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Stability and Convergence Results for the Four-Step Iterative Approximation Scheme in Hyperbolic Metric Spaces

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Abstract
In the present paper, we deal with the four step iterative process named DH-iterative scheme into hyperbolic metric spaces. We modify this iteration into hyperbolic metric spaces where the symmetry condition is satisfied. The weak w^2-stability and data dependence results for contraction mapping in hyperbolic metric spaces are established. Finally, we prove some results related to Δ-convergence and strong convergence theorems for generalized (α, β)-nonexpansive type 1 mappings and we offer a numerical example of generalized (α, β)-nonexpansive type 1 mappings.

Keywords: Fixed point, Stability, Convergence, Hyperbolic metric spaces.

1 Introduction

In fixed point theory, the role played by ambient spaces is paramount . Several problems diverse fields of since are naturally nonlinear. Therefore,transforming the linear version of a given problem into its equivalent nonlinear version is very pertinent.
Moreover, studying various problems in spaces without a linear structure is significant in applied and pure sciences.
Several efforts have been made to introduce a convex-like structure on a metric space .
In 1990, Reich and Shafrir [25] introduced hyperbolic metric spaces .
In 2004, Kohlenbach [19] introduced a more general hyperbolic metric space. AS an example for the convex-like structure is a hyperbolic space . Banach and CAT(0) spaces are well known to be special cases of hyperbolic spaces. Moreover, the class of hyperbolic spaces properly contains a Hilbert ball endowed with hyperbolic spaces , Hadamard manifolds,R-tree,and the cartesian product of Hilbert spaces [13]. Our work will be a carried out in the setting of hyperbolic space studied by Kohlenbach [19].
When we talk a bout the problems emulated into a fixed point, we mean to find a $p^* \in Y$ such that $Gp^* = p^*$, where G is a nonlinear mapping (self or non-self) of an arbitrary space Y. Many researchers have paid very good attention to finding an analytical solution, but this has been almost practically impossible. In view of this, iterative processes have been adopted to find approximate solutions.
The Picard iterative process is one of the very first iterative processes used to approximate a fixed point of a contraction mapping G on a metric space (Y, ρ) . Note that a mapping $G : Y \to Y$ is called a contraction if

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there exists a constant $\mu \in [0, 1)$ such that

$$\rho(Ga, Gb) \leq \mu(a, b), \forall a, b \in Y$$

If $\mu = 1$ in inequality above then G is said to be a nonexpansive mapping. Even though the existence of the fixed point is guaranteed in the case of nonexpansive mapping, the Picard iterative process fails to approximate the fixed point of G. To overcome this problem, researchers of this field developed different iterative processes to approximate fixed points of nonexpansive mappings and other mappings, which are more general than nonexpansive. For example, look at Noor [23], Agarwal et al. [2], Abbas and Nazir [1] CR-iteration [10], Normal-S iteration [27], Picard-S [14], Thakur et al. [34], and M iterative schemes [36]. ,etc.
In 2017, Pant and Shukla [24] introduced the class of generalized α-nonexpansive mappings, which is a larger class of mappings than the classes of nonexpansive, Suzuki generalized nonexpansive and α-nonexpansive mappings.
Very recently, in 2024, Al-baqeri et al [5] introduced new four steps iterative methods called the DH-iterative scheme as follows:

$$\begin{array}{rcl} p_1 & \in & U, \\ z_i & = & G[(1-\alpha_i)p_i + \alpha_i G p_i] \\ w_i & = & G[(1-\beta_i)z_i + \beta_i G z_i] \\ q_i & = & G[(1-\gamma_i)w_i + \gamma_i G w_i] \\ p_{i+1} & = & G(Gq_i) \end{array} \tag{1.1}$$

for $i \geq 1$, where $\{\alpha_i\}$, $\{\beta_i\}$ and $\{\gamma_i\}$ are sequences in (0, 1). They proved that the DH iterative algorithm has a better rate of convergence than most leading algorithms for contractive-like mappings and Reich-Suzuki-type nonexpansive mappings. Under this algorithm, some fixed point convergence results , w^2-stability for contractive-like mappings are studied.
Motivated by the results that mentioned in [5], our work are organized as follows:
(1)Collect some basic definitions and theorems we need its in our studying.
(2)Study DH-iterative algorithm in hyperbolic metric space .
(3)Prove the weak w^2-stability and data dependence results for contraction mapping in hyperbolic metric space.
(4)Establish some results related to the strong and Δ-convergence of the DH-iteration process for generalized (α, β)-nonexpansive type (1) mapping in uniformly convex hyperbolic metric spaces .
(5)Present numerical example of the DH-iteration process for generalized (α, β)-nonexpansive type (1) mapping .

2 Preliminaries

We called $G : X \to Y$ is a mapping if it is satisfied every element in X has only one image in Y by G its denoted $Gx = y$ for all $x \in X$, $y \in Y$.
Now we show some kinds of mappings in metric space

Definition 2.1. *Let (Y, ρ) be a metric space and U be a nonempty subset of Y. A mapping $G : Y \to Y$ is said to be the following:*
(C1)[20] G is called a contraction mapping if there exists a constant $\mu \in [0, 1)$ such that

$$\rho(Ga, Gb) \leq \mu\rho(a, b), \quad \forall \ a, b \in U$$

(C2) [7] G is called a contractive mapping if

$$\rho(Ga, Gb) \leq \rho(a, b), \quad \forall \ a, b \in U, \quad a \neq b$$

(C4)[7] G is called a nonexpensive mapping if $\mu = 1$ *then*

$$\rho(Ga, Gb) \leq \rho(a, b), \quad \forall \ a, b \in U$$

(C5) [11] Quasi-nonexpansive if

$$\rho(Ga, e) \leq \rho(a, e) \ \ forall \ \ a \in U \ \ and \ \ p \in Fix(G)$$

where $Fix(G)$ *is the set of all fixed points of G.*
(C6) [37] Mean non expansive mapping if, for all $a, b \in U$, *there exist* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta \leq 1$ *such that*

$$\rho(Ga, Gb) \leq \alpha\rho(a, b) + \beta\rho(a, Gb)$$

Several extensions and generalizations of nonexpansive mappings have been discussed by many authors due to their importance in terms of applications. For instance, Suzuki in (2008) [26] introduced an interesting generalization of nonexpansive mappings and presented some existence and convergence results. Another common name for such mappings are known as mappings satisfying condition (C).

(C7)[31] Suzuki-generalized nonexpansive (or satisfy condition (C)) if

$$\frac{1}{2}\rho(Ga, a) \leq \rho(a, b) \Rightarrow \rho(Ga, Gb) \leq \rho(a, b) \ \ forall \ \ a, b \in U$$

(C8) [12] Satisfy condition (C_λ) *if*

$$\lambda\rho(Ga, b) \leq \rho(a, b) \Rightarrow \rho(Ga, Gb) \leq \rho(a, b) \ \ forall \ \ a, b \in U$$

(C9) [24] Generalized α*-nonexpansive mapping if,forall* $a, b \in U$ *there exists*
$\alpha \in [0, 1)$ *such that*

$$\frac{1}{2}\rho(Ga, a) \leq \rho(a, b) \Rightarrow \rho(Ga, Gb) \leq \alpha\rho(Ga, b) + \alpha\rho(a, Gb) + (1 - 2\alpha)\rho(a, b).$$

Akutsah and Narain [4] introduced the class of generalized $(\alpha, \beta) - nonexpansive$ type 1 mappings, which generalizes the mappings above, and they gave some basic properties for this class of mappings.

Definition 2.2. *[4]*
Let U *be a nonempty subset of a metric space* (Y, ρ). *A mapping* $G : U \rightarrow U$ *is said to be generalized* $(\alpha, \beta) - nonexpansive$ *type 1 if there exist* $\alpha, \beta, \lambda \in [0, 1)$, *with* $\alpha \leq \beta$ *and* $\alpha + \beta < 1$ *such that*

$$\lambda\rho(Ga, a) \leq \rho(a, b) \Rightarrow \rho(Ga, Gb) \leq \alpha\rho(Ga, b) + \beta\rho(a, Gb) + (1 - (\alpha + \beta))\rho(a, b)$$

for all $a, b \in U$.

