ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2025) 2025:4

New $\left(\frac{G''}{G'+A}\right)$-Expansion Method and New Improved $\left(\frac{G''}{G'+A}\right)$-Expansion Method for Generalized Nonlinear Burgers' Equation

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Abstract
In this paper, we presented two new methods, namley, the $\left(\frac{G''}{G'+A}\right)$-expansion method and the improved $\left(\frac{G''}{G'+A}\right)$-expansion method is proposed for constructing more general and a rich class of new exact traveling wave solutions of nonlinear evolution equations (NLEES).

Mainly, methods have been applied to generalized nonlinear Burgers' (gB) equation for new exact solutions. Moreover, obtained solutions reveal that, these methods are very effective and powerful to handle various nonlinear evolution equations.

Keywords: $\left(\frac{G''}{G'+A}\right)$- expansion method, Improved $\left(\frac{G''}{G'+A}\right)$- expansion, generalized nonlinear Burgers' equation, nonlinear evolution equations, traveling wave solutions.

1 Introduction

Study of nonlinear evolution equations (NLEEs) have demonstrated to be imperative in modelling various nonlinear systems that model numerous and varied natural phenomena. Numerous techniques have been proposed to investigate exact solutions of such equations (NLEEs) such the sine-cosine [1], the tanh-coth method [2, 3], the mapping method [4, 5], the improved F-expansion scheme [6], the Adomain decomposition approach [7, 8], Darboux transformation [9], the $\left(\frac{G'}{G}\right)$-expansion method [10, 11], the improved and generalized $\left(\frac{G'}{G}\right)$ expansion methods [12], the $\left(\frac{G'}{G^2}\right)$-expansion method [13], the $\left(\frac{G'}{G}\diagup\frac{1}{G}\right)$ -expansion method [14], the extended hyperbolic function method [15], the Jacobi elliptic function expansion method [16].Sometimes, numerical techniques are good alternatives to analytical ones, for instance, see [17, 18].

The importance of our present work is, in order to generate many exact traveling wave solutions, new approach $\left(\frac{G''}{G'+A}\right)$-expansion method and new approach improved $\left(\frac{G''}{G'+A}\right)$-expansion method. For illustration and to depict of the proposed methods, the generalized nonlinear Burgers' (gB) equation have been studied and generated abundant and more types of new travelling wave solutions.

Received: 2024-11-21

2 Description of new methods

Consider the general nonlinear partial differential equations (NLPDEs), say, in two variables,

$$\Psi\left(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots\right) = 0, \tag{1}$$

where $v = v(x,t)$ is an unknown function, Ψ is a polynomial in $v(x,t)$ and the subscripts stand for the partial derivatives.

We suppose that the combination of real variables x and t by a complex variable ξ

$$v(x,t) = v(\xi), \quad \xi = ax - ct, \tag{2}$$

where a is the wave number and c is the speed of the traveling wave. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $v = v(\xi)$:

$$\Phi\left(v, -cv', av', c^2v'', -cav'', a^2v'', \ldots\right) = 0, ' \equiv \frac{\partial}{\partial\xi}. \tag{3}$$

According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

2.1 New $\left(\frac{G''}{G'+A}\right)$-expansion method

Suppose that the traveling wave solution of Eq. (3) can be expressed as follows:

$$v(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G''}{G'+A}\right)^i, \tag{4}$$

where the coefficients $a_i (i = 0, 1, 2, \ldots, N)$, a and c are arbitrary constants, and $G = G(\xi)$ satisfies the following auxiliary ordinary differential equation (ODE)

$$G''' + \mu G' + \lambda = 0, \tag{5}$$

then by the help of Eq. (5) we get

$$\left(\frac{G''}{G'+A}\right)' = -\left(\frac{G''}{G'+A}\right)^2 - \mu, \tag{6}$$

where $\lambda = A\mu$; and A are constants, the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (3).

Substituting Eq. (4) into Eq. (3), using Eq. (6) repeatedly, and setting the coefficients of the each order of $\left(\frac{G''}{G'+A}\right)^i$ to zero, we obtain a set of nonlinear algebraic equations for $a_i (i = 0, 1, 2, \ldots, N)$, a, c and μ. With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i (i = 0, 1, 2, \ldots, N)$, a and c.

