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Four New Theorems on Maximum-Hilbert-Type Integral Inequalities

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Abstract
This article presents four distinct theorems on maximum-Hilbert-type integral inequalities. Each theorem introduces innovative maximum-type kernel functions that are either non-symmetric or non-homogeneous, and establishes the corresponding inequalities with sharp constant factors. Comprehensive proofs are provided for all results. In a certain sense, this work builds on and extends the contributions of Huang and Yang in 2011.

Keywords: Non-symmetric kernel functions; Non-homogeneous kernel functions; Hilbert-type integral inequalities; Hölder integral inequality; Fubini-Tonelli integral theorem.

1 Introduction

Hilbert-type integral inequalities play a key role in mathematical analysis, particularly in the study of integral operators and their bounds. They arise naturally in various fields, including harmonic analysis, operator theory, and mathematical physics. Two important and frequently cited examples are presented below.

Classic Hilbert integral inequality: Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^2(x)dx < +\infty, \quad \int_0^{+\infty} g^2(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}. \tag{1}$$

The constant factor π is the best possible.

Classic maximum variation of the Hilbert integral inequality: Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^2(x)dx < +\infty, \quad \int_0^{+\infty} g^2(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{\max(x, y)} f(x)g(y)dxdy \leq 4 \sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}. \tag{2}$$

The constant factor 4 is the best possible.

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The kernel functions associated to these inequalities are

$$k_1(x,y)=\frac{1}{x+y},\quad k_2(x,y)=\frac{1}{\max(x,y)},$$

respectively. Since $k_1(x,y)=k_1(y,x)$ for any $x,y>0$, k_1 is symmetric, and since $k_1(\lambda x,\lambda y)=\lambda^{-1}k_1(x,y)$, it is homogeneous with degree -1. The same holds for k_2.

Over the years, numerous improvements, modifications, and generalizations of Hilbert-type integral inequalities have been developed. These extensions cover a broad spectrum of applications and involve various kernel functions, dimensions, and function spaces. For comprehensive accounts of these developments, we refer the reader to [? ? ? ?].

Significant attention has particularly been devoted to the maximum and minimum versions of Hilbert-type integral inequalities. These variations involve either $\max(x,y)$ or $\min(x,y)$. Progress in this area can be found in [1, 3, 4, 6? –13].

Of these contributions, two particular results are of central interest in the study of maximum-Hilbert-type integral inequalities. The first was established in [7], while the second was presented in [6]. For the sake of completeness, we state both results precisely below, naming them Results 1 and 2, respectively.

Result 1: [7, Theorem 2.2] Let $f,g:[0,+\infty)\to[0,+\infty)$ be two functions such that

$$\int_0^{+\infty}f^2(x)dx<+\infty,\quad \int_0^{+\infty}g^2(y)dy<+\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\max(x,y)}f(x)g(y)dxdy\le 2\sqrt{2}\arctan\left[\frac{1}{\sqrt{2}}\right]\times\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

The constant factor $2\sqrt{2}\arctan\left[1/\sqrt{2}\right]$ is the best possible. The integrand considered here combines the structures of those in Equations (1) and (2).

Result 2: [6, Theorems 1 and 2, and Remark 1] Let $f,g:[0,+\infty)\to[0,+\infty)$ be two functions such that

$$\int_0^{+\infty}f^2(x)dx<+\infty,\quad \int_0^{+\infty}g^2(y)dy<+\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\sqrt{xy}+\max(x,y)}f(x)g(y)dxdy\le 4\log(2)\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

The constant factor $4\log(2)$ is the best possible. The integrand considered here thus introduces the product $\sqrt{xy}$ to that in Equation (2).

The kernel functions associated with these inequalities are

$$k_3(x,y)=\frac{1}{x+y+\max(x,y)},\quad k_4(x,y)=\frac{1}{\sqrt{xy}+\max(x,y)},$$

respectively. Like k_1, and k_2, they are symmetric and homogeneous with degree -1.