Proposition 2.3. *[4]*
(i)If G is a generalized $(\alpha, \beta) - nonexpansive$ *type 1 mapping and has a fixed point, then G is quasi-nonexpansive*
(ii) If G is a generalized $(\alpha, \beta) - nonexpansive$ *type 1 mapping, then for all* $a, b \in Y$.

$$\rho(a, Gb) \leq \frac{2 + \alpha + \beta}{1 - \beta}\rho(a, Ga) + \rho(a, b)$$

(iii) If G is a generalized $(\alpha, \beta) - nonexpansive$ *type 1 mapping, then Fix(G) is closed.*

Definition 2.4. *[20]*
Let (Y, ρ) *be a metric space and* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *be two sequences in Y. We say that these sequences are equivalent if*

$$\lim_{i \to \infty} \rho(a_i, b_i) = 0$$

Timis [35] gave the following definition of weak $w^2 - stability$using equivalent sequences.

Definition 2.5. *([35],Definition 2.4).*
Let (Y, ρ) be a metric space, G be a self-mapping on Y, and $\{a_i\}_{i=1}^{\infty} \subset Y$ be an iterative sequence defined

$$a_1 \in Y$$

$$a_{i+1} = f(G, a_i), \forall i \geq 1$$

where f is a function. Suppose that $\{a_i\}_{i=1}^{\infty}$ converges strongly to $p \in Fix(G)$. If for any equivalent sequence $\{b_i\}_{i=1}^{\infty} \subset Y$ of $\{a_i\}_{i=1}^{\infty}$

$$\lim_{i\to\infty} \rho(b_{i+1}, f(G, b_i)) = 0 \Rightarrow \lim_{i\to\infty} b_i = p^*$$

then the iterative sequence $\{a_i\}_{i=1}^{\infty}$ is said to be weak $w^2 - stable$ with respect to G

In 1990, Reich and Shafrir [25] introduced hyperbolic metric spaces and studied an iteration process for nonexpansive mappings (we gave its definition before) in these spaces. In 2004, Kohlenbach [19] introduced a more general hyperbolic metric space as follows.

Definition 2.6. *Let (Y, ρ) be a metric space, and then (Y, ρ, W) will be the hyperbolic metric space if the function $W : Y \times Y \times [0, 1] \to Y$ is satisfying.*
(i) $\rho(c, W(a, b, \alpha)) \leq (1 - \alpha)\rho(c, a) + \alpha\rho(c, b)$.
(ii)$\rho(W(a, b, \alpha), W(a, b, \beta)) = |\alpha - \beta|\rho(a, b)$.
(iii)$W(a, b, \alpha) = W(b, a, 1 - \alpha)$.
(iv)$\rho(W(a, c, \alpha), W(b, w, \alpha)) \leq (1 - \alpha)\rho(a, b) + \alpha\rho(c, w)$.
for all $a, b, c, w \in Y$ and $\alpha, \beta \in [0, 1]$.

A linear example of a hyperbolic metric space is a Banach space and nonlinear examples are Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric (see [13]) , and CAT(0) spaces in the sense of Gromov (see [7]).

Definition 2.7. *We consider a hyperbolic metric space (Y, ρ, W). If $a, b \in Y$ and $\alpha \in [0, 1]$, then we will use $(1 - \alpha)\, a \oplus ab$ for $W(a, b, \alpha)$.*
(i) A subset U of this hyperbolic metric space is called convex if $a, b \in U$ implies that $W(a, b, \alpha) \in U$. The following equalities hold even for the more general setting of a convex metric space(see [32],Proposition (1.2)):

$$\rho(b, W(a, b, \alpha)) = (1 - \alpha)\rho(a, b) \;\; and \;\; \rho(a, W(a, b, \alpha)) = \alpha\rho(a, b)$$

for all $a, b \in Y$ and $\alpha \in [0, 1]$. As a consequence, we obtain

$$W(a, b, 0) = a \; and \; W(a, b, 1) = b$$

(ii) This hyperbolic metric space is called uniformly convex 5(see [29]) (see [29]) if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists a constant $\delta \in (0, 1]$ such that

$$\rho(W(a, b, \frac{1}{2}), u) \leq (1 - \delta)r$$

for all $u, a, b \in Y$ with $\rho(a, u) \leq r$, $\rho(b, u) \leq r$ and $\rho(a, b) \geq r\varepsilon$.
(iii) A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ is said to be a modulus of uniform convexity if $\delta = \eta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$. Furthermore, the mapping η is called monotone if it decreases with respect to r for a fixed ε.

Definition 2.8. *Let $\{a_i\}_{i=1}^{\infty}$ be a bounded sequence in a nonempty subset X of a metric space (Y, ρ). Then, the mapping $r(., \{a_i\}) : Y \to [0, \infty)$ is defined by*

$$r(a, \{a_i\}) = \limsup_{i\to\infty} \rho(a, \{a_i\}), a \in Y$$

The infimum of $r(., \{a_i\})$ *over* X *is called the asymptotic radius of* $\{a_i\}_{i=1}^{\infty}$ *relative to* X *and is denoted by* $r(X, \{a_i\})$. *A point* $c \in X$ *is said to be an asymptotic center of the sequence* $\{a_i\}_{i=1}^{\infty}$ *relative to* X *if*

$$r(c, \{a_i\}) = inf\{r(a, \{a_i\}) : a \in X\}$$

and the set of all asymptotic centers of $\{a_i\}_{i=1}^{\infty}$ *relative to* X *is denoted by* $A(X, a_i)$.

In 1976, Lim [22] introduced the concept of Δ-convergence, which is an analog of weak convergence, in metric spaces using the asymptotic center.

Definition 2.9. *[22]*
A sequence $\{a_i\}_{i=1}^{\infty}$ *in a metric space* (Y, ρ) *is said to* $\Delta - converge$ *to a point* $a \in Y$ *if* a *is the unique asymptotic center of* $\{u_i\}_{i=1}^{\infty}$ *for every subsequence* $\{u_i\}_{i=1}^{\infty}$ *of* $\{a_i\}_{i=1}^{\infty}$ *In this case, we write* $\Delta - \lim_{i\to\infty} a_i = a$ *and call* a *as* $\Delta - limit$ *of* $\{a_i\}_{i=1}^{\infty}$.

We give three lemmas that will be helpful in proving our main results.

Lemma 2.10. *[21]*
Let (Y, ρ, W) *be a complete uniformly convex hyperbolic metric space with the monotone modulus of uniform convexity* η *and* U *be a nonempty closed and convex subset of* Y. *Then, every bounded sequence* $\{a_i\}_{i=1}^{\infty}$ *in* Y *has a unique asymptotic center relative to* U.

