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Analogue Results of Nevanlinna's Fundamental Theorems for Composite Functions

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Abstract
The theory of value distribution of meromorphic functions depends heavily on the first and second fundamental theorems of R. Nevanlinna. In this paper we prove analogous results of Nevanlinna's fundamental theorems for composite functions of f and g, where f is meromorphic and g is restricted to be an entire function other than a linear polynomial.

Keywords: Entire functions, meromorphic functions, characteristic functions, composite functions, fundamental theorem.

1 Introduction, Definitions and Notations

We first need some important definitions by Nevanlinna [3].

Definition 1.1. The positive logarithmic function $\log^+ x$ for $x \geq 0$ is defined as follows

$$\log^+ x = max(\log x, 0).$$

It is evident that

$$\log x = \log^+ x - \log^+ \frac{1}{x}$$

for all $x > 0$.

For a non-constant meromorphic function $f(z)$ in the disc $|z| \leq R$ $(0 < R < \infty)$ Nevanlinna defined the following functions.

Definition 1.2. $m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$

which is nothing but the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Definition 1.3. The counting function of poles of $f(z)$, denoted by $N(r, f)$ is defined as

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ is the number of poles of $f(z)$ in the disc $|z| \leq t$, multiple poles are counted according to their multiplicities and $n(0, f)$ denotes the multiplicity of poles of $f(z)$ at the origin.

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Definition 1.4. The characteristic function of $f(z)$, denoted by $T(r, f)$ is defined as

$$T(r, f) = m(r, f) + N(r, f) \ .$$

The theory of value distribution of meromorphic functions established by R. Nevanlinna [3], highly rely on the following two fundamental theorems viz. first and second fundamental theorems.

Theorem 1.1. Let $f(z)$ be a meromorphic function in $|z| < R$ $(\leq \infty)$ and a is any complex number. Then for $0 < r < R$

$$T(r, \frac{1}{f-a}) = T(r, f) + \log|c_\lambda| + \epsilon(a, r), \tag{1.1}$$

where c_λ is the first non-zero coefficient of the Laurent expansion of $\frac{1}{f(z)-a}$ at the origin, and

$$|\epsilon(a, r)| \leq \log^+|a| + \log 2.$$

The formula (1.1) can be simply rewritten as

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1), \tag{1.2}$$

which means that, for any complex number a, the difference between $T(r, \frac{1}{f-a})$ and $T(r, f)$ is a bounded quantity.

Theorem 1.2. Let $f(z)$ be a non-constant meromorphic function in $|z| < R$ and a_j $(j = 1, 2, ..., q)$ are q (≥ 2) distinct finite complex numbers. Then for $0 < r < R$

$$m(r, f) + \sum_{j=1}^{q} m(r, \frac{1}{f-a_j}) \leq 2T(r, f) - N_1(r) + S(r, f) \ , \tag{1.3}$$

where

$$N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$$

and

$$S(r, f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}) + O(1) \ .$$

2 Preliminary Results

To reach the main results of this paper we need the following lemmas.

Lemma 2.1. *[1]. If $f(z)$ is meromorphic and $g(z)$ is entire then for all large values of r*

$$T(r, f \circ g) \leq (1 + o(1)) \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) \ . \tag{2.1}$$

Lemma 2.2. *Let $f(z)$ be a transcendental meromorphic function in the complex plane and g(z) be entire. Then for sufficiently large r*

$$T(r, g) = o(\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)) \ . \tag{2.2}$$

Proof. Proof of Lemma 2.2 follows from [1]. □

Since for any entire function $g(z)$ we have $\frac{T(r,g)}{\log M(r,g)} \leq 1$, so (2.2) can be rewritten as

$$T(r, g) = o(T(M(r, g), f)) \ . \tag{2.3}$$

Lemma 2.3. *[3]. If f is a rational function of the form $\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials of degree p and q respectively, then*

$$T(r,f) = d\log r + O(1)\ , \tag{2.4}$$

where $d = max\{p,q\}$.

