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Fixed Point Theorems for Interpolative Ćirić-Reich-Rus-Type Contraction Mappings in b_2-Metric Spaces

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Abstract
This paper introduces new fixed point theorems for Ćirić-Reich-Rus-type contraction mappings in b_2-metric spaces. We provide an example to demonstrate the proven results. Finally, we apply the results to fractional differential equations. The fractional derivative as convolution serves numerous functions. It can represent memory, as in the elasticity hypothesis. The Caputo and Caputo-Fabrizio types can be understood as filters of the local derivative, having power and exponent functions.

Keywords: Fixed point theorem, interpolative mapping, b_2-metric space, functional differential equations

1 Introduction

Kannan [11] developed a discontinuity of contraction mappings that can possess a fixed point on a complete metric space, closing the gap established by Banach [4] for nearly thirty years. Reich [18] demonstrated the fixed point theorem with three metric points, combining Banach and Kanann notions on a full metric space. Ćirić [5] expanded Banach's ideas to establish the fixed point theorem with six metrics. Kannan [11], Reich [18], Ćirić [5] and Rus [19] fixed point theorems have been examined and expanded in several directions.

Later, Karapinar *et al.* [12] blends Reich-Ćirić and Rus ideas to prove an interpolative Reich-Rus-Ćirić in partial metric spaces. Karapinar and Agarwal [13] analysed interpolative Rus-Reich-Ćirić contractions in metric space using simulation functions. Aydi *et al.* [1] showed ω-interpolative Reich-Rus-Ćirić contractions in metric spaces. Mishra et al. [15] established the fixed point theorem for interpolative Reich-Rus-Ćirić and Hardy-Rogers contraction on quasi-partial b-metric space, as well as related fixed point results. For more literature on this direction we refer our readers to [2, 9, 23, 24].

Czerwik's [6] study of a b-metric space yielded numerous fixed point solutions for single-valued mappings. Gahler established the concept of a b-metric in [10], using the example of a triangle's area in $\mathbf{R}^2$. Similarly, fixed point results were found for mappings in such spaces. Unlike other recent metric space generalisations, 2-metric spaces are not topologically identical to metric spaces. The conclusions achieved in 2-metric spaces

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do not easily translate to metric spaces. Mustafa et al. [16] created a new type of generalised metric space known as b_2-metric spaces by combining 2-metric and b-metric notions. Singh *et al.* [21] demonstrated fixed point theorems in a generalised b_2-metric space of (ψ, ϕ)-weakly contractive mapping. For more innovation on this direction we recall [7, 8, 20, 22].

In this paper, we introduce some fixed point theorems under various contractive conditions in a generalized b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces. Then we prove. These include interpolative Rus-Reich-Ćirić contractions conditions. We illustrate these results by appropriate examples, as well as an application to fractional differential equations. The results is focused on extension and generalization of the results from Mustafa *et al.* [16], Karapinar*et al.* [12] and several other works from the literature.

2 Preliminaries

This subsection provides some important definitions, lemmas, propositions, and theorems to help with the proving of the key results.

We use the notion of 2-metric spaces proposed by Gahler in [10] below.

Definition 2.1. *Let X be a non-empty set and $d_2 : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties.*

(i) For $\gamma, \varsigma, \kappa \in X$, $d_2(\gamma, \varsigma, \kappa) = 0$ when at least two of the three points are the same.

(ii) If $\gamma \neq \varsigma$, then there is a point $\kappa \in X$ where $d_2(\gamma, \varsigma, \kappa) \neq 0$.

(iii) Symmetry property for $\gamma, \varsigma, \kappa \in X$,

$$\begin{aligned} d_2(\gamma, \varsigma, \kappa) &= d_2(\gamma, \kappa, \varsigma) = d_2(\varsigma, \gamma, \kappa) = d_2(\varsigma, \kappa, \gamma) \\ &= d_2(\kappa, \gamma, \varsigma) = d_2(\kappa, \varsigma, \gamma). \end{aligned}$$

(iv) Rectangle inequality

$$d_2(\gamma, \varsigma, \kappa) \leq d_2(\gamma, \varsigma, \vartheta) + d_2(\varsigma, \kappa, \vartheta) + d_2(\kappa, \gamma, \vartheta)$$

for $\gamma, \varsigma, \kappa \in X$.

Then d_2 is a 2-metric and (X, d_2) is a 2-metric space.