Lemma 2.11. *[18]*
Let (Y, ρ, W) *be a uniformly convex hyperbolic metric space with the monotone modulus of uniform convexity* η. *Let* $a \in Y$ *and* $\{\sigma_i\}$ *be a sequence in* $[p, q]$ *for some* $p, q \in (0, 1)$. *If* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *are sequences in* Y *such that*

$$\limsup_{i\to\infty} \rho(a_i, a) \leq r, \limsup_{i\to\infty} \rho(b_i, a) \leq r, \lim_{i\to\infty} \rho(W(a_i, b_i, \sigma_i), a) = r$$

for some $r \geq 0$, *then*

$$\lim_{i\to\infty} \rho(a_i, b_i) = 0.$$

Lemma 2.12. *[30]*
Let $\{g_i\}_{i=1}^{\infty}$,$\{r_i\}_{i=1}^{\infty}$ *and* $\{t_i\}_{i=1}^{\infty}$ *be non-negative real sequences with* $r_i \in (0, 1), \forall i \geq 1$ *and* $\sum_{i=1}^{\infty} r_i = \infty$. *Suppose that there exists* $n_0 \in N$ *such that, for all* $n \geq n_0$, *one has the inequality*

$$g_{i+1} \leq (1 - r_i)g_i + r_i t_i.$$

Then, the following inequality holds:

$$0 \leq \limsup_{i\to\infty} g_i \leq \limsup_{i\to\infty} t_i$$

3 Weak w^2-stability result

First,we put DH- iteration in hyperbolic metric spaces as follows:

$$\begin{aligned} p_1 &\in U \\ z_i &= G(W(p_i, Gp_i, \alpha_i)) \\ w_i &= G(W(z_i, Gz_i, \beta_i)) \\ q_i &= G(W(w_i, Gw_i, \gamma_i)) \\ p_{i+1} &= G(Gq_i). \end{aligned} \tag{3.1}$$

where U is nonempty convex subset of a hyperbolic metric space Y,G is a self-mapping on U,and $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ and $\{\gamma_i\}_{i=1}^{\infty}$ are three real sequences in $(0, 1)$.
Now we show the strong convergence theorem.

Theorem 3.1. *Let U is a nonempty closed convex subset of a hyperbolic metric space Y ,G is a self-mapping on U be a contraction mapping with the constant $\mu \in [0,1)$ such that $F_p(G) \neq \emptyset$ and $\{p_i\}_{i=1}^{\infty}$ be the DH-iterative sequence with real sequences $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ and $\{\gamma_i\}_{i=1}^{\infty}$ in $(0,1)$,satisfying $\sum_{i=1}^{\infty}\gamma_i = \infty$. Then, the sequence $\{p_i\}_1^{\infty}$ converges strongly to a fixed point of G.*

Proof. *The contraction mapping G has a fixed point so it is easily shown that this fixed point is unique. Suppose p^* is a unique fixed point of G. From (3.1) we get*

$$\begin{aligned}
\rho(z_i,p^*) &= \rho(G(W(p_i,Gp_i,\alpha_i)),Gp^*)\\
&\leq \mu\rho(W(p_i,Gp_i,\alpha_i),p^*)\\
&\leq \mu[(1-\alpha_i)\rho(p_i,p^*)+\alpha_i\rho(Gp_i,p^*)]\\
&\leq \mu[(1-\alpha_i)\rho(p_i,p^*)+\alpha_i\mu\rho(p_i,p^*)]\\
&= \mu(1-\alpha_i(1-\mu))\rho(p_i,p^*) \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
\rho(w_i,p^*) &= \rho(G(W(z_i,Gz_i,\beta_i)),Gp^*)\\
&\leq \mu\rho(W(z_i,Gz_i,\beta_i),p^*)\\
&\leq \mu[(1-\beta_i)\rho(z_i,p^*)+\beta_i\rho(Gz_i,p^*)]\\
&\leq \mu[(1-\beta_i)\rho(z_i,p^*)+\beta_i\mu\rho(z_i,p^*)]\\
&= \mu(1-\beta_i(1-\mu))\rho(z_i,p^*)\\
\text{by (3.2) we obtain} &\\
&= \mu(1-\beta_i(1-\mu))\mu(1-\alpha_i(1-\mu))\rho(p_i,p^*)\\
&= \mu^2(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i,p^*) \qquad (3.3)
\end{aligned}$$

And,by the same way we obtain

$$\begin{aligned}
\rho(q_i,p^*) &= \rho(G(W(w_i,Gw_i,\gamma_i)),Gp^*)\\
&\leq \mu\rho(W(w_i,Gw_i,\gamma_i),p^*)\\
&\leq \mu[(1-\gamma_i)\rho(w_i,p^*)+\gamma_i\rho(Gw_i,p^*)]\\
&\leq \mu[(1-\gamma_i)\rho(w_i,p^*)+\gamma_i\mu\rho(w_i,p^*)]\\
&= \mu(1-\gamma_i(1-\mu))\rho(w_i,p^*)\\
\text{by (3.3) we get} &\\
&= \mu(1-\gamma_i(1-\mu)(\mu^2(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i,p^*))\\
&= \mu^3(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i,p^*)) \qquad (3.4)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\rho(p_{i+1},p^*) &= \rho(G(Gq_i),Gp^*)\\
&\leq \mu\rho(Gq_i,p^*)\\
&\leq \mu^2\rho(q_i,p^*)\\
\text{by (3.4) we obtain} &\\
&\leq \mu^5((1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i,p^*)) \qquad (3.5)
\end{aligned}$$

Because $\mu \in [0,1)$ and $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ in $(0,1)$, then

$$(1-\alpha_i(1-\mu)) < 1$$

$$(1-\beta_i(1-\mu)) < 1$$

The inequality (3.5) become

$$\rho(p_{i+1}, p^*) \leq \mu^5(1-\gamma_i(1-\mu))\rho(p_i, p^*).$$

Repetition of the above processes gives the following inequalities:

$$\rho(p_{i+1}, p^*) \leq \mu^5(1-\gamma_i(1-\mu))\rho(p_i, p^*)$$

$$\rho(p_i, p^*) \leq \mu^5(1-\gamma_{i-1}(1-\mu))\rho(p_{i-1}, p^*)$$

$$\rho(p_{i-1}, p^*) \leq \mu^5(1-\gamma_{i-2}(1-\mu))\rho(p_{i-2}, p^*)$$

$$.$$

$$.$$

$$\rho(p_2, p^*) \leq \mu^5(1-\gamma_1(1-\mu))\rho(p_1, p^*)$$

Then we get

$$\rho(p_{i+1}, p^*) \leq \rho(p_1, p^*)(\mu^5)^i \prod_{j=1}^{i}(1-\gamma_j(1-\mu))$$

for all $i \in N$ *.Again,since* $\mu \in [0,1)$ *and* $\{\gamma_i\}_{i=1}^{\infty}$ *in* $(0,1)$*,then* $(1-\gamma_j(1-\mu)) < 1.$
It is known that $1-p \leq e^{-p}$ *, for each* $p \in [0,1]$*,then we get*

$$\rho(p_{i+1}, p^*) \leq \rho(p_1, p^*)(\mu^5)^i e^{-(1-\mu)\sum_{j=1}^{i}\gamma_j}. \tag{3.6}$$

Taking the limit of both sides as $i \longrightarrow \infty$ *in (3.6) ,we have*

$$\lim_{i\to\infty} \rho(p_i, p^*) = 0$$

, that is $\{p_i\} \longrightarrow p^*$ *.*□

Now, we prove that the modified iteration process defined by (3.1) is weak w^2-stable with respect to G.

Theorem 3.2. *Suppose that all conditions of theorem (3.1) hold. Then,the iteration process (3.1) is weak* w^2*-stable with respect to* G*.*

Proof. *Let*$\{p_i\}_{i=1}^{\infty}$ *be a sequence generated by (3.1) and* $\{a_i\}_{i=1}^{\infty} \subset U$ *be an equivalent sequence of* $\{p_i\}_{i=1}^{\infty}$ *put*

$$\varepsilon_i = \rho(a_{i+1}, G(Gx_i))$$

where $x_i = G(W(h_i, Gh_i, \gamma i))$*,*$hi = G(W(c_i, Gc_i, \beta_i))$ *and* $c_i = G(W(a_i, Ga_i, \alpha_i))$*. Let* $\lim_{i\to\infty}\varepsilon_i = 0.$

Then ,we have

$$\begin{aligned}
\rho(q_i, x_i) &= \rho(G(W(w_i, Gw_i, \gamma_i)), G(W(h_i, Gh_i, \gamma_i))) \\
&\leq \mu\rho(W(w_i, Gw_i, \gamma_i), W(h_i, Gh_i, \gamma_i)) \\
&\leq \mu[(1-\gamma_i)\rho(w_i, h_i) + \gamma_i\rho(Gw_i, Gh_i)] \\
&\leq \mu[(1-\gamma_i)\rho(w_i, h_i) + \gamma_i\mu\rho(w_i, h_i)] \\
&\leq \mu(1-\gamma_i(1-\mu))\rho(w_i, h_i) \qquad (3.7)
\end{aligned}$$