2.2 New improved $\left(\frac{G''}{G'+A}\right)$- expansion method

Suppose that the travelling wave solution of Eq. (3) can be expressed as follows:

$$v(\xi) = \sum_{i=0}^{N} a_i \left(\frac{\left(\frac{G''}{G'+A}\right)}{1 + \alpha\left(\frac{G''}{G'+A}\right)}\right)^i, \tag{7}$$

where the coefficients $a_i(i = 0, 1, 2, \ldots, N)$, a, c and α are arbitrary constants. Moreover, $G = G(\xi)$ satisfies auxiliary ODE Eq. (5).

We determine the positive integer N, explained previously.

Substituting Eq. (7) into Eq. (3), using Eq. (6) repeatedly, and setting the coefficients of the each order of $\left(\frac{G''}{G'+A}\right)^i$ to zero, we obtain a set of nonlinear algebraic equations for $a_i(i = 0, 1, 2, \ldots, N)$, a, c, α and μ. With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i(i = 0, 1, 2, \ldots, N)$, a and c.

Using the general solution of Eq. (5), we have the following solutions

Family 1. When $\mu < 0$,

$$\left(\frac{G''}{G'+A}\right) = \frac{\sqrt{-\mu}\left(C_1 \cosh\left(\sqrt{-\mu}\,(\xi+h)\right) + C_2 \sinh\left(\sqrt{-\mu}\,(\xi+h)\right)\right)}{C_1 \sinh\left(\sqrt{-\mu}\,(\xi+h)\right) + C_2 \cosh\left(\sqrt{-\mu}\,(\xi+h)\right)}, \tag{8}$$

case (i). If $C_2 \neq 0,\ C_1 = 0$,

$$\left(\frac{G''}{G'+A}\right) = \sqrt{-\mu}\tanh\left(\sqrt{-\mu}\,(\xi+h)\right), \tag{9}$$

case (ii). If $C_1 \neq 0,\ C_2 = 0$,

$$\left(\frac{G''}{G'+A}\right) = \sqrt{-\mu}\coth\left(\sqrt{-\mu}\,(\xi+h)\right). \tag{10}$$

Family 2. When $\mu > 0$,

$$\left(\frac{G''}{G'+A}\right) = \frac{\sqrt{\mu}\left(C_1 \cos\left(\sqrt{\mu}\,(\xi+h)\right) + C_2 \sin\left(\sqrt{\mu}\,(\xi+h)\right)\right)}{C_1 \sin\left(\sqrt{\mu}\,(\xi+h)\right) - C_2 \cos\left(\sqrt{\mu}\,(\xi+h)\right)}, \tag{11}$$

case (i). If $C_2 \neq 0,\ C_1 = 0$,

$$\left(\frac{G''}{G'+A}\right) = -\sqrt{\mu}\tan\left(\sqrt{\mu}\,(\xi+h)\right), \tag{12}$$

case (ii). If $C_1 \neq 0,\ C_2 = 0$,

$$\left(\frac{G''}{G'+A}\right) = \sqrt{\mu}\cot\left(\sqrt{\mu}\,(\xi+h)\right), \tag{13}$$

where h is constant of integration.

3 Application of new methods

In this section, we aim first to apply the our new methods for solve (gB) equation.

3.1 The gB equation

consider the gB equation [19, 20]

$$u_t\,(x,t) + u^r\,(x,t)\,u_x\,(x,t) - \beta u_{xx}\,(x,t) = 0, \tag{14}$$

using the transformation $u(x,t) = u(\xi),\ \xi = ax - ct$ in Eq. (14), we find

$$-cu'(\xi) + au^r(\xi)u'(\xi) - \beta a^2 u''(\xi) = 0. \tag{15}$$

Integrating once Eq. (15), and neglecting the constants of integration, we find

$$-cu(\xi)+\frac{a}{r+1}\left(u(\xi)\right)^{r+1}-\beta a^{2}u'(\xi)=0. \tag{16}$$

Balancing Eq. (16) give:

$$(r+1)N=N+1\Rightarrow N=\frac{1}{r}.$$

To obtain closed form solutions, should be a positive integer. To achieve this goal we use

$$u(\xi)=(w(\xi))^{\frac{1}{r}},2<r\in z^{+} \tag{17}$$

that will carry out Eq. (16) to the ODE

$$-cw(\xi)+\frac{a}{r+1}\left(w(\xi)\right)^{2}-\frac{\beta a^{2}}{r}w'(\xi)=0. \tag{18}$$

Application of new $\left(\frac{G''}{G'+A}\right)$-expansion method

Taking the homogeneous balance between w^2 and w' in Eq. (18), we obtain $N=1$. Therefore, the solution of Eq. (18) is of the form:

$$w(\xi)=a_0+a_1\left(\frac{G''}{G'+A}\right), \tag{19}$$

where a_0 and a_1 are arbitrary constants to be determined.