Mustafa *et al.* [16] defined b_2-metric as the combination of 2-metric and b-metric notions.

Definition 2.2. *Let X be a non-empty set, and $d_2 : X \times X \times X \to [0, \infty)$ be a map that meets the following requirements.*

(i) For $\gamma, \varsigma, \kappa \in X$, $d_2(\gamma, \varsigma, \kappa) = 0$ when at least two of the three points are the same.

(ii) If $\gamma \neq \varsigma$, then there is a point $\kappa \in X$ where $d_2(\gamma, \varsigma, \kappa) \neq 0$.

(iii) Symmetry property for $\gamma, \varsigma, \kappa \in X$,

$$\begin{aligned} d_2(\gamma, \varsigma, \kappa) &= d_2(\gamma, \kappa, \varsigma) = d_2(\varsigma, \gamma, \kappa) = d_2(\varsigma, \kappa, \gamma) \\ &= d_2(\kappa, \gamma, \varsigma) = d_2(\kappa, \varsigma, \gamma). \end{aligned}$$

(iv) s-rectangle inequality: there exists $s \geq 1$ such that

$$d_2(\gamma, \varsigma, \kappa) \leq s[d_2(\gamma, \varsigma, \vartheta) + d_2(\varsigma, \kappa, \vartheta) + d_2(\kappa, \gamma, \vartheta)]$$

for $\gamma, \varsigma, \kappa, \vartheta \in X$.

Then d_2 is a b_2-metric and (X, d_2) is a b_2-metric space. If $s = 1$, the b_2-metric reduces to the 2-metric.

The following example satisfies the properties of b_2-metric.

Example 2.1. *Consider $X = [0, \infty)$ and $d_2(\gamma, \varsigma, \kappa) = [\gamma\varsigma+\varsigma\kappa+\kappa\gamma]^p$ if $\gamma \neq \varsigma \neq \kappa \neq \gamma$. and else $d_2(\gamma, \varsigma, \kappa) = 0$, where $p \geq 1$ is an area number. Given the convexity of the function $S(\gamma) = \gamma^p$ for $\gamma \geq 0$, using Jensen's inequality, we have*

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

Thus, one can show that (X, d_2) is a b_2-metric space with $s \leq 3^{p-1}$.

The following are Cauchy sequence, convergent and completeness property for b_2-metric space.

Definition 2.3. *[16] Let $\{\gamma_n\}$ be a sequence in a b_2-metric space (X, d_2).*

(i) $\{\gamma_n\}$ is said to be b_2-convergent to $x \in X$, written as

$$\lim_{n\to\infty} \gamma_n = \gamma,$$

if for all $\kappa \in X$,

$$d_2(\gamma_n, \gamma, \kappa) = 0.$$

(ii) $\{\gamma_n\}$ is said to be b_2-Cauchy sequence in X if for all $\kappa \in X$,

$$d_2(\gamma_n, \gamma, \kappa) = 0.$$

(iii) (X, d_2) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Refer the following lemma for future use to prove Cauchy convergence.

Lemma 2.1. *[16] Assume (X, d_2) is a b_2-metric space and that $\{\gamma_n\}$ and $\{\varsigma_n\}$ are b_2-convergent to γ and ς, respectively. Finally, we have*

$$\begin{aligned}\frac{1}{s^2} d_2(\gamma, \varsigma, \kappa) &\leq \lim_{n\to\infty} \inf d_2(\gamma_n, \varsigma_n, \kappa), \leq \lim_{n\to\infty} \sup d_2(\gamma_n, \varsigma_n, \kappa), \\ &\leq s^2 d_2(\gamma, \varsigma, \kappa),\end{aligned}$$

for all $\kappa \in X$. In particular, if $\varsigma_n = \varsigma$ constant, then

$$\frac{1}{s} d_2(\gamma, \varsigma, \kappa) \leq \lim_{n\to\infty} \inf d_2(\gamma_n, \varsigma, \kappa), \leq \lim_{n\to\infty} \sup d_2(\gamma_n, \varsigma, \kappa), \leq s d_2(\gamma, \varsigma, \kappa).$$

The following are preliminary results that will be used to develop the main results.

Karapinar et al. [12] demonstrated interpolative Reich-Rus-Ciri contractions on partial metric spaces as follows.

