i=0}^{+\infty}\frac{\eta^i}{(i+\kappa)^\beta}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$=\Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \quad (68)$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and a basic geometric series formula based on $\eta|1-xy|/(1+xy) \in (0,1)$ (almost everywhere), we get

$$\sum_{i=0}^{+\infty}\eta^i\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{i+\kappa-1}}{(1+xy)^{i+\kappa+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]$$
$$=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa+1}}\left\{\sum_{i=0}^{+\infty}\left[\eta\left(\frac{|1-xy|}{1+xy}\right)\right]^i\right\}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa+1}}\times\frac{1}{1-\eta|1-xy|/(1+xy)}\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa}(1+xy-\eta|1-xy|)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy. \quad (69)$$

Combining Equations (68) and (69), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\kappa-1}}{(1+xy)^{\kappa}(1+xy-\eta|1-xy|)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \Gamma(\beta)\Phi(\eta,\beta,\kappa)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 3.7 is demonstrated. □

Proof of Theorem 3.8. Using Theorem 3.1, we have

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. Taking $\alpha = \rho i$ with $i \in \mathbb{N}\backslash\{0\}$, multiplying both sides by $1/i$, summing both sides with respect to the integer i and developing the upper bound using the Lerch transcendent function, we obtain

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\rho i-1}}{(1+xy)^{\rho i+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]$$
$$\leq \left[\sum_{i=1}^{+\infty}\frac{1}{i}\frac{1}{(\rho i)^{\beta}}\Gamma(\beta)\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^{\beta}}\Gamma(\beta)\left[\sum_{i=1}^{+\infty}\frac{1}{i^{\beta+1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^{\beta}}\Gamma(\beta)\left[\sum_{i=0}^{+\infty}\frac{1}{(i+1)^{\beta+1}}\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\rho^{\beta}}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{70}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and a basic logarithmic series formula based on $|1-xy|^{\rho}/(1+xy)^{\rho} \in (0,1)$ (almost everywhere), we get

$$\sum_{i=1}^{+\infty}\frac{1}{i}\left[\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\rho i-1}}{(1+xy)^{\rho i+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{\sum_{i=1}^{+\infty}\frac{1}{i}\left[\left(\frac{|1-xy|}{1+xy}\right)^{\rho}\right]^{i}\right\}\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\left\{-\log\left[1-\left(\frac{|1-xy|}{1+xy}\right)^{\rho}\right]\right\}\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{1}{1-|1-xy|^{\rho}/(1+xy)^{\rho}}\right]\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\rho}}{(1+xy)^{\rho}-|1-xy|^{\rho}}\right]\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy. \tag{71}$$

Combining Equations (70) and (71), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{1}{(1+xy)|1-xy|}\log\left[\frac{(1+xy)^{\rho}}{(1+xy)^{\rho}-|1-xy|^{\rho}}\right]\times$$
$$\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\rho^{\beta}}\Gamma(\beta)\Phi(1,\beta+1,1)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

Theorem 3.8 is proved. □

Proof of Theorem 3.9. Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$ and $\beta > 0$. This can also be written as follows:

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left[\frac{1}{\Gamma(\beta)}\log^{\beta-1}\left(\frac{1+xy}{|1-xy|}\right)\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^{\beta}}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

Taking $\beta = \theta i + \mu$ with $i \in \mathbb{N}$, which is possible because $\mu > 0$, multiplying both sides by ν^i, summing both sides with respect to the integer i and developing the upper bound using a basic geometric series formula taking into account that $\nu \in (0, \alpha^{\theta})$, we have

$$\sum_{i=0}^{+\infty}\nu^i\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left[\frac{1}{\Gamma(\theta i+\mu)}\log^{\theta i+\mu-1}\left(\frac{1+xy}{|1-xy|}\right)\right]f(x)g(y)dxdy$$
$$\leq \left[\sum_{i=0}^{+\infty}\nu^i\frac{1}{\alpha^{\theta i+\mu}}\right]\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha^{\mu}}\left[\sum_{i=0}^{+\infty}\left(\frac{\nu}{\alpha^{\theta}}\right)^i\right]\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$
$$= \frac{1}{\alpha^{\mu}}\left(\frac{1}{1-\nu/\alpha^{\theta}}\right)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$
$$= \frac{\alpha^{\theta-\mu}}{\alpha^{\theta}-\nu}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}. \tag{72}$$

