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Different Upper Bounds for the Integral of the Product of Three Convex Functions

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Abstract
This study extends existing results on integral inequalities by establishing sharp upper bounds for the integral of the product of three convex functions. Various proof techniques, including famous integral inequalities, are used to derive diverse and adaptable bounds. We also formulate an open problem arising from our results.

Keywords: convexity; Hermite-Hadamard integral inequalities; Chebyshev integral inequality; Hölder integral inequality.

1 Introduction

The notion of a convex function, recalled below, is at the center of this study.

Definition 1.1 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ be a function. We say that f is convex if and only if, for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$, we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

If f is twice differentiable, this inequality is equivalent to $f''(x) \geq 0$ for any $x \in [a, b]$.

Convex functions appear in many areas of mathematics. Their well-defined properties allow the derivation of inequalities that are crucial in both theoretical and applied contexts. We refer to [1–11]. For the purposes of this study, we highlight the Hermite-Hadamard integral inequalities, as recalled below.

Theorem 1.2 *(Hermite-Hadamard integral inequalities) Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ be a convex function. Then the following holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}[f(a) + f(b)].$$

These fundamental inequalities have applications in optimization, probability theory and functional analysis. They have been the subject of various generalizations and improvements. We refer to [12–25].

One of the challenges in analysis is to find sharp upper bounds for integrals involving the product of convex functions. These bounds can be useful in deriving approximation results, establishing stability conditions, and analyzing error estimates in numerical methods. When considering the product of two functions, several theoretical results have been established in [23]. Among them, the proposition below is highlighted.

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Proposition 1.3 *([23, Part of Theorem 2.5.]) Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \mapsto [0, +\infty)$ be two convex functions. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}[f(a)+f(b)][g(a)+g(b)].$$

It can be seen as a natural two-function extension of the right-hand side of the Hermite-Hadamard integral inequalities. Our study goes a step further by investigating upper bounds for the integral of the product of three convex functions. Several proof techniques are developed. These include the intermediate use of the Hermite-Hadamard integral inequalities, the Chebyshev integral inequality, the Hölder integral inequality, and several basic convexity inequalities. The upper bounds obtained thus vary in form and applicability, so that they can be adapted to a wide range of mathematical scenarios. An open problem arising from one of our results is also formulated.

The rest of this article is structured as follows: In Section 2, we present the main results. The corresponding proofs are detailed in Section 3. Section 4 concludes the study.

2 Main results

We divide this section into two parts: One part focuses on the results that use intermediate integral inequalities in their proofs, and the other part focuses on the results that use the convexity properties more directly.

2.1 Using intermediate integral inequalities

The proposition below gives a simple upper bound for the integral of the product of three convex functions, making additional assumptions about differentiability and how the functions and their derivatives interact.

Proposition 2.1 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. We assume that f, g, h are twice differentiable and that, for any $x \in [a, b]$, we have*

$$\begin{aligned} &g(x)h(x)f''(x) + 2h(x)f'(x)g'(x) + 2g(x)f'(x)h'(x) + f(x)h(x)g''(x) \\ &+ 2f(x)g'(x)h'(x) + f(x)g(x)h''(x) \geq 0. \end{aligned} \tag{1}$$

Then the following holds:

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{2}[f(a)g(a)h(a) + f(b)g(b)h(b)].$$

The proof is based on the right-hand side of the Hermite-Hadamard integral inequalities. The upper bound obtained has the advantage of being tractable. However, the assumption in Equation (1) can be somewhat difficult to check, motivating further studies under different assumptions.

We now investigate upper bounds for the integral of the product of three convex functions with additional monotonicity assumptions. Before doing so, we propose a new result dealing with only two convex functions, which complements those in [23]. It will be used as an intermediate result in the proof of other propositions.

Proposition 2.2 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \mapsto [0, +\infty)$ be two convex functions. We assume that f and g are monotonic with different types of monotonicity. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{4}[f(a)+f(b)][g(a)+g(b)].$$

The proof is based on the Chebyshev integral inequality and the right-hand side of the Hermite-Hadamard integral inequalities. Since $1/4 < 1/3$, this result shows that, under monotonicity assumptions on f and g, Proposition 1.3 can be improved. This motivates the study of such assumptions when considering three functions, as stated in the proposition below.

Proposition 2.3 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. We assume that f and gh, i.e., the product of g and h, are monotonic with different types of monotonicity. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{6}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

The proof uses the Chebyshev integral inequality, the right-hand side of the Hermite-Hadamard integral inequalities and Proposition 2.2.