Theorem 2.1. *[12] Let (X, p) be an entire metric space. Consider $S : X \to X$ as an interpolative Reich-Rus-Ćirić contraction mapping, such that*

$$p(S\gamma, S\varsigma) \leq \eta [p(\gamma, \varsigma)]^\delta . [p(\gamma, S\gamma)]^\alpha . [p(\varsigma, S\varsigma)]^{1-\alpha-\delta}, \tag{2.1}$$

for all $\gamma, \varsigma \in X \backslash Fix(S)$ where $Fix(S) = \{\gamma \in X, S\gamma = \gamma\}$. Then S has a fixed point in X.

Mustafa *et al.* [16] formulated the following theorem.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set. Assume X has a b_2-metric d_2, making (X, d_2) a complete b_2-metric space. Let $S : X \to X$ be an increasing mapping in comparison to $\preceq$. There is an element $\gamma \in X$ with $\gamma_0 \preceq S\gamma_n$. Let us assume that*

$$sd_2(S\gamma, S\varsigma, \kappa) \leq \beta((d_2(\gamma, \varsigma, \kappa))M(\gamma, \varsigma, \kappa),$$

for all $\kappa \in X$ and all comparable elements $\gamma, \varsigma \in X$, where

$$M(\gamma, \varsigma, \kappa) = \max\Big\{d_2(\gamma, \varsigma, \kappa), \frac{d_2(\gamma, S\gamma, \kappa)d_2(\varsigma, S\varsigma, \kappa)}{1 + d_2(S\gamma, S\varsigma, \kappa)}\Big\}.$$

If S is b_2-continuous, then it contains a fixed point. Furthermore, S's fixed point set is well-ordered if and only if it has a single fixed point.

3 Main Results

The following is the main results.

Theorem 3.1. *Consider (X, d_2) as a whole metric space. Consider $S : X \to X$ as an interpolative Reich-Rus-Ćirić contraction mapping, such that*

$$sd_2(S\gamma, S\varsigma, \kappa) \leq \eta[d_2(\gamma, \varsigma, \kappa)]^{\delta}.[d_2(\gamma, S\gamma, \kappa)]^{\alpha}.[d_2(\varsigma, S\varsigma, \kappa)]^{1-\alpha-\delta}, \tag{3.1}$$

for all $\gamma, \varsigma, \kappa \in X \backslash Fix(S)$ where $Fix(S) = \{\gamma \in X, S\gamma = \gamma\}$ and $s \geq 2, \eta \leq 1$. Then S has a fixed point in X.

Proof. Let$\gamma_0 \in X$ and define a sequence $\{\gamma_n\}_{n\in\mathbf{N}}$ in X by

$$\gamma_n = S\gamma_{n-1},$$

for all $n \in \mathbf{N}$. If $\gamma_n = \gamma_{n+1}$ for all $n \in \mathbf{N}$ the fixed point is archived. On contrary we assume that $\gamma_n \geq \gamma_{n+1}$, and we shall show that the sequence $\{d_2(\gamma_n, \gamma_{n+1}, \kappa)\}_{n\in\mathbf{N}}$ is a decreasing sequence for all $n \in \mathbf{N}$.

By (3.5), $\gamma = \gamma_{n-1}$ and $\varsigma = \gamma_n$, we get

$$\begin{aligned}
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &= sd_2(S\gamma_{n-1}, S\gamma_n, \kappa),\\
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \eta[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\delta}.[d_2(\gamma_{n-1}, S\gamma_{n-1}, \kappa)]^{\alpha}.\\
&\quad [d_2(\gamma_n, S\gamma_n, \kappa)]^{1-\alpha-\delta},\\
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \eta[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\delta}.[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\alpha}\\
&\quad .[d_2(\gamma_n, \gamma_{n+1}, \kappa)]^{1-\alpha-\delta},\\
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \eta[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\delta+\alpha}.[d_2(\gamma_n, \gamma_{n+1}, \kappa)]^{1-\alpha-\delta},\\
s\Big[d_2(\gamma_n, \gamma_{n+1}, \kappa)\Big]^{\alpha+\delta} &\leq \eta[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\delta+\alpha},\\
\Big[d_2(\gamma_n, \gamma_{n+1}, \kappa)\Big]^{\alpha+\delta} &\leq \frac{\eta}{s}[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\delta+\alpha},\\
d_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \Big(\frac{\eta}{s}\Big)^{\frac{1}{\delta+\alpha}} d_2(\gamma_{n-1}, \gamma_n, \kappa).
\end{aligned}$$