The same

$$\begin{aligned}
\rho(w_i, h_i) &= \rho(G(W(z_i, Gz_i, \beta_i)), G(W(c_i, Gc_i, \beta_i))) \\
&\leq \mu\rho(W(z_i, Gz_i, \beta_i), W(c_i, Gc_i, \beta_i)) \\
&\leq \mu[(1-\beta_i)\rho(z_i, c_i) + \beta_i\rho(Gz_i, Gc_i)] \\
&\leq \mu[(1-\beta_i)\rho(z_i, c_i) + \beta_i\mu\rho(z_i, c_i)] \\
&\leq \mu(1-\beta_i(1-\mu))\rho(z_i, c_i) \qquad (3.8)
\end{aligned}$$

Similarly

$$\begin{aligned}
\rho(z_i, c_i) &= \rho(G(W(p_i, Gp_i, \alpha_i)), G(W(a_i, Ga_i, \alpha_i))) \\
&\leq \mu\rho(W(p_i, Gp_i, \alpha_i), W(a_i, Ga_i, \alpha_i)) \\
&\leq \mu[(1-\alpha_i)\rho(p_i, a_i) + \alpha_i\rho(Gp_i, Ga_i)] \\
&\leq \mu[(1-\alpha_i)\rho(p_i, a_i) + \alpha_i\mu\rho(p_i, a_i)] \\
&\leq \mu(1-\alpha_i(1-\mu))\rho(p_i, a_i). \qquad (3.9)
\end{aligned}$$

And

$$\begin{aligned}
\rho(a_{i+1}, p^*) &\leq \rho(a_{i+1}, p_{i+1}) + \rho(p_{i+1}, p^*) \\
&\leq \rho(a_{i+1}, G(Gx_i)) + \rho(G(Gx_i), p_{i+1}) + \rho(p_{i+1}, p^*) \\
&= \varepsilon_i + \rho(G(Gx_i), p_{i+1}) + \rho(p_{i+1}, p^*) \\
&= \varepsilon_i + \rho(G(Gx_i), G(Gq_i)) + \rho(p_{i+1}, p^*) \\
&\leq \varepsilon_i + \mu\rho((Gx_i), (Gq_i)) + \rho(p_{i+1}, p^*) \\
&\leq \varepsilon_i + \mu^2\rho(x_i, q_i) + \rho(p_{i+1}, p^*) \\
&\leq \varepsilon_i + \mu^2\rho(q_i, x_i) + \rho(p_{i+1}, p^*) \\
&\leq \varepsilon_i + \mu^3(1-\gamma_i(1-\mu))\rho(w_i, h_i) + \rho(p_{i+1}, p^*) \\
\textit{By (3.8) we get} & \\
&\leq \varepsilon_i + \mu^4(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))\rho(z_i, c_i) + \rho(p_{i+1}, p^*) \\
\textit{Similary by (3.9) we obtain} & \\
&\leq \varepsilon_i + \mu^5(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, a_i) \\
&+ \rho(p_{i+1}, p^*) \qquad (3.10)
\end{aligned}$$

Put $(1-\gamma_i(1-\mu)) < 1$ *and from theorem (3.1)* $\lim_{i\to\infty}\rho(p_{i+1}, p^*) = 0$ *because* $\{p_i\}_{i=1}^{\infty}$ *and* $\{a_i\}_{i=1}^{\infty}$ *are equivalent sequences then* $\lim_{i\to\infty}\rho(p_i, a_i) = 0$ *also* $\lim_{i\to\infty}\varepsilon_i = 0$. *We take the limit for both sides in (3.10) then we get* $\lim_{i\to\infty}\rho(a_{i+1}, p^*) = 0$ *so* $\{p_i\}_{i=1}^{\infty}$ *is* w^2*-stable with respect to G.*□

4 Data dependence results

Now ,we prove the data dependence result for the modified iteration process (3.1) using the definition of an approximate operator .

Definition 4.1. *([6],p,166)*
let (Y, ρ) *be a metric space and* $G,\bar{G} : Y \to Y$ *be tow operators.*$\bar{G}$ *is called an approximate operator of G,if* $\rho(Ga, \bar{G}a) \leq \epsilon$ *for all* $a \in Y$ *and for a fixed* $\epsilon > 0$.

Theorem 4.2. *Let* Y, U *and* G *be the same as theorem (3.1) and* $\bar{G} : U \to U$ *be an approximate operator of G for given* ϵ *.Let* $\{p_i\}_{i=1}^{\infty}$ *be an iterative sequence generated by (3.1) and define an iterative sequence* $\{\bar{p}_i\}_{i=1}^{\infty}$ *as follows:*

$$\begin{aligned}
\bar{p}_i &\in U \\
\bar{z}_i &= \bar{G}(W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i)) \\
\bar{w}_i &= \bar{G}(W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i)) \\
\bar{q}_i &= \bar{G}(W(\bar{w}_i, \bar{G}\bar{w}_i, \gamma_i)) \\
\bar{p}_{i+1} &= \bar{G}(\bar{G}\bar{q}_i) \qquad (4.1)
\end{aligned}$$

with real sequences $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ *and* $\{\gamma_i\}_{i=1}^{\infty}$ *in* $(0, 1)$ *satisfying* $\alpha_i \geq \frac{1}{2}$, $\forall i \geq 1$, *and* $\sum_{i=1}^{\infty} \alpha_i = \infty$.*If* $p^* = Gp^*$ *and* $\bar{p^*} = G\bar{p^*}$ *such that* $\lim_{i\to\infty} \bar{p}_i = \bar{p^*}$ *then one has*

$$\rho(p^*, \bar{p}^*) \leq \frac{15\epsilon}{(1-\mu)}$$

Where $\mu \in [0, 1)$.

Proof. *From (3.1) and (4.1) we have*

$$\begin{aligned}
\rho(z_i, \bar{z}_i) &= \rho(G(W(p_i, Gp_i, \alpha_i)), \bar{G}(W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i))) \\
&\leq \rho(G(W(p_i, Gp_i, \alpha_i)), G(W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i))) + \rho(G(W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i)), \bar{G}(W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i))) \\
&\leq \mu\rho(W(p_i, Gp_i, \alpha_i), W(\bar{p}_i, \bar{G}\bar{p}_i, \alpha_i)) + \epsilon \\
&\leq \mu[(1-\alpha_i)\rho(p_i, \bar{p}_i) + \alpha_i\rho(Gp_i, \bar{G}\bar{p}_i)] + \epsilon \\
&\leq \mu(1-\alpha_i)\rho(p_i, \bar{p}_i) + \mu\alpha_i(\rho(Gp_i, G\bar{p}_i) + \rho(G\bar{p}_i, \bar{G}\bar{p}_i)) + \epsilon \\
&\leq \mu(1-\alpha_i)\rho(p_i, \bar{p}_i) + \mu^2\alpha_i\rho(p_i, \bar{p}_i) + \mu\alpha_i\epsilon + \epsilon \\
&\leq \mu(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \mu\alpha_i\epsilon + \epsilon
\end{aligned} \tag{4.2}$$