For the main series-double integral term, using the Fubini-Tonelli integral theorem (ensuring the exchange of the sum and integrals) and the two-parameter Mittag-Leffler function, we have

$$\sum_{i=0}^{+\infty}\nu^i\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left[\frac{1}{\Gamma(\theta i+\mu)}\log^{\theta i+\mu-1}\left(\frac{1+xy}{|1-xy|}\right)\right]f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\left\{\sum_{i=0}^{+\infty}\frac{1}{\Gamma(\theta i+\mu)}\left[\nu\log^{\theta}\left(\frac{1+xy}{|1-xy|}\right)\right]^i\right\}\times$$
$$\log^{\mu-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\Psi\left[\nu\log^{\theta}\left(\frac{1+xy}{|1-xy|}\right), \theta, \mu\right]\times$$
$$\log^{\mu-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy. \tag{73}$$

Combining Equations (72) and (73), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|1-xy|^{\alpha-1}}{(1+xy)^{\alpha+1}}\Psi\left[\nu\log^{\theta}\left(\frac{1+xy}{|1-xy|}\right), \theta, \mu\right]\times$$
$$\log^{\mu-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq \frac{\alpha^{\theta-\mu}}{\alpha^{\theta}-\nu}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

Theorem 3.9 is established. □

Proof of Theorem 3.10. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f[h^{-1}(s)]g[k^{-1}(t)]\frac{1}{h'[h^{-1}(s)]}\frac{1}{k'[k^{-1}(t)]}dsdt$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f_{\circeq}(s)g_{\circeq}(t)dsdt, \tag{74}$$

where

$$f_{\circeq}(s) = f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}, \quad g_{\circeq}(t) = g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}.$$

Applying Theorem 3.1 to the functions $f_{\circeq}$ and $g_{\circeq}$, we find that

$$\int_0^{+\infty}\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f_{\circeq}(s)g_{\circeq}(t)dsdt$$
$$\leq \frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty} f_{\circeq}^p(s)ds\right]^{1/p}\left[\int_0^{+\infty} g_{\circeq}^q(t)dt\right]^{1/q}. \tag{75}$$

Let us now express the integral terms of this bound. Using the change of variables $s = h(u)$, we have

$$\int_0^{+\infty} f_{\circeq}^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{76}$$

Similarly, using the change of variables $t = k(v)$, we have

$$\int_0^{+\infty} g_{\circeq}^q(t)dt = \int_0^{+\infty}\left\{g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}\right\}^q dt = \int_c^d g^q(v)\frac{1}{[k'(v)]^q}k'(v)dv$$
$$= \int_c^d g^q(v)\frac{1}{[k'(v)]^{q-1}}dv. \tag{77}$$

Combining Equations (74), (75), (76) and (77), we get

$$\int_c^d \int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha^\beta}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q}.$$

Theorem 3.10 is proved. □

Proof of Theorem 3.11. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$= \int_0^{+\infty}\left\{\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}ds\right\}^p \times$$
$$k'[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}dt$$
$$= \int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f_\approx(s)ds\right]^p dt, \tag{78}$$

where

$$f_\approx(s) = f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}.$$

Applying Theorem 3.2 to the function $f_\approx$, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)f_\approx(s)ds\right]^p dt$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_0^{+\infty} f_\approx^p(s)ds. \tag{79}$$

Let us now determine the integral of this bound. Using the change of variables $s = h(u)$, we have

$$\int_0^{+\infty} f_\approx^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{80}$$

Combining Equations (78), (79) and (80), we get

$$\int_c^d \left\{\int_a^b h^{1/p}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]f(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\Gamma^p(\beta)\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx.$$