Substituting Eq. (19) into Eq. (18), using Eq. (6) repeatedly, and setting the coefficients of the each order of $\left(\frac{G''}{G'+A}\right)^{i}$ to zero, we obtain a set of nonlinear algebraic equations for $a_i (i=0,1,2,\ldots,N)$, a, c, r, β and μ. Solving the system of algebraic equations with the aid of the computer program Maple, we obtain

$$a_0=\frac{\pm\sqrt{-\mu}a\beta\,(r+1)}{r},\ a_1=\frac{-a\beta\,(r+1)}{r},\ c=\frac{\pm2\sqrt{-\mu}a^{2}\beta}{r}. \tag{20}$$

Substituting Eq. (20) into Eq. (19), along with Eq. (8) and simplifying, yields

$$u_{1,2}(\xi)=\left(\frac{\sqrt{-\mu}a\beta\,(r+1)}{r}\right)^{\frac{1}{r}}\left[\pm1-\frac{C_1\cosh\left(\sqrt{-\mu}\,(\xi+h)\right)+C_2\sinh\left(\sqrt{-\mu}\,(\xi+h)\right)}{C_1\sinh\left(\sqrt{-\mu}\,(\xi+h)\right)+C_2\cosh\left(\sqrt{-\mu}\,(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

Substituting Eq. (20) into Eq. (19), along with Eq. (9) and simplifying, yields

$$u_{3,4}(\xi)=\left[\frac{\sqrt{-\mu}a\beta\,(r+1)}{r}\left(1-\tanh\left(\sqrt{-\mu}\,(\xi+h)\right)\right)\right]^{\frac{1}{r}}.$$

Substituting Eq. (20) into Eq. (19), along with Eq. (10) and simplifying, yields

$$u_{5,6}(\xi)=\left[\frac{\sqrt{-\mu}a\beta\,(r+1)}{r}\left(1-\coth\left(\sqrt{-\mu}\,(\xi+h)\right)\right)\right]^{\frac{1}{r}}.$$

Substituting Eq. (20) into Eq. (19), along with Eq. (11) and simplifying, yields

$$u_{7,8}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\pm\sqrt{-\mu} - \frac{\sqrt{\mu}\left(C_1\cos\left(\sqrt{\mu}(\xi+h)\right)+C_2\sin\left(\sqrt{\mu}(\xi+h)\right)\right)}{C_1\sin\left(\sqrt{\mu}(\xi+h)\right)-C_2\cos\left(\sqrt{\mu}(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

Substituting Eq. (20) into Eq. (19), along with Eq. (12) and simplifying, yields

$$u_{9,10}(\xi) = \left[\frac{a\beta(r+1)}{r}\left(\pm\sqrt{-\mu}+\sqrt{\mu}\tan\left(\sqrt{\mu}(\xi+h)\right)\right)\right]^{\frac{1}{r}}.$$

Substituting Eq. (20) into Eq. (19), along with Eq. (13) and simplifying, yields

$$u_{11,12}(\xi) = \left[\frac{a\beta(r+1)}{r}\left(\pm\sqrt{-\mu}-\sqrt{\mu}\cot\left(\sqrt{\mu}(\xi+h)\right)\right)\right]^{\frac{1}{r}},$$

where $\xi = ax \mp \frac{2\sqrt{-\mu}a^2\beta}{r}t$.

Application of new improved $\left(\frac{G''}{G'+A}\right)$-expansion method

Similarly, taking the homogeneous balance between w^2 and w' in Eq. (18), we obtain $N = 1$. Therefore, the solution of Eq. (18) is of the form:

$$w(\xi) = a_0 + a_1\left(\frac{\left(\frac{G''}{G'+A}\right)}{1+\alpha\left(\frac{G''}{G'+A}\right)}\right), \tag{21}$$

where a_0 and a_1 are arbitrary constants to be determined.