By (4.2) , we have

$$\begin{aligned}
\rho(w_i, \bar{w}_i) &= \rho(G(W(z_i, Gz_i, \beta_i)), \bar{G}(W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i))) \\
&\leq \rho(G(W(z_i, Gz_i, \beta_i)), G(W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i))) + \rho(G(W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i)), \bar{G}(W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i))) \\
&\leq \mu\rho(W(z_i, Gz_i, \beta_i)), (W(\bar{z}_i, \bar{G}\bar{z}_i, \beta_i)) + \epsilon \\
&\leq \mu[(1-\beta_i)\rho(z_i, \bar{z}_i) + \beta_i\rho(Gz_i, \bar{G}\bar{z}_i)] + \epsilon \\
&\leq \mu(1-\beta_i)\rho(z_i, \bar{z}_i) + \mu\beta_i(\rho(Gz_i, G\bar{z}_i) + \rho(G\bar{z}_i, \bar{G}\bar{z}_i)) + \epsilon \\
&\leq \mu(1-\beta_i)\rho(z_i, \bar{z}_i) + \mu\beta_i(\mu\rho(z_i, \bar{z}_i) + \epsilon) + \epsilon \\
&\leq \mu(1-\beta_i)\rho(z_i, \bar{z}_i) + \mu^2\beta_i\rho(z_i, \bar{z}_i) + \mu\beta_i\epsilon + \epsilon \\
&\leq \mu(1-\beta_i(1-\mu))\rho(z_i, \bar{z}_i) + \mu\beta_i\epsilon + \epsilon \\
&\leq \mu(1-\beta_i(1-\mu))[\mu(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \mu\alpha_i\epsilon + \epsilon] + \mu\beta_i\epsilon + \epsilon \\
&\leq \mu^2(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \mu^2\alpha_i\epsilon(1-\beta_i(1-\mu)) \\
&+ \mu\epsilon(1-\beta_i(1-\mu)) + \mu\beta_i\epsilon + \epsilon.
\end{aligned} \tag{4.3}$$

Similarly,using (4.3) we get

$$\begin{aligned}
\rho(q_i, \bar{q}_i) &= \rho(G(W(w_i, Gw_i, \gamma_i)), \bar{G}(W(\bar{w}_i, \bar{G}\bar{w}_i, \gamma_i))) \\
&\leq \rho(G(W(w_i, Gw_i, \gamma_i)), G(W(\bar{w}_i, \bar{G}\bar{w}_i, \gamma_i))) + \rho(G(W(\bar{w}_i, \bar{G}\bar{w}_i, \gamma_i)), \bar{G}(W(\bar{w}, \bar{G}\bar{w}_i, \gamma_i))) \\
&\leq \mu\rho(W(w_i, Gw_i, \gamma_i)), (W(\bar{w}_i, \bar{G}\bar{w}_i, \gamma_i)) + \epsilon \\
&\leq \mu[(1-\gamma_i)\rho(w_i, \bar{w}_i) + \gamma_i\rho(Gw_i, \bar{G}\bar{w}_i)] + \epsilon \\
&\leq \mu(1-\gamma_i)\rho(w_i, \bar{w}_i) + \mu\gamma_i[\rho(Gw_i, G\bar{w}_i) + \rho(G\bar{w}_i, \bar{G}\bar{w}_i)] + \epsilon \\
&\leq \mu(1-\gamma_i)\rho(w_i, \bar{w}_i) + \mu\gamma_i[\mu\rho(w_i, \bar{w}_i) + \epsilon] + \epsilon \\
&\leq \mu(1-\gamma_i)\rho(w_i, \bar{w}_i) + \mu^2\gamma_i\rho(w_i, \bar{w}_i) + \mu\gamma_i\epsilon + \epsilon \\
&\leq \mu(1-\gamma_i(1-\mu))\rho(w_i, \bar{w}_i) + \mu\gamma_i\epsilon + \epsilon \\
&\leq \mu(1-\gamma_i(1-\mu))[\mu^2(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) \\
&+ \mu^2\alpha_i\epsilon(1-\beta_i(1-\mu)) + \mu\epsilon(1-\beta_i(1-\mu)) + \mu\beta_i\epsilon + \epsilon] + \mu\gamma_i\epsilon + \epsilon \\
&\leq \mu^3(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i)
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \mu^3\alpha_i\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) + \mu^2\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) \\
+ & \quad \mu^2\beta_i\epsilon(1-\gamma_i(1-\mu)) + \mu\epsilon(1-\gamma_i(1-\mu)) + \mu\gamma_i\epsilon + \epsilon.
\end{aligned} \tag{4.4}
$$

Finally ,using (4.4) we obtain

$$
\begin{aligned}
\rho(p_{i+1}, \bar{p}_{i+1}) &= \rho(G(Gq_i), \bar{G}(\bar{G}\bar{q}_i)) \\
&\leq \rho(G(Gq_i), G(\bar{G}\bar{q}_i)) + \rho(G(\bar{G}\bar{q}_i), \bar{G}(\bar{G}\bar{q}_i)) \\
&\leq \mu\rho(Gq_i, \bar{G}\bar{q}_i) + \epsilon \\
&\leq \mu[\rho(Gq_i, G\bar{q}_i) + \rho(G\bar{q}_i, \bar{G}\bar{q}_i)] + \epsilon \\
&\leq \mu[\mu\rho(q_i, \bar{q}_i) + \epsilon] + \epsilon \\
&\leq \mu^2\rho(q_i, \bar{q}_i) + \mu\epsilon + \epsilon \\
&\leq \mu^2[\mu^3(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) \\
&+ \mu^3\alpha_i\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) + \mu^2\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) \\
&+ \mu^2\beta_i\epsilon(1-\gamma_i(1-\mu)) + \mu\epsilon(1-\gamma_i(1-\mu)) + \mu\gamma_i\epsilon + \epsilon] + \mu\epsilon + \epsilon \\
&\leq \mu^5(1-\gamma_i(1-\mu))(1-\beta_i(1-\mu))(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) \\
&+ \mu^5\alpha_i\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) + \mu^4\epsilon(1-\beta_i(1-\mu))(1-\gamma_i(1-\mu)) \\
&+ \mu^4\beta_i\epsilon(1-\gamma_i(1-\mu)) + \mu^3\epsilon(1-\gamma_i(1-\mu)) + \mu^3\gamma_i\epsilon + \mu^2\epsilon + \mu\epsilon + \epsilon
\end{aligned} \tag{4.5}
$$

Since $\mu \in [0, 1)$ *also* $\alpha_i, \beta_i, \gamma_i \in (0, 1)$ *we get that*

$$
\begin{aligned}
(1-\gamma_i(1-\mu)) &< 1 \\
(1-\beta_i(1-\mu)) &< 1 \\
\mu^5, \mu^4, \mu^3, \mu^2, \mu &< 1 \\
\mu^5\gamma_i, \mu^5\beta_i &< 1
\end{aligned} \tag{4.6}
$$

and $(1-\alpha_i+\mu\alpha_i) \leq 2\alpha_i$. *And by assumption* $\alpha_i \geq \frac{1}{2}$, *we get* $(1-\alpha_i) \leq \alpha_i$.
Now we use all assumption ,then we get

$$
\begin{aligned}
&\leq \mu^5(1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \mu^5\alpha_i\epsilon + \mu^4\epsilon + \mu^4\epsilon + \mu^3\epsilon + \mu^3\epsilon \\
&+ \mu^2\epsilon + \mu\epsilon + \epsilon \\
&\leq (1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \alpha_i\epsilon + 7\epsilon \\
&= (1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \alpha_i\epsilon + 7(1-\alpha_i+\alpha_i)\epsilon
\end{aligned}
$$

$Becouse (1-\alpha_i) \leq \alpha_i then :$

$$
\begin{aligned}
&\leq (1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \alpha_i\epsilon + 14\alpha_i\epsilon \\
&= (1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + 15\alpha_i\epsilon \\
&= (1-\alpha_i(1-\mu))\rho(p_i, \bar{p}_i) + \alpha_i(1-\mu)\frac{15\epsilon}{(1-\mu)}.
\end{aligned} \tag{4.7}
$$

Now suppose $g_i = \rho(p_i, \bar{p}_i)$, $r_i = \alpha_i(1-\mu)$ *and* $t_i = \frac{15\epsilon}{(1-\mu)}$.
By lemma (2.10) we obtain

$$0 \leq \limsup_{i\to\infty} g_i \leq \limsup_{i\to\infty} t_i$$

We know that from Theorem (3.1) $\lim_{i\to\infty} p_i = p^*$ *and by the assumption in the hypotheses,we have* $\lim_{i\to\infty} \bar{p}_i = \bar{p}^*$, *then we get*

$$\rho(p^*, \bar{p}^*) \leq \frac{15\epsilon}{(1-\mu)}$$

. □

5 Convergence results

In this section, we discuss and study several preparatory results, which are needed to develop our convergence theorems.