Building on this theoretical framework, this article establishes four distinct theorems on maximum-Hilbert-type integral inequalities. Each theorem involves a new kernel function, specifically

$$k_5(x,y)=\frac{1}{x+\max(x,y)},\quad k_6(x,y)=\frac{1}{xy+\max(xy,1)},$$

$$k_7(x,y)=\frac{1}{x+\sqrt{xy}+\max(x,y)},\quad k_8(x,y)=\frac{1}{1+\sqrt{xy}+\max(xy,1)},$$

respectively. These kernel functions are either non-symmetric or non-homogeneous, distinguishing them from the classical kernel functions, k_1, k_2, k_3 and k_4, that have been studied previously in the literature. We establish the corresponding Hilbert-type integral inequality with a sharp constant factor for each of them. Detailed proofs are provided for all results. In this sense, the present work complements and extends the contributions made by Huang and Yang in [6].

The rest of the article is organized as follows: The four theorems are presented in Sections 2, 3, 4, and 5, respectively. A conclusion is provided in Section 6.

2 First theorem

The first maximum-Hilbert-type integral inequality is presented in the theorem below.

Theorem 2.1 *Let $p>1$, $q=p/(p-1)$, and $f,g:[0,+\infty)\to[0,+\infty)$ be two functions such that*

$$\int_0^{+\infty}x^{p/2-1}f(x)dx<+\infty,\quad \int_0^{+\infty}y^{q/2-1}g(y)dy<+\infty.$$

Then we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+\max(x,y)}f(x)g(y)dxdy\\&\le\left(\frac{\pi}{2}+1\right)\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Proof. Decomposing the integrand in a suitable manner and using the Hölder integral inequality, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+\max(x,y)}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}x^{-1/(2q)}y^{1/(2p)}y^{-1/(2p)}\frac{1}{x+\max(x,y)}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}y^{-1/(2p)}\frac{1}{[x+\max(x,y)]^{1/p}}f(x)\\&\times y^{1/(2p)}x^{-1/(2q)}\frac{1}{[x+\max(x,y)]^{1/q}}g(y)dxdy\\&\le A^{1/p}B^{1/q},\end{aligned}\tag{3}$$

where

$$A=\int_0^{+\infty}\int_0^{+\infty}x^{p/(2q)}y^{-1/2}\frac{1}{x+\max(x,y)}f^p(x)dxdy$$

and

$$B=\int_0^{+\infty}\int_0^{+\infty}y^{q/(2p)}x^{-1/2}\frac{1}{x+\max(x,y)}g^q(y)dxdy.$$

Let us now investigate A and B.

For A, by the Fubini-Tonelli integral theorem, we have

$$A = \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \left(\frac{y}{x}\right)^{-1/2} \frac{1}{1+\max(1, y/x)} \frac{1}{x} dy \right] dx.$$

For the central integral, making the change of variables $u = y/x$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \left(\frac{y}{x}\right)^{-1/2} \frac{1}{1+\max(1, y/x)} \frac{1}{x} dy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{u}(1+\max(1, u))} du \\
&= \int_0^1 \frac{1}{\sqrt{u}(1+1)} du + \int_1^{+\infty} \frac{1}{\sqrt{u}(1+u)} du \\
&= \left[\sqrt{u}\right]_{u=0}^{u=1} + \left[2\arctan[\sqrt{u}]\right]_{u=1}^{u\to+\infty} \\
&= 1 - 0 + 2 \times \frac{\pi}{2} - 2\arctan(1) = 1 + \pi - 2 \times \frac{\pi}{4} = \frac{\pi}{2} + 1.
\end{aligned}$$

This and $q = p/(p-1)$ yield

$$\begin{aligned}
A &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left(\frac{\pi}{2} + 1\right) dx \\
&= \left(\frac{\pi}{2} + 1\right) \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \qquad (4)
\end{aligned}$$