Under different monotonicity assumptions, the result below suggests an alternative upper bound.

Proposition 2.4 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. We assume that f and gh are monotonic with different types of monotonicity, and that g and h are monotonic with different types of monotonicity. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{8}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

The proof uses the Chebyshev integral inequality and the right-hand side of the Hermite-Hadamard integral inequalities. Compared to the upper bound in Proposition 2.3, since $1/8 < 1/6$, the one obtained is sharper.

We now present new results based on different proof techniques involving the Hölder integral inequality and its generalization, starting with the proposition below.

Proposition 2.5 *Let $a, b \in \mathbb{R}$ with $a < b$, $p, q > 1$ such that $1/p + 1/q = 1$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx$$
$$\leq \frac{1}{2^{1/p}3^{1/q}}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^q(a)+h^q(b)]^{1/q}.$$

The roles of f, g and h can be exchanged, giving two more similar inequalities.

The proof consists in using the Hölder integral inequality, the right-hand side of the Hermite-Hadamard integral inequalities, and Proposition 1.3. The upper bound obtained deals with the power of the functions involved at a and b, making it original in this context. For this reason, it is difficult to compare with those obtained in the previous propositions.

In the same spirit, the result below suggests an alternative.

Proposition 2.6 *Let $a, b \in \mathbb{R}$ with $a < b$, $p, q, r > 1$ such that $1/p+1/q+1/r = 1$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. Then the following holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx$$
$$\leq \frac{1}{2}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^r(a)+h^r(b)]^{1/r}.$$

The roles of f, g and h can be exchanged, giving five more similar inequalities.

The proof is based on the generalized Hölder integral inequality and the right-hand side of the Hermite-Hadamard integral inequalities. Again, the upper bound obtained deals with the power of the functions involved at a and b. The flexibility in the choice of p, q and r makes it adaptable to different mathematical scenarios with such an integral of the product of convex functions.

An original upper bound for the integral of the product of three convex functions is given in the proposition below.

Proposition 2.7 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx\\&\leq [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] - \frac{1}{2}[f(a)+f(b)][h(a)+h(b)]g\left(\frac{a+b}{2}\right)\\&-\frac{1}{2}[f(a)+f(b)][g(a)+g(b)]h\left(\frac{a+b}{2}\right) - \frac{1}{2}[g(a)+g(b)][h(a)+h(b)]f\left(\frac{a+b}{2}\right).\end{aligned}$$

The proof uses the convexity nature of the functions involved, an appropriate decomposition, the left-hand side of the Hermite-Hadamard integral inequalities and Proposition 1.3.

The rest of the section is devoted to results that make direct use of the convexity of the function, without the need for intermediate integral results.

2.2 Without using intermediate integral inequalities

The proposition below can be presented as the analogue of Proposition 1.3 but with three functions instead of two.

Proposition 2.8 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx\\&\leq \frac{1}{12}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] + \frac{1}{6}[f(a)g(a)h(a)+f(b)g(b)h(b)].\end{aligned}$$

The proof is based on a change of variables, the basic definition of a convex function, an appropriate decomposition and several integral developments.

Remark 2.9 *In Proposition 2.8, if f, g, h are concave instead of convex, then the final inequality is reversed.*

A consequence of this proposition is the corollary below, that offers a very simple upper bound.

Corollary 2.10 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g, h : [a, b] \mapsto [0, +\infty)$ be three convex functions. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{4}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

This result can be seen as a natural three-function extension of the right-hand side of the Hermite-Hadamard integral inequalities.

3 Proofs

This section contains the detailed proofs of the new results. It is important to note that the positivity of the functions involved, i.e., f, g and h, will be used implicitly at many stages of development without being mentioned each time for the sake of redundancy.

Proof of Proposition 2.1. Let us consider the function $k : [a, b] \mapsto [0, +\infty)$ defined by $k(x) = f(x)g(x)h(x)$, $x \in [a, b]$. It follows from the differentiation rules for the product of functions and Equation (1) that, for any $x \in [a, b]$,

$$\begin{aligned}k''(x) &= [f(x)g(x)h(x)]''\\&= g(x)h(x)f''(x) + 2h(x)f'(x)g'(x) + 2g(x)f'(x)h'(x) + f(x)h(x)g''(x)\\&+ 2f(x)g'(x)h'(x) + f(x)g(x)h''(x) \geq 0.\end{aligned}$$

Therefore, k is convex. Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the function k, we get