Substituting Eq. (21) into Eq. (18), using Eq. (6) repeatedly, and setting the coefficients of the each order of $\left(\frac{G''}{G'+A}\right)^i$ to zero, we obtain a set of nonlinear algebraic equations for $a_i (i = 0, 1, 2, ..., N)$, a, c, r, β, α and μ. Solving the system of algebraic equations with the aid of the computer program Maple, we obtain

$$a_0 = \frac{a\beta(r+1)\left(\alpha\mu\pm\sqrt{-\mu}\right)}{r},\ a_1 = \frac{-a\beta(r+1)\left(\alpha^2\mu+1\right)}{r},\ c = \frac{\pm 2\sqrt{-\mu}a^2\beta}{r}. \tag{22}$$

Substituting Eq. (22) into Eq. (21), along with Eq. (8) and simplifying, yields

$$u_{1,2}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu\pm\sqrt{-\mu}\right) - \frac{\sqrt{-\mu}\left(\alpha^2\mu+1\right)\left(C_1\cosh\left(\sqrt{-\mu}(\xi+h)\right)+C_2\sinh\left(\sqrt{-\mu}(\xi+h)\right)\right)}{A_1\sinh\left(\sqrt{-\mu}(\xi+h)\right)+A_2\cosh\left(\sqrt{-\mu}(\xi+h)\right)}\right]^{\frac{1}{r}},$$

where $A_1 = C_1 + \alpha C_2\sqrt{-\mu}$, $A_2 = C_2 + \alpha C_1\sqrt{-\mu}$.

Substituting Eq. (22) into Eq. (21), along with Eq. (9) and simplifying, yields

$$u_{3,4}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu\pm\sqrt{-\mu}\right) - \frac{\sqrt{-\mu}\left(\alpha^2\mu+1\right)\tanh\left(\sqrt{-\mu}(\xi+h)\right)}{1+\alpha\sqrt{-\mu}\tanh\left(\sqrt{-\mu}(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

Substituting Eq. (22) into Eq. (21), along with Eq. (10) and simplifying, yields

$$u_{5,6}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu \pm \sqrt{-\mu}\right) - \frac{\sqrt{-\mu}\left(\alpha^2\mu+1\right)\coth\left(\sqrt{-\mu}(\xi+h)\right)}{1+\alpha\sqrt{-\mu}\coth\left(\sqrt{-\mu}(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

Substituting Eq. (22) into Eq. (21), along with Eq. (11) and simplifying, yields

$$u_{7,8}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu \pm \sqrt{-\mu}\right) - \frac{\sqrt{\mu}\left(\alpha^2\mu+1\right)\left(C_1\cos\left(\sqrt{\mu}(\xi+h)\right)+C_2\sin\left(\sqrt{\mu}(\xi+h)\right)\right)}{A_1\sin\left(\sqrt{\mu}(\xi+h)\right)+A_2\cos\left(\sqrt{\mu}(\xi+h)\right)}\right]^{\frac{1}{r}},$$

where $A_1 = C_1 + \alpha C_2\sqrt{\mu}$, $A_2 = \alpha C_1\sqrt{\mu} - C_2$.
Substituting Eq. (22) into Eq. (21), along with Eq. (12) and simplifying, yields

$$u_{9,10}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu \pm \sqrt{-\mu}\right) + \frac{\sqrt{\mu}\left(\alpha^2\mu+1\right)\tan\left(\sqrt{\mu}(\xi+h)\right)}{1-\alpha\sqrt{\mu}\tan\left(\sqrt{\mu}(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

Substituting Eq. (22) into Eq. (21), along with Eq. (13) and simplifying, yields

$$u_{11,12}(\xi) = \left(\frac{a\beta(r+1)}{r}\right)^{\frac{1}{r}} \left[\left(\alpha\mu \pm \sqrt{-\mu}\right) - \frac{\sqrt{\mu}\left(\alpha^2\mu+1\right)\cot\left(\sqrt{\mu}(\xi+h)\right)}{1+\alpha\sqrt{\mu}\cot\left(\sqrt{\mu}(\xi+h)\right)}\right]^{\frac{1}{r}}.$$

where $\xi = ax \mp \frac{2\sqrt{-\mu}a^2\beta}{r}t$.

4 Conclusion

In this paper, new approach of $\left(\frac{G''}{G'+A}\right)$-expansion method and new improved $\left(\frac{G''}{G'+A}\right)$-expansion method with linear auxiliary equation have been proposed to construct many new and more general exact solutions of NLEEs. The methods have been successfully implemented to find new for generalized nonlinear Burgers' (gB) equation. The results show that this method is a powerful Mathematical tool for obtaining new exact solutions for our equation. It is also a promising method to solve other NLEEs.