For B, by the Fubini-Tonelli integral theorem, we have

$$B = \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \left(\frac{x}{y}\right)^{-1/2} \frac{1}{x/y+\max(x/y, 1)} \frac{1}{y} dx \right] dy.$$

For the central integral, making the change of variables $v = x/y$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \left(\frac{x}{y}\right)^{-1/2} \frac{1}{x/y+\max(x/y, 1)} \frac{1}{y} dx \\
&= \int_0^{+\infty} \frac{1}{\sqrt{v}(v+\max(v, 1))} dv \\
&= \int_0^1 \frac{1}{\sqrt{v}(v+1)} dv + \int_1^{+\infty} \frac{1}{\sqrt{v}(v+v)} dv \\
&= \left[2\arctan[\sqrt{v}]\right]_{v=0}^{v=1} + \left[-v^{-1/2}\right]_{v=1}^{v\to+\infty} \\
&= 2\arctan(1) - 0 + 0 - (-1) = 2 \times \frac{\pi}{4} + 1 = \frac{\pi}{2} + 1.
\end{aligned}$$

This and $p = q/(q-1)$ give

$$\begin{aligned}
B &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left(\frac{\pi}{2} + 1\right) dy \\
&= \left(\frac{\pi}{2} + 1\right) \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (5)
\end{aligned}$$

It follows from Equations (3), (4) and (5) and $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+\max(x, y)} f(x) g(y) dx dy \\
&\le \left[\left(\frac{\pi}{2} + 1\right) \int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\left(\frac{\pi}{2} + 1\right) \int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q} \\
&\le \left(\frac{\pi}{2} + 1\right) \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This completes the proof. □

Following the procedure outlined in [6, Theorem 2], and considering the same extremal functions, we are able to prove that the constant factor $\pi/2 + 1$ is the best possible. The kernel function considered is thus

$$k_5(x, y) = \frac{1}{x + \max(x, y)},$$

which is obviously non-symmetric. However, it is homogeneous with degree -1. Its nature differs from that of the standard kernel functions, k_1, k_2, k_3 and k_4.

Of course the theory still holds with the following modified kernel function:

$$k_5^*(x, y) = \frac{1}{y + \max(x, y)},$$

giving the following integral inequality:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{y+\max(x,y)}f(x)g(y)dxdy\\&\le\left(\frac{\pi}{2}+1\right)\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

For the sake of redundancy, the details are omitted.

3 Second theorem

The second maximum-Hilbert-type integral inequality theorem is presented below.

Theorem 3.1 *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} x^{p/2-1}f(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1}g(y)dy < +\infty.$$

Then we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+\max(xy,\,1)}f(x)g(y)dxdy\\&\le\left(\frac{\pi}{2}+1\right)\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Proof. Decomposing the integrand in a suitable manner and using the Hölder integral inequality, we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+\max(xy,\,1)}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}x^{-1/(2q)}y^{1/(2p)}y^{-1/(2p)}\frac{1}{1+\max(xy,\,1)}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}y^{-1/(2p)}\frac{1}{[1+\max(xy,\,1)]^{1/p}}f(x)\\&\times y^{1/(2p)}x^{-1/(2q)}\frac{1}{[1+\max(xy,\,1)]^{1/q}}g(y)dxdy\\&\le C^{1/p}D^{1/q},\end{aligned}\tag{6}$$

where

$$C = \int_0^{+\infty}\int_0^{+\infty} x^{p/(2q)}y^{-1/2}\frac{1}{1+\max(xy,\,1)}f^p(x)dxdy$$

and

$$D = \int_0^{+\infty} \int_0^{+\infty} y^{q/(2p)} x^{-1/2} \frac{1}{1+\max(xy,\,1)} g^q(y) dx dy.$$

Let us now examine C and D.