Lemma 5.1. *let G be a generalized (α, β)-nonexpansive type (1) mapping define on a nonempty convex subset U of a hyperbolic metric space Y with $F_p(G) \neq \emptyset$ .If $p^* \in F_p(G)$ and $\{p_i\}_{i=1}^{\infty}$ is the iterative sequence define by (3.1) ,then $\lim_{i\to\infty} \rho(p_i, p^*)$ exists.*

Proof. *Using proposition (2.3) i,we get*

$$\begin{aligned} \rho(z_i, p^*) &= \rho(G(W(p_i, Gp_i, \alpha_i)), p^*) \\ &\leq \rho(W(p_i, Gp_i, \alpha_i), p^*) \\ &\leq (1-\alpha_i)\rho(p_i, p^*) + \alpha_i \rho(Gp_i, p^*) \\ &\leq (1-\alpha_i)\rho(p_i, p^*) + \alpha_i \rho(p_i, p^*) \\ &= \rho(p_i, p^*). \end{aligned} \tag{5.1}$$

Similarly

$$\begin{aligned} \rho(w_i, p^*) &= \rho(G(W(z_i, Gz_i, \beta_i)), p^*) \\ &\leq \rho(W(z_i, Gz_i, \beta_i), p^*) \\ &\leq (1-\beta_i)\rho(z_i, p^*) + \beta_i \rho(Gz_i, p^*) \\ &\leq (1-\beta_i)\rho(z_i, p^*) + \beta_i \rho(z_i, p^*) \\ &= \rho(z_i, p^*) \leq \rho(p_i, p^*). \end{aligned} \tag{5.2}$$

Similarly, we get

$$\begin{aligned} \rho(q_i, p^*) &= \rho(G(W(w_i, Gw_i, \gamma_i)), p^*) \\ &\leq \rho(W(w_i, Gw_i, \gamma_i), p^*) \\ &\leq (1-\gamma_i)\rho(w_i, p^*) + \gamma_i \rho(Gw_i, p^*) \\ &\leq (1-\gamma_i)\rho(w_i, p^*) + \gamma_i \rho(W_i, p^*) \\ &= \rho(w_i, p^*) \leq \rho(z_i, p^*) \leq \rho(p_i, p^*). \end{aligned} \tag{5.3}$$

Finally, we get

$$\begin{aligned} \rho(p_{i+1}, p^*) &= \rho(G(Gq_i), p^*) \\ &\leq \rho(Gq_i, p^*) \\ &= \rho(q_i, p^*) \leq \rho(w_i, p^*) \leq \rho(z_i, p^*) \leq \rho(p_i, p^*) \end{aligned} \tag{5.4}$$

Hence, we obtain

$$\rho(p_{i+1}, p^*) \leq \rho(p_i, p^*)$$

This shows that $\{\rho(p_i, p^*)\}_{i=1}^{\infty}$ is decreasing sequence and bounded from the below for each $p^* \in F_p(G)$ .So,we obtain that $\lim_{i\to\infty} \rho(p_i, p^*)$ exists for any $p^* \in F_p(G)$. □

Theorem 5.2. *let U be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space Y with the monotone modulus of uniform convexity η and $G : U \to U$ be a generalized (α, β)-nonexpansive type (1) mapping.Let $\{p_i\}_{i=1}^{\infty}$ be the iterative sequence define by (3.1) with real sequences $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ and $\{\gamma_i\}_{i=1}^{\infty}$ in $(0, 1)$ .Then, $F_p(G) \neq \emptyset$ if and only if $\{p_i\}_{i=1}^{\infty}$ is bounded sequence and $\lim_{i\to\infty} \rho(p_i, Gp_i) = 0$.*

Proof. *Let $F_p(G) \neq \emptyset$ and $p^* \in F_p(G)$ by Lemma (5.1), $\lim_{i\to\infty} \rho(p_i, p^*)$ exists and $\{p_i\}_{i=1}^{\infty}$ is bounded.Therefore,we can consider that*

$$\lim_{i\to\infty} \rho(p_i, p^*) = r \quad for \;\; some \; r \geq 0 \tag{5.5}$$

By proposition (2.3)(i) then we get

$$\rho(Gp_i, p^*) \leq \rho(p_i, p^*)$$

and we take lim sup of both sides of the inequality above ,we get that

$$\limsup_{i\to\infty} \rho(Gp_i, p^*) \leq r \tag{5.6}$$

and we have by (5.1)

$$\rho(z_i, p^*) \leq \rho(p_i, p^*)$$

and taking lim sup of both sides of the inequality above ,we get that

$$\limsup_{i\to\infty} \rho(z_i, p^*) \leq r \tag{5.7}$$

From the relation (5.4), it follows that

$$\begin{aligned}
\rho(p_{i+1}, p^*) &= \rho(G(Gq_i), p^*) \\
&\leq \rho(Gq_i, p^*) \\
&\leq \rho(q_i, p^*) \\
&\leq (1-\gamma_i)\rho(w_i, p^*) + \gamma_i \rho(Gw_i, p^*) \\
&\leq (1-\gamma_i)\rho(w_i, p^*) + \gamma_i \rho(w_i, p^*) \\
&= \rho(w_i, p^*) \\
&\leq (1-\beta_i)\rho(z_i, p^*) + \beta_i \rho(Gz_i, p^*) \\
&\leq (1-\beta_i)\rho(p_i, p^*) + \beta_i \rho(z_i, p^*) \\
&= \rho(p_i, p^*) - \beta_i \rho(p_i, p^*) + \beta_i \rho(z_i, p^*)
\end{aligned} \tag{5.8}$$

Since $\beta_i \in (0, 1)$,the last inequality leads to

$$\rho(p_{i+1}, p^*) - \rho(p_i, p^*) \leq \frac{\rho(p_{i+1}, p^*) - \rho(p_i, p^*)}{\beta_i} \leq \rho(z_i, p^*) - \rho(p_i, p^*)$$

which implies that

$$\rho(p_{i+1}, p^*) \leq \rho(z_i, p^*)$$

thus by (5.5) we get

$$r \leq \liminf_{i\to\infty} \rho(z_i, p^*) \tag{5.9}$$

from (5.7)and (5.9) we have

$$\lim_{i\to\infty} \rho(z_i, p^*) = r \tag{5.10}$$

from (5.1) we have

$$\rho(z_i, p^*) \leq \rho(W(p_i, Gp_i, \alpha_i)), p^*) \leq \rho(p_i, p^*)$$

from (5.5)and (5.10) we get

$$\lim_{i\to\infty} \rho(W(p_i, Gp_i, \alpha_i), p^*) = r. \tag{5.11}$$

Also from (5.5), (5.6),(5.11) and lemma (2.11) we obtain

$$\lim_{i\to\infty} \rho(p_i, Gp_i) = 0$$

Conversely,assume that $\{p_i\}_{i=0}^{\infty}$ *is bounded and* $\lim_{i\to\infty} \rho(p_i, Gp_i) = 0$ *suppose* $p^* \in A(U, \{p_i\})$ *Using proposition (2.3)(ii), we get*

$$\begin{aligned} r(Gp^*, \{p_i\}) &= \limsup_{i\to\infty} \rho(p_i, Gp^*) \\ &\le \frac{2+\alpha+\beta}{1-\beta} \limsup_{i\to\infty} \rho(p_i, Gp_i) + \limsup_{i\to\infty} \rho(p_i, p^*) \\ &= \limsup_{i\to\infty} \rho(p_i, p^*) \\ &= r(p^*, \{p_i\}) \end{aligned} \tag{5.12}$$

This implies that $Gp^* \in A(U, \{p_i\})$. *Because the sequence* $\{p_i\}_{i=1}^{\infty}$ *is bounded and we use lemma (2.10) then* $A(U, \{p_i\})$ *consists of exactly one point .Hence ,we have* $Gp^* = p^*$. *Thus,* $F_p(G) \neq \emptyset$.□

Considering the previous two results, we are now ready to prove the Δ-convergence theorem of the modified iterative sequence $\{p_i\}_{i=1}^{\infty}$ defined by (3.1) for a generalized (α, β)- nonexpansive type 1 mapping.