Theorem 3.11 is established. □

Proof of Theorem 3.12. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]F(x)G(y)h'(x)k'(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F[h^{-1}(s)]G[k^{-1}(t)]\times$$
$$h'[h^{-1}(s)]k'[k^{-1}(t)]\frac{1}{h'[h^{-1}(s)]}\frac{1}{k'[k^{-1}(t)]}dsdt$$
$$= \int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F_{\eqsim}(s)G_{\eqsim}(t)dsdt, \tag{81}$$

where

$$F_{\eqsim}(s) = F[h^{-1}(s)], \quad G_{\eqsim}(t) = G[k^{-1}(t)].$$

Using the change of variables $r = h^{-1}(u)$, we have

$$F_{\eqsim}(s) = F[h^{-1}(s)] = \int_a^{h^{-1}(s)} f(r)dr = \int_0^s f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}du = \int_0^s f_{\eqsim}(u)du,$$

where

$$f_{\eqsim}(u) = f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}.$$

Similarly, using the change of variables $r = k^{-1}(v)$, we get

$$G_{\eqsim}(t) = G[k^{-1}(t)] = \int_c^{k^{-1}(t)} g(r)dr = \int_0^t g[k^{-1}(v)]\frac{1}{k'[k^{-1}(v)]}dv = \int_0^t g_{\eqsim}(v)dv,$$

where

$$g_{\eqsim}(v) = g[k^{-1}(v)]\frac{1}{k'[k^{-1}(v)]}.$$

Applying Theorem 3.3 to the functions $f_{\eqsim}$ and $g_{\eqsim}$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} s^{1/p-1}t^{1/q-1}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F_{\eqsim}(s)G_{\eqsim}(t)dsdt$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left[\int_0^{+\infty} f_{\eqsim}^p(s)ds\right]^{1/p}\left[\int_0^{+\infty} g_{\eqsim}^q(t)dt\right]^{1/q}. \tag{82}$$

Let us now express the integral terms of this bound. Using the change of variables $s = h(u)$, we have

$$\int_0^{+\infty} f_{\eqsim}^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{83}$$

Similarly, using the change of variables $t = k(v)$, we have

$$\int_0^{+\infty} g_{\eqsim}^q(t)dt = \int_0^{+\infty}\left\{g[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}\right\}^q dt = \int_c^d g^q(v)\frac{1}{[k'(v)]^q}k'(v)dv$$
$$= \int_c^d g^q(v)\frac{1}{[k'(v)]^{q-1}}dv. \tag{84}$$

Combining Equations (81), (82), (83) and (84), we obtain

$$\int_c^d \int_a^b h^{1/p-1}(x)k^{1/q-1}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]F(x)G(y)h'(x)k'(y)dxdy$$
$$\leq \frac{p^2}{\alpha^\beta(p-1)}\Gamma(\beta)\left\{\int_a^b f^p(x)\frac{1}{[h'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d g^q(y)\frac{1}{[k'(y)]^{q-1}}dy\right\}^{1/q}.$$

Theorem 3.12 is demonstrated. □

Proof of Theorem 3.13. Using the changes of variables $s = h(x)$ and $t = k(y)$, so that $x = h^{-1}(s)$ and $y = k^{-1}(t)$, and $dx = [1/\{h'[h^{-1}(s)]\}]ds$ and $dy = [1/\{k'[k^{-1}(t)]\}]dt$, we obtain

$$\int_c^d \left\{\int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]F(x)h'(x)dx\right\}^p k'(y)dy$$
$$= \int_0^{+\infty}\left\{\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F[h^{-1}(s)]h'[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}ds\right\}^p \times$$
$$k'[k^{-1}(t)]\frac{1}{k'[k^{-1}(t)]}dt$$
$$= \int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F_\circ(s)ds\right]^p dt, \tag{85}$$

where

$$F_\circ(s) = F[h^{-1}(s)].$$

Using the change of variables $r = h^{-1}(u)$, we have

$$F_\circ(s) = F[h^{-1}(s)] = \int_a^{h^{-1}(s)} f(r)dr = \int_0^s f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}du = \int_0^s f_\circ(u)du,$$