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx = \frac{1}{b-a}\int_a^b k(x)dx \leq \frac{1}{2}[k(a)+k(b)]$$
$$= \frac{1}{2}[f(a)g(a)h(a)+f(b)g(b)h(b)].$$

This ends the proof of Proposition 2.1. □

Proof of Proposition 2.2. Since f and g are monotonic with different types of monotonicity, the Chebyshev integral inequality ensures that

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]. \tag{2}$$

Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the functions f and g, we get

$$\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right] \leq \frac{1}{2}[f(a)+f(b)]\times\frac{1}{2}[g(a)+g(b)]$$
$$= \frac{1}{4}[f(a)+f(b)][g(a)+g(b)]. \tag{3}$$

Combining Equations (2) and (3), we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{4}[f(a)+f(b)][g(a)+g(b)].$$

This completes the proof of Proposition 2.2. □

Proof of Proposition 2.3. Since f and gh are monotonic with different types of monotonicity, the Chebyshev integral inequality ensures that

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx = \frac{1}{b-a}\int_a^b f(x)[g(x)h(x)]dx$$
$$\leq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)h(x)dx\right]. \tag{4}$$

Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the function f and Proposition 1.3 to the functions g and h, we get

$$\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)h(x)dx\right]$$
$$\leq \frac{1}{2}[f(a)+f(b)]\times\frac{1}{3}[g(a)+g(b)][h(a)+h(b)]$$
$$= \frac{1}{6}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]. \tag{5}$$

Combining Equations (4) and (5), we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{6}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

This concludes the proof of Proposition 2.3. □

Proof of Proposition 2.4. Since f and gh are monotonic with different types of monotonicity, the Chebyshev integral inequality ensures that

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx = \frac{1}{b-a}\int_a^b f(x)[g(x)h(x)]dx$$
$$\leq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)h(x)dx\right]. \tag{6}$$

Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the function f and Proposition 2.2 to the functions g and h, which is possible because they are monotonic with different types of monotonicity, we obtain

$$\left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)h(x)dx\right]$$
$$\leq \frac{1}{2}[f(a)+f(b)] \times \frac{1}{4}[g(a)+g(b)][h(a)+h(b)]$$
$$= \frac{1}{8}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]. \tag{7}$$

Combining Equations (6) and (7), we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{8}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

This completes the proof of Proposition 2.4. □

Proof of Proposition 2.5. Applying the Hölder integral inequality with the parameters p and q, and the functions f and gh, we find that

$$\int_a^b f(x)g(x)h(x)dx = \int_a^b f(x)[g(x)h(x)]dx$$
$$\leq \left[\int_a^b f^p(x)dx\right]^{1/p}\left[\int_a^b g^q(x)h^q(x)dx\right]^{1/q}.$$

Dividing by $b-a$ and using $1/p+1/q=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \leq \frac{1}{b-a}\left[\int_a^b f^p(x)dx\right]^{1/p}\left[\int_a^b g^q(x)h^q(x)dx\right]^{1/q}$$
$$= \left[\frac{1}{b-a}\int_a^b f^p(x)dx\right]^{1/p}\left[\frac{1}{b-a}\int_a^b g^q(x)h^q(x)dx\right]^{1/q}. \tag{8}$$

Furthermore, since $p, q > 1$ and f, g and h are convex, f^p, g^q and h^q are also convex (as a composition of an increasing convex function with a convex function). Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the function f^p, and Proposition 1.3 to the functions g^q and h^q, we get

$$\left[\frac{1}{b-a}\int_a^b f^p(x)dx\right]^{1/p}\left[\frac{1}{b-a}\int_a^b g^q(x)h^q(x)dx\right]^{1/q}$$
$$\leq \left\{\frac{1}{2}[f^p(a)+f^p(b)]\right\}^{1/p}\left\{\frac{1}{3}[g^q(a)+g^q(b)][h^q(a)+h^q(b)]\right\}^{1/q}$$
$$= \frac{1}{2^{1/p}3^{1/q}}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^q(a)+h^q(b)]^{1/q}. \tag{9}$$

Combining Equations (8) and (9), we obtain

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx\\&\leq \frac{1}{2^{1/p}3^{1/q}}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^q(a)+h^q(b)]^{1/q}.\end{aligned}$$

This completes the proof of Proposition 2.5. □

Proof of Proposition 2.6. Applying the generalized Hölder integral inequality with the parameters p, q and r, and the functions f, g and h, we obtain

$$\int_a^b f(x)g(x)h(x)dx \leq \left[\int_a^b f^p(x)dx\right]^{1/p}\left[\int_a^b g^q(x)dx\right]^{1/q}\left[\int_a^b h^r(x)dx\right]^{1/r}.$$