In fact, (2.4) can be rewritten as

$$T(r,f) = O(\log r)\ , \tag{2.5}$$

if and only if f is a rational function.

Lemma 2.4. *Let $f(z)$ be a rational function and $g(z)$ be entire. Then*

$$T(r,g) = O(T(M(r,g),f))\ . \tag{2.6}$$

Proof. Using (2.5) and since $T(r,g) \le \log M(r,g)$, so

$$\begin{aligned}\frac{T(r,g)}{T(M(r,g),f)} &= \frac{T(r,g)}{O(\log M(r,g))}\\ &= \frac{T(r,g)}{\log M(r,g)\ O(1)}\\ &\le k,\ \ a\ constant.\end{aligned}$$

Thus we can write

$$T(r,g) = O(T(M(r,g),f))\ .$$

□

Note 2.1. Combining (2.3) and (2.6) for any meromorphic function f and entire function g, we can write

$$T(r,g) = O(T(M(r,g),f))\ . \tag{2.7}$$

Lemma 2.5. *[3]. Let $f(z)$ be a non-constant meromorphic function in $|z| < R$ and $a_j (j = 1,2,\ldots,q)$ are q distinct finite complex numbers. Then*

$$m(r, \sum_{j=1}^{q} \frac{1}{f - a_j}) = \sum_{j=1}^{q} m(r, \frac{1}{f - a_j}) + O(1)\ , \tag{2.8}$$

for $0 < r < R$.

Lemma 2.6. *[3]. Let $f(z)$ be a non-constant meromorphic function in the complex plane. Then*

$$m(r, \frac{f'}{f}) = S(r,f) = o(T(r,f))\ . \tag{2.9}$$

3 Main Results

In this section we prove the following theorems, which are analogous of Nevanlinna's fundamental theorems for composite functions.

Theorem 3.1. *Let $f(z)$ be a meromorphic function in $|z| < R\ (\le \infty)$ and $g(z)$ be an entire function other than a linear polynomial. For any complex number a and for* $0 < r < R$

$$T(r, \frac{1}{f\circ g - a}) = T(r_1, f) + T(r_1, g) + O(1) \tag{3.1}$$

where $r_1 = M(r,g)$.

Proof. Applying (1.2) for the meromorphic function $f \circ g$ and then using (2.1) and (2.7) we have

$$\begin{aligned}
T(r, \frac{1}{f \circ g - a}) &= T(r, f \circ g) + O(1) \\
&\leq (1 + o(1)) \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) + O(1) \ , \\
&\leq (1 + o(1)) T(r_1, f) + O(1) \ , \quad \textit{since} \ \frac{T(r, g)}{\log M(r, g)} \leq 1 \ \textit{and} \ r_1 = M(r, g) \\
&= T(r_1, f) + o(T(r_1, f)) + O(1) \\
&\leq T(r_1, f) + O(T(r_1, f)) + O(1) \\
&= T(r_1, f) + T(r, g) + O(1) \\
&\leq T(r_1, f) + T(r_1, g) + O(1) \ , \\
&\quad \textit{since} \ r \leq r_1 = M(r, g) \ \textit{as} \ g \ \textit{is entire other than a linear polynomial},
\end{aligned}$$

which completes the proof of the theorem. □

Note 3.1. It can be noted that when $g(z) = z$ then

$$r_1 = r \quad \textit{and} \quad T(r_1, g) = \log r \ .$$

So, in this case (3.1) takes the form

$$T(r, \frac{1}{f - a}) = T(r, f) + \log r + O(1). \tag{3.2}$$

Now if f is rational then by using (2.5), we can rewrite (3.2) as

$$\begin{aligned}
T(r, \frac{1}{f - a}) &= O(\log r) + \log r + O(1) \\
&= O(\log r) + O(1) \\
&= T(r, f) + O(1) \ ,
\end{aligned}$$

which coincides with the first fundamental theorem (1.1).