Repeating the above procedure through induction, we get

$$d_2(\gamma_n, \gamma_{n+1}, \kappa) \leq \Big(\frac{\eta}{s}\Big)^{\frac{n}{\delta+\alpha}} d_2(\gamma_{n-1}, \gamma_n, \kappa). \tag{3.2}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$d_2(\gamma_n, \gamma_{n+1}, \kappa) \leq 0,$$

which is a contradiction. Therefore, the sequence $\{d_2(\gamma_n, \gamma_{n+1}, \kappa)\}_{n\in\mathbf{N}}$ is decreasing. □

We use the rectangular inequality and (3.5) to show that $\{\gamma_n\}$ is a b_2-Cauchy sequence.

$$\begin{aligned} d_2(\gamma_n, \gamma_m, \kappa) &\le sd_2(\gamma_n, \gamma_m, \gamma_{m+1}) + sd_2(\gamma_n, \kappa, \gamma_{m+1}) + \\ &\quad sd_2(\kappa, \gamma_n, \gamma_{m+1}), \\ &\le sd_2(\gamma_n, \gamma_m, \gamma_{m+1}) + \\ &\quad s^2[d_2(\gamma_m, \gamma_{m+1}, \kappa) + d_2(\gamma_{n+1}, \gamma_{m+1}, \kappa) + \\ &\quad d_2(\gamma_m, \gamma_{m+1}, \gamma_{n+1})] + sd_2(\kappa, \gamma_n, \gamma_{m+1}), \\ &\le sd_2(\gamma_n, \gamma_m, \gamma_{m+1}) + \\ &\quad s^2 d_2(\gamma_m, \gamma_{m+1}, \kappa) + s^2 d_2(\gamma_{n+1}, \gamma_{m+1}, \kappa) + \\ &\quad s^2 d_2(\gamma_m, \gamma_{m+1}, \gamma_{n+1}) + sd_2(\kappa, \gamma_n, \gamma_{m+1}). \end{aligned}$$

By using (i) of Definition 2.2 and the concepts in Lemma 2.1, we obtain

$$\begin{aligned} d_2(\gamma_n, \gamma_m, \kappa) &\le s^2 d_2(\gamma_{n+1}, \gamma_{m+1}, \kappa), \\ &\le s^2 d_2(\mathrm{S}\gamma_n, \mathrm{S}\gamma_m, \kappa). \end{aligned}$$

Using (3.5), with $\gamma = \gamma_n$ and $\varsigma = \gamma_m$, we have

$$\begin{aligned} \frac{1}{s} d_2(\gamma_n, \gamma_m, \kappa) &\le sd_2(\mathrm{S}\gamma_n, \mathrm{S}\gamma_m, \kappa). \\ &\le s\eta [d_2(\gamma_n, \gamma_m, \kappa)]^{\delta}.[d_2(\gamma_n, \mathrm{S}\gamma_n, \kappa)]^{\alpha}. \\ &\quad [d_2(\gamma_m, \mathrm{S}\gamma_m, \kappa)]^{1-\alpha-\delta}, \\ &\le s\eta [d_2(\gamma_n, \gamma_m, \kappa)]^{\delta}.[d_2(\gamma_n, \gamma_{n+1}, \kappa)]^{\alpha}. \\ &\quad [d_2(\gamma_m, \gamma_{m+1}, \kappa)]^{1-\alpha-\delta}, \\ \frac{1}{s} d_2(\gamma_n, \gamma_m, \kappa) &\le 0, \\ d_2(\gamma_n, \gamma_m, \kappa) &= 0, \end{aligned}$$

we deduce that

$$\lim_{n\to\infty} d_2(\gamma_n, \gamma_m, \kappa) = 0,$$

This is a contradiction. Thus, $\{\gamma_n\}$ is a b_2-Cauchy sequence in X. Because (X, d_2) is a b_2-complete, the sequence $\{\gamma_n\}$ b_2-converges to some $\vartheta \in X$, that is, $\lim_{n\to\infty} d_2(\gamma_n, \vartheta, \kappa) = 0$.