For C, by the Fubini-Tonelli integral theorem, we have

$$C = \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\max(xy,\,1)]} x dy \right] dx.$$

For the central integral, making the change of variables $u = xy$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\max(xy,\,1)]} x dy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{u}[1+\max(u,\,1)]} du \\
&= \int_0^{1} \frac{1}{\sqrt{u}(1+1)} du + \int_1^{+\infty} \frac{1}{\sqrt{u}(1+u)} du \\
&= \left[\sqrt{u}\right]_{u=0}^{u=1} + \left[2\arctan[\sqrt{u}]\right]_{u=1}^{u\to+\infty} \\
&= 1 - 0 + 2 \times \frac{\pi}{2} - 2\arctan(1) = 1 + \pi - 2 \times \frac{\pi}{4} = \frac{\pi}{2} + 1.
\end{aligned}$$

This and $q = p/(p-1)$ yield

$$\begin{aligned}
C &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left(\frac{\pi}{2}+1\right) dx \\
&= \left(\frac{\pi}{2}+1\right) \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \qquad (7)
\end{aligned}$$

For D, by the Fubini-Tonelli integral theorem, we have

$$D = \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\max(xy,\,1)]} y dx \right] dy.$$

For the central integral, making the change of variables $v = xy$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\max(xy,\,1)]} y dx \\
&= \int_0^{+\infty} \frac{1}{\sqrt{v}[1+\max(v,\,1)]} dv \\
&= \int_0^{1} \frac{1}{\sqrt{v}(1+1)} dv + \int_1^{+\infty} \frac{1}{\sqrt{v}(1+v)} dv \\
&= \left[\sqrt{v}\right]_{v=0}^{v=1} + \left[2\arctan[\sqrt{v}]\right]_{v=1}^{v\to+\infty} \\
&= 1 - 0 + 2 \times \frac{\pi}{2} - 2\arctan(1) = 1 + \pi - 2 \times \frac{\pi}{4} = \frac{\pi}{2} + 1.
\end{aligned}$$

This and $p = q/(q-1)$ give

$$\begin{aligned}
D &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left(\frac{\pi}{2}+1\right) dy \\
&= \left(\frac{\pi}{2}+1\right) \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (8)
\end{aligned}$$

It follows from Equations (6), (7) and (8) and $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{1+\max(xy,\,1)}f(x)g(y)dxdy\\
&\le\left[\left(\frac{\pi}{2}+1\right)\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\left(\frac{\pi}{2}+1\right)\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}\\
&\le\left(\frac{\pi}{2}+1\right)\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof. □

Following the procedure outlined in [6, Theorem 2], and considering the same extremal functions, we are able to prove that the constant factor $\pi/2 + 1$ is the best possible. The kernel function considered is thus

$$k_6(x,y) = \frac{1}{1+\max(xy,\,1)},$$

which is clearly symmetric. However, it is non-homogeneous, which makes it particularly interesting in the context of the existing literature on the topic.

4 Third theorem

The third maximum-Hilbert-type integral inequality theorem is presented below. The kernel function is more technical than those of the previous theorems.

Theorem 4.1 *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that*

$$\int_0^{+\infty}x^{p/2-1}f(x)dx < +\infty, \quad \int_0^{+\infty}y^{q/2-1}g(y)dy < +\infty.$$

Then we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+\sqrt{xy}+\max(x,y)}f(x)g(y)dxdy\\
&\le\left[2\log\left(\frac{3}{2}\right)+\frac{2\pi}{3\sqrt{3}}\right]\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Proof. Decomposing the integrand in a suitable manner and using the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+\sqrt{xy}+\max(x,y)}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}x^{-1/(2q)}y^{1/(2p)}y^{-1/(2p)}\frac{1}{x+\sqrt{xy}+\max(x,y)}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}x^{1/(2q)}y^{-1/(2p)}\frac{1}{[x+\sqrt{xy}+\max(x,y)]^{1/p}}f(x)\\
&\times y^{1/(2p)}x^{-1/(2q)}\frac{1}{[x+\sqrt{xy}+\max(x,y)]^{1/q}}g(y)dxdy\\
&\le E^{1/p}F^{1/q},
\end{aligned}\tag{9}$$