Dividing by $b-a$ and using $1/p+1/q+1/r=1$, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx\\&\leq \frac{1}{b-a}\left[\int_a^b f^p(x)dx\right]^{1/p}\left[\int_a^b g^q(x)dx\right]^{1/q}\left[\int_a^b h^r(x)dx\right]^{1/r}\\&=\left[\frac{1}{b-a}\int_a^b f^p(x)dx\right]^{1/p}\left[\frac{1}{b-a}\int_a^b g^q(x)dx\right]^{1/q}\left[\frac{1}{b-a}\int_a^b h^r(x)dx\right]^{1/r}.\end{aligned}\tag{10}$$

Furthermore, since $p,q,r>1$ and f, g and h are convex, f^p, g^q and h^r are also convex. Applying the right-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the functions f^p, g^q and h^r, and using again $1/p+1/q+1/r=1$, we find that

$$\begin{aligned}&\left[\frac{1}{b-a}\int_a^b f^p(x)dx\right]^{1/p}\left[\frac{1}{b-a}\int_a^b g^q(x)dx\right]^{1/q}\left[\frac{1}{b-a}\int_a^b h^r(x)dx\right]^{1/r}\\&\leq \left\{\frac{1}{2}[f^p(a)+f^p(b)]\right\}^{1/p}\left\{\frac{1}{2}[g^q(a)+g^q(b)]\right\}^{1/q}\left\{\frac{1}{2}[h^r(a)+h^r(b)]\right\}^{1/r}\\&=\frac{1}{2}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^r(a)+h^r(b)]^{1/r}.\end{aligned}\tag{11}$$

Combining Equations (10) and (11), we obtain

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx\\&\leq \frac{1}{2}[f^p(a)+f^p(b)]^{1/p}[g^q(a)+g^q(b)]^{1/q}[h^r(a)+h^r(b)]^{1/r}.\end{aligned}$$

This completes the proof of Proposition 2.6. □

Proof of Proposition 2.7. For this proof, we need a simple inequality, which is satisfied by arbitrary convex functions, as described below for f. For any $x\in[a,b]$, there exists $\lambda\in[0,1]$ such that $x=\lambda a+(1-\lambda)b$. Therefore, using appropriate decompositions and the convexity of f, we have

$$\begin{aligned}&f(a+b-x)=f(a+b-\lambda a-(1-\lambda)b)=f(\lambda b+(1-\lambda)a)\leq \lambda f(b)+(1-\lambda)f(a)\\&=f(a)+f(b)-\lambda f(a)-(1-\lambda)f(b)\leq f(a)+f(b)-f(\lambda a+(1-\lambda)b)\\&=f(a)+f(b)-f(x).\end{aligned}$$

Applying this inequality to f, g and h then developing the resulting product, we get

$$\begin{aligned}
&\int_a^b f(x)g(x)h(x)dx \\
&\le \int_a^b [f(a)+f(b)-f(x)][g(a)+g(b)-g(x)][h(a)+h(b)-h(x)]dx \\
&= \int_a^b \big\{[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] \\
&- [f(a)+f(b)][g(a)+g(b)]h(x) - [f(a)+f(b)][h(a)+h(b)]g(x) \\
&- [g(a)+g(b)][h(a)+h(b)]f(x) + [f(a)+f(b)]g(x)h(x) \\
&+ [g(a)+g(b)]f(x)h(x) + [h(a)+h(b)]f(x)g(x) - f(x)g(x)h(x)\big\}dx \\
&= [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)](b-a) \\
&- [f(a)+f(b)][g(a)+g(b)]\int_a^b h(x)dx - [f(a)+f(b)][h(a)+h(b)]\int_a^b g(x)dx \\
&- [g(a)+g(b)][h(a)+h(b)]\int_a^b f(x)dx + [f(a)+f(b)]\int_a^b g(x)h(x)dx \\
&+ [g(a)+g(b)]\int_a^b f(x)h(x)dx + [h(a)+h(b)]\int_a^b f(x)g(x)dx - \int_a^b f(x)g(x)h(x)dx.
\end{aligned}$$