When f is transcendental then since $\log r = o(T(r, f))$, so (3.2) can be written as

$$T(r, \frac{1}{f - a}) = [1 + o(1)] \ T(r, f) + O(1) \ .$$

Theorem 3.2. *Let $f(z)$ be a non-constant meromorphic function in $|z| < R$ and $g(z)$ be an entire function other than a linear polynomial. If a_j $(j = 1, 2, \ldots, q)$ are $q(\geq 2)$ distinct finite complex numbers then for $0 < r < R$, we have*

$$m(r, f \circ g) + \sum_{j=1}^{q} m(r, \frac{1}{f \circ g - a_j}) \leq 2 \ T(r_1, f) + 3 \ T(r_1, g) - N_2(r) + S_1(r) \ , \tag{3.3}$$

where

$$N_2(r) = N(r_1, f) - N(r_1, f') + N(r, f \circ g) + N(r, \frac{1}{(f \circ g)'}) \tag{3.4}$$

and

$$S_1(r) = m(r_1, \frac{f'}{f}) + m(r_1, \frac{g'}{g}) + m(r, \sum_{j=1}^{q} \frac{(f \circ g)'}{f \circ g - a_j}) + O(1) \tag{3.5}$$

with $r_1 = M(r, g)$.

Proof. Let

$$F(z) = \sum_{j=1}^{q} \frac{1}{f \circ g - a_j} \ . \tag{3.6}$$

Then by using (2.8), we have from (3.6)

$$m(r, F) = \sum_{j=1}^{q} m(r, \frac{1}{f \circ g - a_j}) + O(1) \ . \tag{3.7}$$

Now using (3.1) and (1.2) we get

$$\begin{aligned}
m(r, F) &\le m(r, (f \circ g)' F) + m(r, \frac{1}{(f \circ g)'}) \\
&= m(r, (f \circ g)' F) + T(r, \frac{1}{(f \circ g)'}) - N(r, \frac{1}{(f \circ g)'}) \\
&= m(r, (f \circ g)' F) + T(r, \frac{1}{(f' \circ g)\, g'}) - N(r, \frac{1}{(f \circ g)'}) \\
&\le m(r, (f \circ g)' F) + T(r, \frac{1}{(f' \circ g)}) + T(r, \frac{1}{g'}) - N(r, \frac{1}{(f \circ g)'}) \\
&= m(r, (f \circ g)' F) + T(r_1, f') + T(r_1, g) + T(r, g') - N(r, \frac{1}{(f \circ g)'}) + O(1)
\end{aligned} \tag{3.8}$$

where $r_1 = M(r, g)$.

Now

$$\begin{aligned}
T(r_1, f') &= m(r_1, f') + N(r_1, f') \\
&\le m(r_1, f) + m(r_1, \frac{f'}{f}) + N(r_1, f') \\
&= T(r_1, f) - N(r_1, f) + m(r_1, \frac{f'}{f}) + N(r_1, f') \ .
\end{aligned} \tag{3.9}$$

Since g' is entire and $r \le r_1 = M(r, g)$ so by using (2.9) we get

$$\begin{aligned}
T(r, g') &= m(r, g') \\
&\le m(r, g) + m(r, \frac{g'}{g}) \\
&= T(r, g) + m(r, \frac{g'}{g}) \\
&\le T(r_1, g) + o(T(r, g)) \\
&= T(r_1, g) + T(r, g)\, o(1) \\
&\le T(r_1, g) + T(r_1, g)\, o(1) \\
&= T(r_1, g) + o(T(r_1, g)) \\
&= T(r_1, g) + S(r_1, g) \\
&= T(r_1, g) + m(r_1, \frac{g'}{g}).
\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) in (3.8) we get

$$m(r, F) \le m(r, (f \circ g)' F) + T(r_1, f) - N(r_1, f) + m(r_1, \frac{f'}{f}) + N(r_1, f')$$

$$+\,2\,T(r_1,g)+m(r_1,\frac{g^{'}}{g})-N(r,\frac{1}{(f\circ g)^{'}})+O(1)\;. \tag{3.11}$$