Now to show that ϑ is a fixed point of S. Using the rectangle inequality, we get

$$\begin{aligned} d_2(\mathrm{S}\vartheta, \vartheta, \kappa) &\le sd_2(\mathrm{S}\vartheta, \mathrm{S}\gamma_m, \vartheta) + sd_2(\vartheta, \kappa, \mathrm{S}\gamma_m) + sd_2(\kappa, \mathrm{S}\vartheta, \mathrm{S}\gamma_m), \\ d_2(\mathrm{S}\vartheta, \vartheta, \kappa) &\le sd_2(\mathrm{S}\vartheta, \mathrm{S}\gamma_m, \vartheta) + sd_2(\vartheta, \kappa, \mathrm{S}\gamma_m) + \\ &\quad \eta [d_2(\vartheta, \gamma_n, \kappa)]^{\delta}.[d_2(\vartheta, \mathrm{S}\vartheta, \kappa)]^{\alpha}.[d_2(\gamma_n, \mathrm{S}\gamma_n, \kappa)]^{1-\alpha-\delta}. \end{aligned}$$

Letting $n \to \infty$ and using the continuity of S and (i) of Definition 2.2, we have

$$\begin{aligned} d_2(\mathrm{S}\vartheta, \vartheta, \kappa) &\le sd_2(\mathrm{S}\vartheta, \mathrm{S}\vartheta, \vartheta) + sd_2(\vartheta, \kappa, \mathrm{S}\vartheta) + \\ &\quad \eta [d_2(\vartheta, \vartheta, \kappa)]^{\delta}.[d_2(\vartheta, \mathrm{S}\vartheta, \kappa)]^{\alpha}.[d_2(\vartheta, \mathrm{S}\vartheta, \kappa)]^{1-\alpha-\delta}, \\ d_2(\mathrm{S}\vartheta, \vartheta, \kappa) &\le sd_2(\vartheta, \kappa, \mathrm{S}\vartheta), \end{aligned}$$

$$
\begin{aligned}
(1-s)d_2(S\vartheta, \vartheta, \kappa) &\leq 0. \\
d_2(S\vartheta, \vartheta, \kappa) &\leq 0,
\end{aligned}
$$

we have

$$S\vartheta = \vartheta.$$

Thus, ϑ is a fixed point of S.

Assume that S contains two fixed locations. In contrast, ϑ and θ are distinct fixed points of S. Using (3.5), we have

$$
\begin{aligned}
sd_2(S\vartheta, S\theta, \kappa) &\leq \eta[d_2(\vartheta, \theta, \kappa)]^{\delta}.[d_2(\vartheta, S\vartheta, \kappa)]^{\alpha}.[d_2(\theta, S\theta, \kappa)]^{1-\alpha-\delta}, \\
sd_2(\vartheta, \theta, \kappa) &\leq \eta[d_2(\vartheta, \theta, \kappa)]^{\delta}.[d_2(\vartheta, \vartheta, \kappa)]^{\alpha}.[d_2(\theta, \theta, \kappa)]^{1-\alpha-\delta}, \\
sd_2(\vartheta, \theta, \kappa) &\leq 0, \\
d_2(\vartheta, \theta, \kappa) &\leq 0,
\end{aligned}
$$

which is a contradiction, that $\vartheta = \theta$. Hence ϑ is a unique fixed point of S.

The following theorem demonstrates some novel results. We extend Mustafa *et al.* [16]'s Theorem 2.2 for interpolative mapping in b_2-metric spaces.

Theorem 3.2. *Assume (X, d_2) is a complete b_2 metric space. Let $S : X \to X$ represent an increasing interpolative mapping. Then there exists an element $\gamma \in X$ where $\gamma_0 \leq S\gamma_n$. Suppose that*

$$sd_2(S\gamma, S\varsigma, \kappa) \leq \mu[d_2(\gamma, \varsigma, \kappa)]^{\sigma}.\Big[\frac{d_2(\gamma, S\gamma, \kappa)d_2(\varsigma, S\varsigma, \kappa)}{d_2(\gamma, S\gamma, \kappa)}\Big]^{1-\sigma}. \tag{3.3}$$

for all $\kappa \in X$ and all comparable elements $\gamma, \varsigma \in X$. If S is b_2-continuous, then S has one and only fixed point.