where

$$f_\circ(u) = f[h^{-1}(u)]\frac{1}{h'[h^{-1}(u)]}.$$

Applying Theorem 3.4 to the function $f_\circ$, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty} s^{1/p-1}t^{1/q}\frac{|1-st|^{\alpha-1}}{(1+st)^{\alpha+1}}\log^{\beta-1}\left(\frac{1+st}{|1-st|}\right)F_\circ(s)ds\right]^p dt$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f_\circ^p(s)ds. \tag{86}$$

Let us now determine the integral of this bound. Using the change of variables $s = h(u)$, we find that

$$\int_0^{+\infty} f_\circ^p(s)ds = \int_0^{+\infty}\left\{f[h^{-1}(s)]\frac{1}{h'[h^{-1}(s)]}\right\}^p ds = \int_a^b f^p(u)\frac{1}{[h'(u)]^p}h'(u)du$$
$$= \int_a^b f^p(u)\frac{1}{[h'(u)]^{p-1}}du. \tag{87}$$

Combining Equations (85), (86) and (87), we obtain

$$\int_c^d \left\{\int_a^b h^{1/p-1}(x)k^{1/q}(y)\frac{|1-h(x)k(y)|^{\alpha-1}}{[1+h(x)k(y)]^{\alpha+1}}\log^{\beta-1}\left[\frac{1+h(x)k(y)}{|1-h(x)k(y)|}\right]F(x)h'(x)dx\right\}^p k'(y)dy$$
$$\leq \frac{1}{\alpha^{\beta p}}\left(\frac{p}{p-1}\right)^p\Gamma^p(\beta)\int_0^{+\infty} f^p(x)\frac{1}{[h'(x)]^{p-1}}dx.$$

Theorem 3.13 is proved. □

Proof of Theorem 3.14. We can write

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\left[x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)\right]^{\lambda}\times$$
$$\left[x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{\theta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)\right]^{1-\lambda}dxdy. \tag{88}$$

It follows from the Hölder integral inequality applied to the parameter $1/\lambda > 1$, and Theorems 2.1 and 3.1 with a suitable definition of the parameters that

$$\int_0^{+\infty}\int_0^{+\infty}\left[x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)\right]^{\lambda}\times$$
$$\left[x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)\right]^{1-\lambda}dxdy$$
$$\leq\left[\int_0^{+\infty}\int_0^{+\infty}x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]^{\lambda}\times$$
$$\left[\int_0^{+\infty}\int_0^{+\infty}x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{\theta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]^{1-\lambda}$$
$$\leq\left\{\frac{1}{\alpha^{\beta}}\Gamma(\beta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{\lambda}\times$$
$$\left\{\frac{1}{\gamma^{\theta}}\Gamma(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{1-\lambda}$$
$$=\frac{1}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}. \tag{89}$$

Combining Equations (88) and (89), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p}y^{1/q}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy$$
$$\leq\frac{1}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

Theorem 3.14 is established. □

Proof of Theorem 3.15. We can write

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\left[x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)F(x)G(y)\right]^{\lambda}\times$$
$$\left[x^{1/p-1}y^{1/q-1}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{\theta-1}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)\right]^{1-\lambda}dxdy. \tag{90}$$

It follows from the Hölder integral inequality applied to the parameter $1/\lambda > 1$, and Theorems 2.3 and 3.3 with a suitable definition of the parameters that