Theorem 5.3. *let* U *be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space* Y *with the monotone modulus of uniform convexity* η *and* $G : U \to U$ *be a generalized* (α, β)*-nonexpansive type* (1) *mapping with* $F_p(G) \neq \emptyset$. *Let* $\{p_i\}_{i=1}^{\infty}$ *be the iterative sequence (3.1) with real sequences* $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ *and* $\{\gamma_i\}_{i=1}^{\infty}$ *in* $(0, 1)$ *.Then, the sequence* $\{p_i\}_{i=1}^{\infty}$ $\Delta - convergence$ *to a fixed point of* G.

Proof. *From Lemma (2.10), the sequence* $\{p_i\}_{i=1}^{\infty}$ *has a unique asymptotic center* $A(U, \{p_i\}) = \{p\}$. *Suppose* $\{v_i\}_{i=1}^{\infty}$ *be any subsequence of* $\{p_i\}_{i=1}^{\infty}$ *such that* $A(U, \{v_i\}) = \{v\}$.*by theorem (5.2), we get*

$$\lim_{i\to\infty} \rho(v_i, Gv_i) = 0. \tag{5.13}$$

It follows similarly from the proof of Theorem (5.2) that v *is a fixed point of* G.*Then,we claim that the fixed point* v *is the unique asymptotic center for each subsequence* $\{v_i\}_{i=1}^{\infty}$ *of* $\{p_i\}_{i=1}^{\infty}$. *On the contrary ,we suppose that* $p \neq v$ *from lemma (5.1), we deduce that* $\lim_{i\to\infty} \rho(p_i, v)$ *exists .Therefore,by the uniqueness of the asymptotic center we can see that*

$$\begin{aligned} \limsup_{i\to\infty} \rho(v_i, v) &< \limsup_{i\to\infty} \rho(v_i, p) \\ &\le \limsup_{i\to\infty} \rho(p_i, p) \\ &< \limsup_{i\to\infty} \rho(p_i, v) \\ &= \limsup_{i\to\infty} \rho(v_i, v) \end{aligned}$$

then we get $p = v$ *this is a contradiction.So ,a fixed point* v *of* G *is the unique asymptotic center for each subsequence* $\{v_i\}_{i=1}^{\infty}$ *of* $\{p_i\}_{i=1}^{\infty}$.*This proves that the sequence* $\{p_i\}_{i=1}^{\infty}$ $\Delta - converges$ *to a fixed point of* G. □

Next , we prove tow strong convergence results for a generalized (α, β)- nonexpansive type 1 mapping.

Theorem 5.4. . *Under the assumptions of Theorem (5.3),if* U *is a compact subset of* Y*,then the sequence* $\{p_i\}_{i=1}^{\infty}$ *converges strongly to a fixed point of* G.

Proof. *We consider an element* $p^* \in U$. *The compactness of* U *implies that there exists a subsequence* $\{p_{i_n}\}_{i=1}^{\infty}$ *of* $\{p_i\}_{i=1}^{\infty}$ *such that* $\lim_{n\to\infty} \rho(p_{i_n}, p^*) = 0$.
According to proposition (2.3)(ii) ,we have

$$\lim_{n\to\infty} \rho(p_{i_n}, Gp^*) \le \frac{2+\alpha+\beta}{1-\beta} \lim_{n\to\infty} \rho(p_{i_n}, Gp_{i_n}) + \lim_{n\to\infty} \rho(p_{i_n}, p^*)$$

By theorem (5.2), we get $\lim_{n\to\infty} \rho(p_{i_n}, Gp_{i_n}) = 0$*.Then, we have* $Gp^* = p^*$ *that is* $p^* \in F_p(G)$*.Using (5.1), we obtain* $\lim_{i\to\infty} \rho(p_i, p^*)$ *exists and hence* $\{p_i\}$ *converges strongly to a fixed point* p^*.□

Theorem 5.5. . *let* Y, U, G *and* $\{p_i\}_{i=1}^{\infty}$ *be the same as in theorem (5.3).Then ,the sequence*$\{p_i\}_{i=1}^{\infty}$ *converges strongly to a fixed point of G if and only if*

$$\liminf_{i\to\infty} \rho(p_i, F_p(G)) = 0 \ \ or \ \ \limsup_{i\to\infty} \rho(p_i, F_p(G)) = 0,$$

where $\rho(p, F_p(G)) = \inf\{\rho(p, p^*) : p^* \in F_p(G)\}$.

Proof. *If the sequence* $\{p_i\}_{i=1}^{\infty}$ *converges strongly to* $p^* \in F_p(G)$*,then* $\lim_{i\to\infty} \rho(p_i, p^*) = 0$ *.Becouse* $0 \leq \rho(p_i, F_p(G)) \leq \rho(p_i, p^*)$*, we get* $\lim_{i\to\infty} \rho(p_i, F_p(G)) = 0$.
For the converse part,let $\liminf_{i\to\infty} \rho(p_i, F_p(G)) = 0$*.By Lemma (5.1) we get* $\lim_{i\to\infty} \rho(p_i, F_p(G))$ *exists and hence* $\lim_{i\to\infty} \rho(p_i, F_p(G)) = 0$*.Therefore,there exist a subsequence* $\{p_{i_n}\}_{i=1}^{\infty}$ *of* $\{p_i\}_{i=1}^{\infty}$ *and a sequence* $\{p_n^*\}_{n=1}^{\infty}$ *in* $F_p(G))$ *such that*

$$\rho(p_{i_n}, p_n^*) < \frac{1}{2^n} \ \ for \ all \ \ n \geq 1$$

By the proof of lemma (5.1) ,we get

$$\rho(p_{i_{n+1}}, p_{n+1}^*) \leq \rho(p_{i_n}, p_n^*) < \frac{1}{2^n}$$

thus,

$$\begin{aligned} \rho(p_{n+1}^*, p_n^*) &\leq \rho(p_{n+1}^*, p_{i_{n+1}}) + \rho(p_{i_{n+1}}, p_n^*) \\ &< \frac{1}{2^{n+1}} + \frac{1}{2^n} < \frac{1}{2^{n+1}} \\ &\to 0 \ \ as \ \ \to \infty \end{aligned}$$

Hence, $\{p_n^*\}_{i=1}^{\infty}$ *is a Cauchy sequence in* $F_p(G)$*.From proposition (2.3)(iii), we have* $F_p(G)$ *is closed and so* $\{p_n^*\}$ *converges strongly to* $p^* \in F_p(G)$*.On the other hand,we get*

$$\rho(p_{i_n}, p^*) \leq \rho(p_{i_n}, p_n^*) + \rho(p_n^*, p^*)$$

taking the limit of both sides of this inequality, we get that$\{p_{i_n}\}_{i=1}^{\infty}$ *converges strongly to* $p^* \in F_p(G)$*. Because* $\lim_{i\to\infty} \rho(p_i, p^*)$ *exists by lemma(5.1),*p^* *is strong limit of* $\{p_i\}_{i=1}^{\infty}$. □

In 1974, Senter and Dotson [28] introduced a mapping satisfying Condition (I), which is stated as follows: A mapping $G : U \to U$ is said to satisfy Condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(b) > 0$ for all $b \in (0, \infty)$, such that$\rho(p, Gp) \geq f(\rho(p, F_p(G)))$ for all $p \in U$.
Now, we present the final strong convergence result using Condition (I).