Proof. Starting with the given γ_0, using a Picard sequence $\gamma_m = S^n\gamma_0$. Since $\gamma_0 \geq S\gamma_0$ and S is an increasing function we obtain the following

$$\gamma_0 \leq S\gamma_0 \leq S^2\gamma_0 \leq \cdots \leq S^n\gamma_0 \leq S^{n+1}\gamma_0 \leq \ldots$$

We commerce by showing that $\lim_{n\to\infty} d_2(\gamma_n, \gamma_{n+1}, \kappa) \geq 0$, since $\gamma_n \leq \gamma_{n+1}$ for each $n \in \mathbf{N}$, then by (3.3) we have

$$
\begin{aligned}
sd_2(S\gamma_{n-1}, S\gamma_n, \kappa) &\leq \mu[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}. \\
&\quad \Big[\frac{d_2(\gamma_{n-1}, S\gamma_{n-1}, \kappa)d_2(\gamma_n, S\gamma_n, \kappa)}{d_2(\gamma_{n-1}, S\gamma_{n-1}, \kappa)}\Big]^{1-\sigma}, \\
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \mu[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}. \\
&\quad \Big[\frac{d_2(\gamma_{n-1}, \gamma_n, \kappa)d_2(\gamma_n, \gamma_{n+!}, \kappa)}{d_2(\gamma_{n-1}, \gamma_n, \kappa)}\Big]^{1-\sigma}, \\
sd_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \mu[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}.\Big[d_2(\gamma_n, \gamma_{n+!}, \kappa)\Big]^{1-\sigma}, \\
d_2(\gamma_n, \gamma_{n+1}, \kappa) &\leq \frac{\mu}{s}[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}.\Big[d_2(\gamma_n, \gamma_{n+!}, \kappa)\Big]^{1-\sigma}, \\
\Big[d_2(\gamma_n, \gamma_{n+!}, \kappa)\Big]^{1-(1-\sigma)} &\leq \frac{\mu}{s}[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}, \\
\Big[d_2(\gamma_n, \gamma_{n+!}, \kappa)\Big]^{\sigma} &\leq \frac{\mu}{s}[d_2(\gamma_{n-1}, \gamma_n, \kappa)]^{\sigma}, \\
d_2(\gamma_n, \gamma_{n+!}, \kappa) &\leq \Big(\frac{\mu}{s}\Big)^{\frac{1}{\sigma}} d_2(\gamma_{n-1}, \gamma_n, \kappa),
\end{aligned}
$$

proceeding with a reparation of same steps as above, eventually we squire the following results for n^{th} steps

$$d_2(\gamma_n, \gamma_{n+!}, \kappa) \quad \leq \quad \left(\frac{\mu}{s}\right)^{\frac{1n}{\sigma}} d_2(\gamma_{n-1}, \gamma_n, \kappa).$$

For $n \longrightarrow \infty$ in the above inequality we have

$$d_2(\gamma_n, \gamma_{n+1}, \kappa) \quad \leq \quad 0,$$

which is a contradiction. Therefore the sequence $d_2(\gamma_n, \gamma_{n+!}, \kappa) = 0$. The other steps of the proof of fixed point, uniqueness and b_2-Cauchy convergent is the same as in Theorem 3.1. Hence this complete this proofs. □

Due to the novelty of Theorem 3.1 we introduce the following corollary's.

Corollary 3.1. *Let (X, d_2) be a b_2-metric spaces. Let $\mathrm{S} : X \longrightarrow X$ be non decreasing continuous interpolative contractive mapping suppose that there exists $k_1 \in [0, 1]$ and $L \geq 1$ such that*

$$\begin{aligned} sd_2(\mathrm{S}\gamma, \mathrm{S}\varsigma, \kappa) \quad \leq \quad & L\Big[\max\{d_2(\gamma, \varsigma, \kappa), d_2(\gamma, \mathrm{S}\gamma, \kappa), d_2(\varsigma, \mathrm{S}\varsigma, \kappa), \\ & \frac{d_2(\gamma, \mathrm{S}\varsigma, \kappa) + d_2(\varsigma, \mathrm{S}\gamma, \kappa)}{2}\}\Big]^{k_1}. \\ & \Big[\min\{d_2(\gamma, \mathrm{S}\gamma, \kappa), d_2(\varsigma, \mathrm{S}\gamma, \kappa)\}\Big]^{1-k_1}, \end{aligned}$$

for all elements $\gamma, \varsigma, \kappa \in X$. Then S *have one and only one fixed point.*