Combining the two integral terms involving fgh, we obtain

$$\begin{aligned}
&2\int_a^b f(x)g(x)h(x)dx \\
&\le [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)](b-a) \\
&- [f(a)+f(b)][g(a)+g(b)]\int_a^b h(x)dx - [f(a)+f(b)][h(a)+h(b)]\int_a^b g(x)dx \\
&- [g(a)+g(b)][h(a)+h(b)]\int_a^b f(x)dx + [f(a)+f(b)]\int_a^b g(x)h(x)dx \\
&+ [g(a)+g(b)]\int_a^b f(x)h(x)dx + [h(a)+h(b)]\int_a^b f(x)g(x)dx.
\end{aligned}$$

Dividing by $b-a$, we get

$$\begin{aligned}
&\frac{2}{b-a}\int_a^b f(x)g(x)h(x)dx \\
&\le [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] \\
&- [f(a)+f(b)][g(a)+g(b)]\frac{1}{b-a}\int_a^b h(x)dx - [f(a)+f(b)][h(a)+h(b)]\frac{1}{b-a}\int_a^b g(x)dx \\
&- [g(a)+g(b)][h(a)+h(b)]\frac{1}{b-a}\int_a^b f(x)dx + [f(a)+f(b)]\frac{1}{b-a}\int_a^b g(x)h(x)dx \\
&+ [g(a)+g(b)]\frac{1}{b-a}\int_a^b f(x)h(x)dx + [h(a)+h(b)]\frac{1}{b-a}\int_a^b f(x)g(x)dx. \qquad (12)
\end{aligned}$$

Applying the left-hand side of the Hermite-Hadamard integral inequalities in Theorem 1.2 to the functions f, g and h, and Proposition 1.3 to the functions g and h, f and h, and f and g, we get

$$
\begin{aligned}
&- [f(a)+f(b)][g(a)+g(b)]\frac{1}{b-a}\int_a^b h(x)dx - [f(a)+f(b)][h(a)+h(b)]\frac{1}{b-a}\int_a^b g(x)dx \\
&- [g(a)+g(b)][h(a)+h(b)]\frac{1}{b-a}\int_a^b f(x)dx + [f(a)+f(b)]\frac{1}{b-a}\int_a^b g(x)h(x)dx \\
&+ [g(a)+g(b)]\frac{1}{b-a}\int_a^b f(x)h(x)dx + [h(a)+h(b)]\frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\leq -[f(a)+f(b)][g(a)+g(b)]h\left(\frac{a+b}{2}\right) - [f(a)+f(b)][h(a)+h(b)]g\left(\frac{a+b}{2}\right) \\
&- [g(a)+g(b)][h(a)+h(b)]f\left(\frac{a+b}{2}\right) + [f(a)+f(b)]\frac{1}{3}[g(a)+g(b)][h(a)+h(b)] \\
&+ [g(a)+g(b)]\frac{1}{3}[f(a)+f(b)][h(a)+h(b)] + [h(a)+h(b)]\frac{1}{3}[f(a)+f(b)][g(a)+g(b)] \\
&= -[f(a)+f(b)][g(a)+g(b)]h\left(\frac{a+b}{2}\right) - [f(a)+f(b)][h(a)+h(b)]g\left(\frac{a+b}{2}\right) \\
&- [g(a)+g(b)][h(a)+h(b)]f\left(\frac{a+b}{2}\right) + [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]. \qquad (13)
\end{aligned}
$$

Combining Equations (12) and (13), we find that

$$
\begin{aligned}
&\frac{2}{b-a}\int_a^b f(x)g(x)h(x)dx \\
&\leq 2[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] - [f(a)+f(b)][h(a)+h(b)]g\left(\frac{a+b}{2}\right) \\
&- [f(a)+f(b)][g(a)+g(b)]h\left(\frac{a+b}{2}\right) - [g(a)+g(b)][h(a)+h(b)]f\left(\frac{a+b}{2}\right).
\end{aligned}
$$

We thus get

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \\
&\leq [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] - \frac{1}{2}[f(a)+f(b)][h(a)+h(b)]g\left(\frac{a+b}{2}\right) \\
&- \frac{1}{2}[f(a)+f(b)][g(a)+g(b)]h\left(\frac{a+b}{2}\right) - \frac{1}{2}[g(a)+g(b)][h(a)+h(b)]f\left(\frac{a+b}{2}\right).
\end{aligned}
$$

This concludes the proof of Proposition 2.7. □

Proof Proposition 2.8. Using the change of variables $x = (1-t)a + tb$ with respect to t, and the convexity of the functions involved, we obtain

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \\
&= \int_0^1 f((1-t)a+tb)g((1-t)a+tb)h((1-t)a+tb)dt \\
&\leq \int_0^1 [(1-t)f(a)+tf(b)][(1-t)g(a)+tg(b)][(1-t)h(a)+th(b)]dt.
\end{aligned}
$$