where

$$E=\int_0^{+\infty}\int_0^{+\infty}x^{p/(2q)}y^{-1/2}\frac{1}{x+\sqrt{xy}+\max(x,y)}f^p(x)dxdy$$

and

$$F = \int_0^{+\infty} \int_0^{+\infty} y^{q/(2p)} x^{-1/2} \frac{1}{x + \sqrt{xy} + \max(x, y)} g^q(y) dx dy.$$

Let us now investigate E and F.

For E, by the Fubini-Tonelli integral theorem, we have

$$E = \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \left(\frac{y}{x}\right)^{-1/2} \frac{1}{1 + \sqrt{y/x} + \max(1, y/x)} \frac{1}{x} dy \right] dx.$$

For the central integral, making the change of variables $u = y/x$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \left(\frac{y}{x}\right)^{-1/2} \frac{1}{1 + \sqrt{y/x} + \max(1, y/x)} \frac{1}{x} dy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{u}[1 + \sqrt{u} + \max(1, u)]} du \\
&= \int_0^1 \frac{1}{\sqrt{u}[1 + \sqrt{u} + 1]} du + \int_1^{+\infty} \frac{1}{\sqrt{u}[1 + \sqrt{u} + u]} du \\
&= \int_0^1 \frac{1}{\sqrt{u}[2 + \sqrt{u}]} du + \int_1^{+\infty} \frac{1}{\sqrt{u}\left\{3/4 + [\sqrt{u} + 1/2]^2\right\}} du \\
&= \left[2 \log[2 + \sqrt{u}]\right]_{u=0}^{u=1} + \left[\frac{4}{\sqrt{3}} \arctan\left\{ \frac{2}{\sqrt{3}} \left[\sqrt{u} + \frac{1}{2} \right] \right\} \right]_{u=1}^{u \to +\infty} \\
&= 2\log(3) - 2\log(2) + \frac{4}{\sqrt{3}} \times \frac{\pi}{2} - \frac{4}{\sqrt{3}} \arctan[\sqrt{3}] \\
&= 2 \log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

This and $q = p/(p-1)$ give

$$\begin{aligned}
E &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[2 \log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] dx \\
&= \left[2 \log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \qquad (10)
\end{aligned}$$

For F, by the Fubini-Tonelli integral theorem, we have

$$F = \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \left(\frac{x}{y}\right)^{-1/2} \frac{1}{x/y + \sqrt{x/y} + \max(x/y,\, 1)} \frac{1}{y} dx \right] dy.$$

For the central integral, making the change of variables $v = x/y$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \left(\frac{x}{y}\right)^{-1/2} \frac{1}{x/y + \sqrt{x/y} + \max(x/y,\, 1)} \frac{1}{y} dx \\
&= \int_0^{+\infty} \frac{1}{\sqrt{v}[v + \sqrt{v} + \max(v,\, 1)]} dv \\
&= \int_0^{1} \frac{1}{\sqrt{v}[v + \sqrt{v} + 1]} dv + \int_1^{+\infty} \frac{1}{\sqrt{v}[v + \sqrt{v} + v]} dv \\
&= \int_0^{1} \frac{1}{\sqrt{v}\left\{3/4 + [\sqrt{v} + 1/2]^2\right\}} dv + \int_1^{+\infty} \left\{\frac{1}{v} - \frac{2}{\sqrt{v}[2\sqrt{v} + 1]}\right\} dv \\
&= \left[\frac{4}{\sqrt{3}} \arctan\left\{\frac{2}{\sqrt{3}}\left[\sqrt{v} + \frac{1}{2}\right]\right\}\right]_{v=0}^{v=1} + \left[\log(v) - 2\log[2\sqrt{v} + 1]\right]_{v=1}^{v\to+\infty} \\
&= \frac{4}{\sqrt{3}} \arctan[\sqrt{3}] - \frac{4}{\sqrt{3}} \arctan\left[\frac{1}{\sqrt{3}}\right] + 2\log\left(\frac{3}{2}\right) \\
&= 2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