Proof. Proof follows from Theorem 3.1. □

Corollary 3.2. *Let (X, d_2) be a b_2-complete b_2-metric spaces. A mapping $\mathrm{S} : X \longrightarrow X$ is a weakly interpolative contractive mapping if there exists $\mu \in [0, 1]$ and $L \geq 1$ such that*

$$\begin{aligned} sd_2(\mathrm{S}\gamma, \mathrm{S}\varsigma, \kappa) \quad \leq \quad & L\Big[\max\Big\{d_2(\gamma, \varsigma, \kappa), \frac{d_2(\gamma, \mathrm{S}\gamma, \kappa)d_2(\varsigma, \mathrm{S}\varsigma, \kappa)}{1 + d_2(\gamma, \varsigma, \kappa)}, \\ & \frac{d_2(\gamma, \mathrm{S}\gamma, \kappa)d_2(\varsigma, \mathrm{S}\varsigma, \kappa)}{1 + d_2(\mathrm{S}\gamma, \mathrm{S}\varsigma, \kappa)}\Big\}\Big]^{\mu}.\Big[\min\{d_2(\gamma, \mathrm{S}\gamma, \kappa), \\ & d_2(\gamma, \mathrm{S}\varsigma, \kappa), d_2(\varsigma, \mathrm{S}\varsigma, \kappa), d_2(\varsigma, \mathrm{S}\gamma, \kappa)\}\Big]^{1-\mu}, \end{aligned}$$

for all elements $\gamma, \varsigma, \kappa \in X$. Then S *have one and only one fixed point.*

Proof. Again the proof follows from Theorem 3.1. □

We give an example for demonstration of Theorem 3.1.

Example 3.1. *Let $X = [0, 1]$ be a set with a b_2-metric defined as $d_2(\gamma, \varsigma, \kappa) = [\gamma\varsigma + \varsigma\kappa + \kappa\gamma]^p$ if $\gamma \neq \varsigma \neq \kappa \neq \gamma$, otherwise $d_2(\gamma, \varsigma, \kappa) = 0$, where $p = s = 2$. Define the mapping $\mathrm{S} : X \longrightarrow X$ by*

$$\mathrm{S}\gamma = \frac{1}{\gamma}.\gamma \geq 0. \tag{3.4}$$

In demonstration of Theorem 3.1, we proof the following example.

We begin by finding the b_2-metrics according to inequality 3.5.

For $\gamma = 2$, $\varsigma = 3$, $\kappa = 1$ and $s = 2$, we have

$$\begin{aligned}
d_2(\gamma, \varsigma, \kappa) &= \|\gamma\varsigma + \varsigma\kappa + \kappa\gamma\|^2, \\
&\leq \|2 \times 3 + 3 \times 1 + 1 \times 2\|^2, \\
&\leq \|11\|^2 = 121. \\
d_2(\gamma, S\gamma, \kappa) &= \|\gamma.S\gamma + S\gamma.\kappa + \kappa\gamma\|^2, \\
&\leq \|2 \times 0.5 + 0.5 \times 1 + 1 \times 2\|^2, \\
&\leq \|3.5\|^2 = 12.25. \\
d_2(\varsigma, S\varsigma, \kappa) &= \|\varsigma.S\varsigma + S\varsigma.\kappa + \kappa\varsigma\|^2, \\
&\leq \|3 \times 0.5 + 0.5 \times 1 + 1 \times 3\|^2, \\
&\leq \|5\|^2 = 25, \\
d_2(S\gamma, S\varsigma, \kappa) &= \|S\gamma.S\varsigma + S\varsigma.\kappa + \kappa S\gamma\|^2, \\
&\leq \|0.33 \times 0.5 + 0.5 \times 1 + 1 \times 0.33\|^2, \\
&\leq \|1\|^2 = 1,
\end{aligned}$$

By using inequality 3.5 and $\delta = 0.5$, $\alpha = 0.2$, we obtain

$$\begin{aligned}
2 \times 1 &\leq \eta[121]^{12.25}.[25]^{0.2}.[2]^{0.3}, \\
2 &\leq \eta[11].[1.65].[2.63], \\
2 &\leq 47.67\eta.
\end{aligned}$$

For $\eta \leq 1$ the above inequality satisfy all the conditions in Theorem 3.1. Hence S has a unique fixed point $\kappa = 1$.