$$\int_0^{+\infty}\int_0^{+\infty}\left[x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)\right]^{\lambda}\times$$
$$\left[x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)\right]^{1-\lambda}dxdy$$
$$\leq\left[\int_0^{+\infty}\int_0^{+\infty}x^{1/p}y^{1/q}\frac{|x-y|^{\alpha-1}}{(x+y)^{\alpha+1}}\log^{\beta-1}\left(\frac{x+y}{|x-y|}\right)f(x)g(y)dxdy\right]^{\lambda}\times$$
$$\left[\int_0^{+\infty}\int_0^{+\infty}x^{1/p}y^{1/q}\frac{|1-xy|^{\gamma-1}}{(1+xy)^{\gamma+1}}\log^{\theta-1}\left(\frac{1+xy}{|1-xy|}\right)f(x)g(y)dxdy\right]^{1-\lambda}$$
$$\leq\left\{\frac{p^2}{\alpha^{\beta}(p-1)}\Gamma(\beta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{\lambda}\times$$
$$\left\{\frac{p^2}{\gamma^{\theta}(p-1)}\Gamma(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{1-\lambda}$$
$$=\frac{p^2}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}(p-1)}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}. \tag{91}$$

Combining Equations (90) and (91), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} x^{1/p-1}y^{1/q-1}\frac{|x-y|^{\lambda(\alpha-1)}}{(x+y)^{\lambda(\alpha+1)}}\frac{|1-xy|^{(1-\lambda)(\gamma-1)}}{(1+xy)^{(1-\lambda)(\gamma+1)}}\log^{\lambda(\beta-1)}\left(\frac{x+y}{|x-y|}\right)\times$$
$$\log^{(1-\lambda)(\theta-1)}\left(\frac{1+xy}{|1-xy|}\right)F(x)G(y)dxdy$$
$$\leq\frac{p^2}{\alpha^{\lambda\beta}\gamma^{(1-\lambda)\theta}(p-1)}\Gamma^{\lambda}(\beta)\Gamma^{1-\lambda}(\theta)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

Theorem 3.15 is proved. □

5 Conclusion and perspectives

In conclusion, this article presents a generalized perspective on the Hardy-Hilbert integral inequality by incorporating parametric power and power-logarithmic functions. The proposed modifications have the advantage of optimizing the associated constants in the factor, while maintaining the tractability necessary for further mathematical exploration. The proofs are fully detailed, innovating on some technical aspects through the use of integral and power series with respect to the parameters involved. Furthermore, by considering special functions and mixed approaches, our results provide a broader framework for future research, including potential applications in functional analysis, operator theory and related mathematical fields. A logical continuation of this article is the exploration of three-dimensional versions of our theorems. The addition of a dimensionality complicates the situation, requiring more effort and new techniques, which we plan to develop for a future article.

Conflicts of interest: The author declares that he has no competing interests.

Funding: The author has not received any funding.

References

[1] V. Adiyasuren, T. Batbold, M. Krnić, Multiple Hilbert-type inequalities involving some differential operators, Banach J. Math. Anal. 10 (2016), 320–337.

[2] V. Adiyasuren, T. Batbold, M. Krnić, Hilbert-type inequalities involving differential operators, the best constants and applications, Math. Inequal. Appl. 18 (2015), 111–124.

[3] T.M. Apostol, Lerch's transcendent, in F.W.J. Olver, D.M. Lozier, R.F. Boisvert, C.W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge Univ. Press, 2010.

[4] L.E. Azar, The connection between Hilbert and Hardy inequalities, J. Inequal. Appl. 2013 (2013), 1–10.

[5] T. Batbold, Y. Sawano, Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces, Math. Inequal. Appl. 20 (2017), 263–283.

[6] A. Bényi, C.T. Oh, Best constant for certain multilinear integral operator, J. Inequal. Appl. 2006 (2006), 1–12.

[7] Q. Chen, B.C. Yang, A survey on the study of Hilbert-type inequalities, J. Inequal. Appl. 2015 (2015), 1–29.

[8] C. Chesneau, Some four-parameter trigonometric generalizations of the Hilbert integral inequality, Asia Math. 8 (2024), 45–59.

[9] C. Chesneau, General inequalities of the Hilbert integral type using the method of switching to polar coordinates, Hilbert J. Math. Anal. 3 (2024), 7–26.

[10] C. Chesneau, Contributions to the fractional Hardy integral inequality, Univ. J. Math. Appl. 8 (2025), 21–29.