Theorem 5.6. *Let U be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space Y with the monotone modulus of uniform convexity* η *and* $G : U \to U$ *be a generalized* (α, β)*-nonexpansive type* (1) *mapping with* $F_p(G) \neq \emptyset$*. If G satisfies condition (1) and* $\{p_i\}_{i=1}^{\infty}$ *is the iterative sequence define by (3.1) with real sequences* $\{\alpha_i\}_{i=1}^{\infty}$*,* $\{\beta_i\}_{i=1}^{\infty}$ *and* $\{\gamma_i\}_{i=1}^{\infty}$ *in* $(0, 1)$ *,then* $\{p_i\}_{i=1}^{\infty}$ *converges strongly to a point of* $F_p(G)$

Proof. *From theorem (5.2),we have* $\lim_{i\to\infty} \rho(p_i, Gp_i) = 0$*. Then ,by condition (1),we get* $\lim_{i\to\infty} f(\rho((p_i, F_p(G))) \leq \lim_{i\to\infty} \rho(p_i, Gp_i) = 0$*, that is,* $\lim_{i\to\infty} f(\rho(p_i, F_p(G))) = 0$*. Because* $f : [0, \infty) \to [0, \infty)$ *is a function with* $f(0) = 0$ *and* $f(b) > 0$ *for all* $b \in (0, \infty)$*, we have* $\lim_{i\to\infty} \rho((p_i, F_p(G)) = 0$*. By pervious theorem we obtain* $\{p_i\}_{i=1}^{\infty}$ *converges strongly to a point of* $F_p(G)$ □

Remark 5.7. . *In this section, we used the generalized* (α, β)*-nonexpansive type* (1) *mapping which contains the class of generalized* α*-nonexpansive mapping on the hyperbolic metric space. Thus, Theorems 5.3-5.6 generalize the results of [33, 36] in two ways: (1) the class of underlying space, and (2) the class of mappings.*

6 Numerical Example

In this section, we construct the following example of a generalized (α, β)-nonexpansive type 1 mapping.

Example 6.1. *Let $Y = R$ with the usual metric and $X = [0, \infty)$. Define a mapping $G : X \to X$.*

$$Ga = \begin{cases} 0 \;\; if \;\; a \in [0, \frac{12}{10}) \\ \frac{10a}{24} \;\; if \;\; [\frac{12}{10}, \infty) \end{cases} \tag{6.1}$$

Clearly, $a = 0$ is the fixed point of G. Then, the following:
(i) Because G is not continuous at the point $a = \frac{12}{10}$, G is not a nonexpansive mapping.
(ii) Let $a = \frac{8}{10}$ and $b = \frac{12}{10}$. Then,

$$\frac{1}{2} \mid a - Ga \mid = \frac{4}{10} \leq \frac{4}{10} = \mid a - b \mid$$

On the other hand ,

$$\mid Ga - Gb \mid = \frac{1}{2} \geq \frac{4}{10} = \mid a - b \mid$$

Thus, G is not a Suzuki-generalized nonexpansive mapping.
(iii) Let $a = \frac{8}{10}$ and $b = \frac{12}{10}$. Then,

$$\mid Ga - Gb \mid \leq \alpha \mid a - b \mid + \beta \mid a - Gb \mid$$
$$\frac{1}{2} \leq \frac{4\alpha}{10} + \frac{3\beta}{10}$$
$$5 \leq 4\alpha + 3\beta$$

Therefore, the implications fail to be satisfied, which leads to the conclusion that G is not a mean nonexpansive mapping.
(iv) Now, we prove that G is a generalized (α, β)-nonexpansive type 1 mapping. For this purpose,let $\lambda = \frac{1}{4}$, $\alpha = \frac{11}{24}$, $\beta = \frac{12}{24}$, and consider the following cases:

- *Case A: $a \in [0, \frac{12}{10})$. Then,$\lambda \mid a - Ga \mid = \frac{1a}{4} \leq \mid a - b \mid$, which gives two possibilities (1) Let $a < b$. Then, $\frac{1a}{4} \leq b - a \Rightarrow b \geq \frac{5}{4}a \Rightarrow b \in [0, \frac{3}{2})$.*

(a) If $b \in [0, \frac{12}{10})$, then we have

$$\mid Ga - Gb \mid \leq \alpha \mid Ga - b \mid + \beta \mid a - Gb \mid + (1 - (\alpha + \beta)) \mid a - b \mid$$

.

$$0 \leq \frac{11}{24} \mid b \mid + \frac{12}{24} \mid a \mid + \frac{1}{24} \mid a - b \mid$$

(b) If $b \in [\frac{12}{10}, \frac{3}{2})$,then we have

$$\mid Ga - Gb \mid = \frac{10}{24} \mid b \mid \leq \frac{11}{24} \mid b \mid + \frac{12}{24} \mid a - \frac{10}{24}b \mid + \frac{1}{24} \mid a - b \mid$$

(2) Let $a > b$. Then,$\frac{1a}{4} \leq a - b \Rightarrow b \leq \frac{3}{4}a \Rightarrow b \in [0, \frac{9}{10}) \subset [0, \frac{12}{10})$, which is already included in case (1)(a).

- *Case B: $a \in [\frac{12}{10}, \infty)$. Then,$\lambda \mid a - Ga \mid = \frac{1}{4} \mid a - \frac{10}{24}a \mid = \frac{7}{48}a \leq \mid a - b \mid$, which gives two possibilities:*

(1) Let $a < b$. Then, $\frac{7}{48}a \leq b - a \Rightarrow b \geq \frac{55}{48}a \Rightarrow b \in [\frac{11}{8}, \infty) \subset [\frac{12}{10}, \infty)$. So

$$\begin{aligned} \mid Ga - Gb \mid &= \frac{10}{24} \mid a - b \mid \\ &< \frac{11}{24}(\mid \frac{34}{24}a - \frac{34}{24}b \mid) + \frac{1}{24} \mid a - b \mid \\ &\leq \frac{11}{24} \mid \frac{10}{24}a - b \mid + \frac{11}{24} \mid a - \frac{10}{24}b \mid + \frac{1}{24} \mid a - b \mid \\ &\leq \frac{11}{24} \mid \frac{10}{24}a - b \mid + \frac{12}{24} \mid a - \frac{10}{24}b \mid + \frac{1}{24} \mid a - b \mid \end{aligned}$$

(2) Let $a > b$. Then, $\frac{7}{48}a \leq a - b \Rightarrow b \leq \frac{41}{48}a \Rightarrow b \in [\frac{41}{40}, \infty)$.
(a) If $b \in [\frac{41}{40}, \frac{12}{10})$ *, then we have*

$$|Ga - Gb| = \frac{10}{24} | a | \leq \frac{11}{24} | \frac{10}{24}a - b | + \frac{12}{24} | a | + \frac{1}{24} | a - b |$$

(b) If $b \in [\frac{12}{10}, \infty)$ *is already included in case (1).*
Hence,G is a generalized $(\frac{11}{24}, \frac{12}{24})$*-nonexpansive type 1 mapping with* $F_p(G) \neq \phi$

7 Conclusions

In this paper, we have modified the our newly introduced iterative algorithm (1.1) into the hyperbolic metric spaces and established the weak w^2-stability and data dependence results for contraction mappings .We derived some convergence results for(α, β)-nonexpansive type 1 mappings using this modified iterative scheme.
Finally, as future works for this paper, we appointed the following:-
Using similar approaches of this article, the generalized (α, β)-nonexpansive type 2 mappings,which is introduced by d by Akutsah and Narain [4], can be studied in hyperbolic metric spaces .

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