4 An application to nonlinear fractional differential equation in b_2-metric space

The fractional derivative as convolution serves numerous functions. It can represent memory, as in the elasticity hypothesis. The Caputo and Caputo-Fabrizio types can be understood as filters of the local derivative, having power and exponent functions (refer to [25]).

The purpose of this section is to provide an application of Theorem 3.1 to get a unique solution of a non-linear fractional differential equation, where we can apply an interpolative Reich-Rus-Ćirić contraction mapping in b_2-metric spaces.

The following are basic definitions of fractional calculus from [14, 17].

Definition 4.1. *For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order ϑ is defined by*

$$D^{\vartheta} g(t) = \frac{1}{\Gamma(n - \vartheta)} \int_0^t (t - s)^{n - \vartheta - 1} g^{(n)}(s) ds,$$

$(n - 1 < \vartheta < n, n = [0] + 1)$, where $[\vartheta]$ denote the iteger part of the real number ϑ.

Definition 4.2. *[17] The Riemann-Liouville fractional derivative of order ϑ for continuous function defined $g(t)$ is defined by*

$$D^{\vartheta} g(t) = \frac{1}{\Gamma(n - \vartheta)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{n - \vartheta + 1} g(s) ds,$$

where $[\vartheta]$ denotes the integer part of the real number ϑ and $n = [\vartheta] + 1$, provided that the right hand side is pointwise defined on $(0, \infty)$.

Lemma 4.1. *The Banach space* $(X, ||.||)$*, denoted by* $C = C([0, 1], X)$ *is a continuous function from* $[0, 1]$ *into* X*.*

$$d_2(\gamma, \varsigma, \kappa) = \|\gamma\varsigma + \varsigma\kappa + \kappa\gamma\|^p = \|\gamma - \varsigma\|^p, p \geq 2,$$

The following non-linear fractional differential equation with integral boundary valued conditions inspired by Baleanu *et al.* [3].

$$\begin{cases} D^{\vartheta}\gamma(t) = g(t, \gamma(t)),\ 0 < t < 1,\ 1 < \vartheta \leq 2 \\ \gamma(0) = 0, \\ \gamma(1) = \int_0^{\sigma} \gamma(s)ds,\ 0 < \sigma < 1, \end{cases} \tag{4.1}$$

where D^{ϑ} denotes the Riemann-Liouville fractional derivative of order ϑ and $g : [0, 1]\times X \to X$ is a continuous function.

Theorem 4.1. *Suppose that*

(i) $\zeta : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function endowed with the following conditions*

$$\|g(t, \gamma) - g(t, \varsigma)\| = \eta\|\gamma - \varsigma\|^p$$

for all $t \in [0, 1]$ *and* $\gamma, \varsigma \in \mathbb{R}$*, where*

$$d_2(\gamma, \varsigma, \kappa) = \|\gamma\varsigma + \varsigma\kappa + \kappa\gamma\|^p = \|\gamma - \varsigma\|^p = \|\gamma - \varsigma\|^p, p \geq 2,$$

and

$$d_2(\gamma, \varsigma, \kappa) = [d_2(\gamma, \varsigma, \kappa)]^{\delta}.[d_2(\gamma, \mathrm{S}\gamma, \kappa)]^{\alpha}.[d_2(\varsigma, \mathrm{S}\varsigma, \kappa)]^{1-\alpha-\delta}.$$

(ii) there exists a constant η *such that*

$$\eta = \leq 1.$$

Then, the differential equation (4.1) *has a unique solution.*

Proof. Define a mapping $\mathrm{S} : X \to X$ by

$$\begin{aligned} \mathrm{S}\gamma(t) = {} & \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1} g(s, \gamma(s))ds - \\ & \frac{2t}{(2-\sigma^2)\Gamma(\vartheta)}\int_0^1 (1-s)^{\vartheta-1} g(s, \gamma(s))ds + \\ & \frac{2t}{(2-\sigma^2)\Gamma(\vartheta)}\int_0^{\sigma}\left(\int_0^s (s-\tau)^{\vartheta-1} g(\tau, \gamma(\tau))d\tau\right)ds,\ (t \in [0, 1]). \end{aligned}$$

□

5 Conclusion

This paper aims to obtain fresh conclusions for fixed point theorems for Ćirić-Reich-Rus-type contraction mapping in b_2-Metric spaces. Also, provide an example to demonstrate the proven results. Finally, it presents an application of fractional differential equations.

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