[11] C. Chesneau, Study of two three-parameter non-homogeneous variants of the Hilbert integral inequality, Lobachevskii J. Math. 45 (2024), 4931–4953.

[12] C. Chesneau, Study of several new variations of the Hardy-Hilbert integral inequality, Pan-Am. J. Math. 4 (2025), 1–40.

[13] H. Du, Y. Miao, Several new Hardy-Hilbert's inequalities, Filomat 25 (2011), 153–162.

[14] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, The two-parametric Mittag-Leffler function, in R. Gorenflo et al., Mittag-Leffler Functions, Related Topics and Applications, Springer Monographs in Mathematics, Springer, Berlin, 2014.

[15] G.H. Hardy, Note on a theorem of Hilbert concerning series of positive terms, Proc. Lond. Math. Soc. 23 (1925), 45–46.

[16] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Univ. Press, Cambridge, 1934.

[17] Y. Hong, On multiple Hardy-Hilbert integral inequalities with some parameters, J. Inequal. Appl. 2006 (2006), 1–11.

[18] Z. Huang, B.C. Yang, A multidimensional Hilbert-type integral inequality, J. Inequal. Appl. 2015 (2015), 1–13.

[19] S.W. Jian, F.Z. Yang, All-sided generalization about Hardy-Hilbert integral inequalities, Acta Math. Sin. (Chin. Ser.) 44 (2001), 619–626.

[20] Y. Li, Y. Qian, B. He, On further analogs of Hilbert's inequality, Int. J. Math. Math. Sci. 2007 (2007), 1–6.

[21] W.T. Sulaiman, On Hardy-Hilbert's integral inequality, J. Inequal. Pure Appl. Math. 5 (2004), 1–9.

[22] W.T. Sulaiman, On three inequalities similar to Hardy-Hilbert's integral inequality, Acta Math. Univ. Comenianae 76 (2007), 273–278.

[23] W.T. Sulaiman, A different type of Hardy-Hilbert's integral inequality, AIP Conf. Proc. 1046 (2008), 135–141.

[24] W.T. Sulaiman, General new forms of Hardy-Hilbert's integral inequality via new ideas, Int. J. Contemp. Math. Sci. 3(22) (2008), 1059–1067.

[25] W.T. Sulaiman, New types of Hardy-Hilbert's integral inequality, Gen. Math. Notes 2(2) (2011), 111–118.

[26] B. Sun, A multiple Hilbert-type integral inequality with the best constant factor, J. Inequal. Appl. 2007 (2007), 1–14.

[27] D.C. Ullrich, A simple elementary proof of Hilbert's inequality, Am. Math. Mon. 120 (2013), 161–164.

[28] J.S. Xu, Hardy-Hilbert's inequalities with two parameters, Adv. Math. 36 (2007), 63–76.

[29] B.C. Yang, On Hilbert's integral inequality, J. Math. Anal. Appl. 220 (1998), 778–785.

[30] B.C. Yang, On Hardy-Hilbert's integral inequality, J. Math. Anal. Appl. 261 (2001), 295–306.

[31] B.C. Yang, A multiple Hardy-Hilbert integral inequality, Chin. Ann. Math. Ser. A 24 (2003), 743–750.

[32] B.C. Yang, On the norm of an integral operator and applications, J. Math. Anal. Appl. 321 (2006), 182–192.

[33] B.C. Yang, On the norm of a Hilbert's type linear operator and applications, J. Math. Anal. Appl. 325 (2007), 529–541.

[34] B.C. Yang, The norm of operator and Hilbert-type inequalities, Science Press, Beijing, 2009.

[35] B.C. Yang, Hilbert-type integral inequalities, Bentham Sci. Publ., UAE, 2009.

[36] B.C. Yang, M. Krnić, On the norm of a multi-dimensional Hilbert-type operator, Sarajevo J. Math. 7 (2011), 223–243.

[37] W.Y. Zhong, B.C. Yang, On a multiple Hilbert-type integral inequality with the symmetric kernel, J. Inequal. Appl. 2007 (2007), 1–17.