Now we use (3.7) in (3.11) to get

$$\sum_{j=1}^{q} m(r,\frac{1}{f\circ g-a_j}) \le T(r_1,f)+2\,T(r_1,g)-[N(r_1,f)-N(r_1,f^{'})+N(r,\frac{1}{(f\circ g)^{'}})]$$
$$+\,[m(r_1,\frac{f^{'}}{f})+m(r_1,\frac{g^{'}}{g})+m(r,(f\circ g)^{'}F)+O(1)]\;.$$

Now adding $m(r,f\circ g)$ in both sides of it and then using (3.1) gives

$$\begin{aligned} m(r,f\circ g)+\sum_{j=1}^{q} m(r,\frac{1}{f\circ g-a_j}) &\le m(r,f\circ g)+T(r_1,f)+2\,T(r_1,g)-[N(r_1,f)\\ &\quad -N(r_1,f^{'})+N(r,\frac{1}{(f\circ g)^{'}})]+[m(r_1,\frac{f^{'}}{f})\\ &\quad +m(r_1,\frac{g^{'}}{g})+m(r,(f\circ g)^{'}F)+O(1)]\\ &\le T(r,f\circ g)+T(r_1,f)+2\,T(r_1,g)-[N(r_1,f)\\ &\quad -N(r_1,f^{'})+N(r,f\circ g)+N(r,\frac{1}{(f\circ g)^{'}})]\\ &\quad +[m(r_1,\frac{f^{'}}{f})+m(r_1,\frac{g^{'}}{g})+m(r,\sum_{j=1}^{q}\frac{(f\circ g)^{'}}{f\circ g-a_j})\\ &\quad +O(1)]\\ &= 2\,T(r_1,f)+3\,T(r_1,g)-N_2(r)+S_1(r). \end{aligned}$$

where

$$N_2(r)=N(r_1,f)-N(r_1,f^{'})+N(r,f\circ g)+N(r,\frac{1}{(f\circ g)^{'}})$$

and

$$S_1(r)=m(r_1,\frac{f^{'}}{f})+m(r_1,\frac{g^{'}}{g})+m(r,\sum_{j=1}^{q}\frac{(f\circ g)^{'}}{f\circ g-a_j})+O(1)\;,$$

which completes the proof of the theorem. □

Note 3.2. Here we note that when $g(z)=z$ then

$$r_1=r\;,\;\; m(r_1,\frac{g^{'}}{g})=\log^{+}\frac{1}{r}\longrightarrow 0\; for\ large\ r\ and\ T(r_1,g)=\log r\;.$$

So, in this case (3.3), (3.4) and (3.5) take the following forms

$$m(r,f)+\sum_{j=1}^{q} m(r,\frac{1}{f-a_j}) \le 2T(r,f)+3\log r-N_2(r)+S_1(r)\;, \tag{3.12}$$

where

$$N_2(r)=2N(r,f)-N(r,f^{'})+N(r,\frac{1}{f^{'}})=N_1(r)$$

and

$$S_1(r)=m(r,\frac{f^{'}}{f})+m(r,\sum_{j=1}^{q}\frac{f^{'}}{f-a_j})+O(1)=S(r,f)\;,$$

which is almost same as (1.3) except for the presence of an additional term $3\log r$ in the right hand side of (3.12).

Concluding Remark

Here the composition of functions f and g are so taken that f is a meromorphic function and g is restricted to be an entire function except a linear polynomial. Also the result corresponding to first fundamental theorem is extended to composition of only two functions has been discussed. But the scope for finding the same when composition of n such functions are taken remains open.

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