This and $p = q/(q-1)$ yield

$$\begin{aligned}
F &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}\right] dy \\
&= \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}\right] \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (11)
\end{aligned}$$

It follows from Equations (9), (10) and (11) and $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x + \sqrt{xy} + \max(x, y)} f(x) g(y) dx dy \\
&\le \left\{\left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}\right] \int_0^{+\infty} x^{p/2-1} f^p(x) dx\right\}^{1/p} \\
&\times \left\{\left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}\right] \int_0^{+\infty} y^{q/2-1} g^q(y) dy\right\}^{1/q} \\
&\le \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}\right] \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This completes the proof. □

Following the procedure outlined in [6, Theorem 2], and considering the same extremal functions, we are able to prove that the constant factor $2\log(3/2) + 2\pi/[3\sqrt{3}]$ is the best possible. The kernel function considered is thus

$$k_7(x, y) = \frac{1}{x + \sqrt{xy} + \max(x, y)},$$

which is obviously not symmetric. It is however homogeneous with degree -1.

5 Fourth theorem

The fourth and last maximum-Hilbert-type integral inequality theorem is presented below. It can be presented as a variation of the previous theorem.

Theorem 5.1 *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} x^{p/2-1} f(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g(y)dy < +\infty.$$

Then we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} f(x)g(y)dxdy \\
&\leq \left[2\log\left(\frac{3}{2}\right)+\frac{2\pi}{3\sqrt{3}}\right]\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

Proof. Decomposing the integrand in a suitable manner and using the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/(2q)}x^{-1/(2q)}y^{1/(2p)}y^{-1/(2p)} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} x^{1/(2q)}y^{-1/(2p)} \frac{1}{[1+\sqrt{xy}+\max(xy,\,1)]^{1/p}} f(x) \\
&\times y^{1/(2p)}x^{-1/(2q)} \frac{1}{[1+\sqrt{xy}+\max(xy,\,1)]^{1/q}} g(y)dxdy \\
&\leq G^{1/p}H^{1/q}, \qquad (12)
\end{aligned}$$

where

$$G = \int_0^{+\infty}\int_0^{+\infty} x^{p/(2q)}y^{-1/2} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} f^p(x)dxdy$$

and

$$H = \int_0^{+\infty}\int_0^{+\infty} y^{q/(2p)}x^{-1/2} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} g^q(y)dxdy.$$

Let us now examine G and H.

For G, by the Fubini-Tonelli integral theorem, we have

$$G = \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x)\left[\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\sqrt{xy}+\max(xy,\,1)]} xdy\right]dx.$$

For the central integral, making the change of variables $u = xy$ and using the Chasles integral relation, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\sqrt{xy}+\max(xy,\,1)]} xdy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{u}[1+\sqrt{u}+\max(u,\,1)]} du \\
&= \int_0^{1} \frac{1}{\sqrt{u}[1+\sqrt{u}+1]} du + \int_1^{+\infty} \frac{1}{\sqrt{u}[1+\sqrt{u}+u]} du \\
&= \int_0^{1} \frac{1}{\sqrt{u}[2+\sqrt{u}]} du + \int_1^{+\infty} \frac{1}{\sqrt{u}\left\{3/4+[\sqrt{u}+1/2]^2\right\}} du \\
&= \left[2\log[2+\sqrt{u}]\right]_{u=0}^{u=1} + \left[\frac{4}{\sqrt{3}}\arctan\left\{\frac{2}{\sqrt{3}}\left[\sqrt{u}+\frac{1}{2}\right]\right\}\right]_{u=1}^{u\to+\infty} \\
&= 2\log(3) - 2\log(2) + \frac{4}{\sqrt{3}}\times\frac{\pi}{2} - \frac{4}{\sqrt{3}}\arctan[\sqrt{3}] \\
&= 2\log\left(\frac{3}{2}\right)+\frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