References

[1] N. Azizi, R. Pourgholi, Applications of Sine–Cosine wavelets method for solving the generalized Hirota–Satsuma coupled KdV equation, Math. Sci. 17 (2023), 503-516.

[2] S.N. Ananna, et al., Solitary wave structures of a family of 3D fractional WBBM equation via the tanh–coth approach, Part. Diff. Equ. Appl. Math. 5 (2022), 100237.

[3] F.M. Al-Askar, et al., The Impact of the Wiener process on the analytical solutions of the stochastic (2+1)-dimensional breaking soliton equation by using tanh–coth method, Mathematics 10 (2022), 817.

[4] A. Maher, H.M. El-Hawary, M.S. Al-Amry, New exact solutions for new model nonlinear partial differential equation, J. Appl. Math. 2013 (2013), 767380.

[5] A. Maher, H.M. El-Hawary, M.S. Al-Amry, The mapping method for solving a new model of nonlinear partial differential equation, Int. J. Pure Appl. Math. 95 (2014), 181-208.

[6] X.-Q. Zhao, H.-Y. Zhi, An Improved F-Expansion Method and Its Application to Coupled Drinfel'd–Sokolov–Wilson Equation, Comm. Theor. Phys. 50 (2008), 309-316.

[7] Q. Wang, Numerical solutions for fractional KdV–Burgers equation by Adomian decomposition method, Appl. Math. Comp. 182 (2006), 1048-1055.

[8] O. Gonzalez-Gaxiola, et al., Highly dispersive optical solitons with non-local law of refractive index by Laplace-Adomian decomposition, Opt. Quant. Elec. 53 (2021), 1-12.

[9] C.-Q. Dai, Y.-Y. Wang, J.-F. Zhang, Managements of scalar and vector rogue waves in a partially nonlocal nonlinear medium with linear and harmonic potentials, Nonlinear Dyn. 102 (2020), 379-391.

[10] A. Aniqa, J. Ahmad, Soliton solution of fractional Sharma-Tasso-Olever equation via an efficient (G'/G)-expansion method, Ain Shams Eng. J. 13 (2022), 101528.

[11] S.K. Mohanty, et al., An efficient technique of (G'/G)-expansion method for modified KdV and Burgers equations with variable coefficients, Results Phys. 37 (2022), 105504.

[12] S.K. Mohanty, O.V. Kravchenko, A.N. Dev, Exact traveling wave solutions of the Schamel Burgers' equation by using generalized-improved and generalized (G'/G)-expansion methods, Results in Physics 33 (2022), 105124.

[13] M.S. Al-Amry, E.F. Abdullah, Exact solutions for a new models of nonlinear partial differential equations Using (G'/G^2)-Expansion Method, Univ. Aden J. Nat. Appl. Sci. 23.1 (2019), 189-199.

[14] M.A. Iqbal, et al., New soliton solutions of the mZK equation and the Gerdjikov-Ivanov equation by employing the double (G'/G ÷ 1/G)-expansion method, Results Phys. 47 (2023), 106391.

[15] M.S. Al-Amry, E.F. Al-Abdali, New Exact Solutions for Generalized of Combined with Negative Calogero-Bogoyavlenskii Schiff and Generalized Yu–Toda–Sassa–Fukuyama Equations, Univ. Aden J. Nat. Appl. Sci. 28 (2024), 25-30.

[16] J.A. Haider, S. Asghar, S. Nadeem, Travelling wave solutions of the third-order KdV equation using Jacobi elliptic function method, Int. J. Mod. Phys. B 37 (2023), 2350117.

[17] J. Alzabut, et al., Existence, uniqueness and synchronization of a fractional tumor growth model in discrete time with numerical results, Results Phys. 54 (2023), 107030.

[18] H. Khan, et al., Nonlinear dynamics of a piecewise modified ABC fractional-order leukemia model with symmetric numerical simulations, Symmetry 15 (2023), 1338.

[19] D. Li, C. Zhang, M. Ran, A linear finite difference scheme for generalized time fractional Burgers equation, Appl. Math. Model. 40 (2016), 6069-6081.

[20] H. Bulut, H.M. Baskonus, Y. Pandir, The modified trial equation method for fractional wave equation and time fractional generalized Burgers equation, Abstr. Appl. Anal. 2013 (2013), 1-12.