Developing the integrand, using the basic integral results $\int_0^1 t^2(1-t)dt = 1/12$, $\int_0^1 t(1-t)^2 dt = 1/12$, $\int_0^1 t^3 dt = 1/4$ and $\int_0^1 (1-t)^3 dt = 1/4$, and factorizing, we find that

$$\begin{aligned}
&\int_0^1 [(1-t)f(a)+tf(b)][(1-t)g(a)+tg(b)][(1-t)h(a)+th(b)]dt \\
&= \int_0^1 \big[t^2(1-t)h(a)f(b)g(b)+t^2(1-t)g(a)f(b)h(b)+t^2(1-t)f(a)g(b)h(b) \\
&+t(1-t)^2 g(a)h(a)f(b)+t(1-t)^2 f(a)h(a)g(b)+t(1-t)^2 f(a)g(a)h(b) \\
&+(1-t)^3 f(a)g(a)h(a)+t^3 f(b)g(b)h(b)\big]dt \\
&= [h(a)f(b)g(b)+g(a)f(b)h(b)+f(a)g(b)h(b)]\int_0^1 t^2(1-t)dt \\
&+[g(a)h(a)f(b)+f(a)h(a)g(b)+f(a)g(a)h(b)]\int_0^1 t(1-t)^2 dt \\
&+f(a)g(a)h(a)\int_0^1 (1-t)^3 dt + f(b)g(b)h(b)\int_0^1 t^3 dt \\
&= \frac{1}{12}[h(a)f(b)g(b)+g(a)f(b)h(b)+f(a)g(b)h(b) \\
&+g(a)h(a)f(b)+f(a)h(a)g(b)+f(a)g(a)h(b)] \\
&+\frac{1}{4}[f(a)g(a)h(a)+f(b)g(b)h(b)] \\
&= \frac{1}{12}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]+\frac{1}{6}[f(a)g(a)h(a)+f(b)g(b)h(b)].
\end{aligned}$$

This ends the proof of Proposition 2.8. □

Proof of Corollary 2.10. It is direct that

$$f(a)g(a)h(a)+f(b)g(b)h(b) \le [f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].$$

Using this and Proposition 2.8, we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)h(x)dx \\
&\le \frac{1}{12}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]+\frac{1}{6}[f(a)g(a)h(a)+f(b)g(b)h(b)] \\
&\le \frac{1}{12}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)]+\frac{1}{6}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)] \\
&= \frac{1}{4}[f(a)+f(b)][g(a)+g(b)][h(a)+h(b)].
\end{aligned}$$

This ends the proof of Corollary 2.10 □

4 Conclusion

In this study, we have contributed to the topic of integral inequalities under convexity assumptions. Specifically, we have made eight propositions and one corollary about upper bounds for the integral of the product of three convex functions. These upper bounds are of different kinds, reflecting the diversity of proof techniques. Some are based on existing integral inequalities, such as the Hermite-Hadamard integral inequalities, the Chebyshev integral inequality, the Hölder integral inequality, and others on the basic definition of convexity. Our results thus improve the understanding of the convexity in integral inequalities and provide useful tools for various mathematical applications.

Clearly, some of our results can be refined by using improved versions of the right-hand side of the Hermite-Hadamard integral inequalities, where they are used as intermediate tools in the corresponding proofs. We can think of the version studied in [24], which states that, in the setting of Theorem 1.2,

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{4}[f(a)+f(b)].$$

However, the refinements obtained lead to less tractable integral inequalities due to the excessive length of the formulas.

Based on our study, an open problem can be considered, as formulated below. Let $n \in \mathbb{N}\backslash\{0\}$, $a, b \in \mathbb{R}$ with $a < b$ and $f_1, \ldots, f_n : [a,b] \mapsto [0,+\infty)$ be n convex functions. Then the following inequality can be discussed:

$$\frac{1}{b-a}\int_a^b \left[\prod_{i=1}^{n} f_i(x)\right]dx \leq \frac{1}{n+1}\prod_{i=1}^{n}[f_i(a)+f_i(b)].$$

Indeed, it is true for $n = 1$ thanks to the Hermite-Hadamard integral inequality (see Theorem 1.2), it is true for $n = 2$ thanks to Proposition 1.3 and it is now true for $n = 3$ thanks to Corollary 2.10. An open problem is to prove it rigorously for any n, in full generality.

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