This and $q = p/(p-1)$ give

$$\begin{aligned} G &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] dx \\ &= \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \end{aligned} \tag{13}$$

For H, by the Fubini-Tonelli integral theorem, we have

$$H = \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\sqrt{xy}+\max(xy,\,1)]} y dx \right] dy.$$

For the central integral, making the change of variables $v = xy$ and using the Chasles integral relation, we have

$$\begin{aligned} &\int_0^{+\infty} \frac{1}{\sqrt{xy}[1+\sqrt{xy}+\max(xy,\,1)]} y dx \\ &= \int_0^{+\infty} \frac{1}{\sqrt{v}[1+\sqrt{v}+\max(v,\,1)]} dv \\ &= \int_0^{1} \frac{1}{\sqrt{v}[1+\sqrt{v}+1]} dv + \int_1^{+\infty} \frac{1}{\sqrt{v}[1+\sqrt{v}+v]} dv \\ &= \int_0^{1} \frac{1}{\sqrt{v}[2+\sqrt{v}]} dv + \int_1^{+\infty} \frac{1}{\sqrt{v}\left\{3/4+[\sqrt{v}+1/2]^2\right\}} dv \\ &= \left[2\log[2+\sqrt{v}] \right]_{v=0}^{v=1} + \left[\frac{4}{\sqrt{3}} \arctan\left\{ \frac{2}{\sqrt{3}} \left[\sqrt{v} + \frac{1}{2} \right] \right\} \right]_{v=1}^{v\to+\infty} \\ &= 2\log(3) - 2\log(2) + \frac{4}{\sqrt{3}} \times \frac{\pi}{2} - \frac{4}{\sqrt{3}} \arctan[\sqrt{3}] \\ &= 2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}}. \end{aligned}$$

This and $p = q/(q-1)$ give

$$\begin{aligned} H &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] dy \\ &= \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \end{aligned} \tag{14}$$

It follows from Equations (12), (13) and (14) and $1/p + 1/q = 1$ that

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1}{1+\sqrt{xy}+\max(xy,\,1)} f(x) g(y) dx dy \\ &\leq \left\{ \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \int_0^{+\infty} x^{p/2-1} f^p(x) dx \right\}^{1/p} \\ &\times \left\{ \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \int_0^{+\infty} y^{q/2-1} g^q(y) dy \right\}^{1/q} \\ &\leq \left[2\log\left(\frac{3}{2}\right) + \frac{2\pi}{3\sqrt{3}} \right] \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy \right]^{1/q}. \end{aligned}$$

This concludes the proof. □

Following the procedure outlined in [6, Theorem 2], and considering the same extremal functions, we are able to prove that the constant factor $2\log(3/2) + 2\pi/[3\sqrt{3}]$ is the best possible. The kernel function considered is thus

$$k_8(x,y) = \frac{1}{1+\sqrt{xy}+\max(xy,\,1)},$$

which is clearly symmetric. However, it is non-homogeneous, which makes it particularly original in the field of maximum-Hilbert-type integral inequalities.

6 Conclusion and perspectives

In this article, we establish four new maximum-Hilbert-type integral inequalities involving non-symmetric or non-homogeneous kernel functions. These results extend the classical framework developed in [6, 7], offering more precise insights into integral inequalities with mixed structures. Future work could explore analogous inequalities in higher dimensions or for weighted function spaces, in which maximum-type kernel functions could play a pivotal role. Furthermore, the potential applications of these inequalities to operator theory and probability, particularly in the study of maximal functions and correlation inequalities, warrant further investigation.

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