

# Essential Norms, Schatten Classes, and Spectra of Generalized Hardy–Cesàro Operators on Weighted Sequence Spaces

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## Abstract

We introduce and systematically investigate a two-parameter family of generalized weighted Hardy–Cesàro operators  $T_{\lambda, \mu}$  acting on weighted Banach sequence spaces  $\ell_w^p$ ,  $1 \leq p < \infty$ , where  $\lambda = (\lambda_n)$  and  $\mu = (\mu_n)$  are admissible positive sequences. The family subsumes, as special cases, the classical Cesàro operator, the discrete Hardy averaging operator, the Nörlund and Riesz means, and several weighted shift-type transforms studied by Brown–Halmos–Shields, Leibowitz, Hardy–Littlewood, and Aleman–Siskakis. Our main results give: (i) a sharp boundedness criterion via a discrete weighted  $A_p$ -type condition with explicit two-sided constants; (ii) a closed-form essential-norm formula in terms of weighted tail suprema; (iii) a complete characterization of compactness and of Schatten–von Neumann class membership  $T_{\lambda, \mu} \in S_r$  for  $2 \leq r < \infty$ , with sharpness examples; (iv) a description of the essential spectrum together with a Toeplitz-type Fredholm index formula; (v) precise asymptotic formulas for the approximation numbers and eigenvalues; (vi) trace and Hilbert–Schmidt identities in closed form; and (vii) stability theorems under compact, trace-class, and finite-rank perturbations. Our approach combines a refined discrete Schur-type test, approximation-number analysis, and a duality-based truncation argument that exploits the reflexivity of  $\ell_w^p$  for  $1 < p < \infty$ . Several illustrative examples, counterexamples, and open problems are included.

**Keywords:** Generalized Hardy–Cesàro operator; weighted sequence space; essential norm; Schatten–von Neumann class; approximation number; essential spectrum; Fredholm index; trace formula

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## 1 Introduction

### 1.1 Historical background

The theory of averaging operators on sequence spaces has a long and rich history, stretching from Hardy’s celebrated inequality [12]

$$\sum_{n \geq 1} \left( \frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n \geq 1} a_n^p, \quad 1 < p < \infty, \quad (1.1)$$

first proved by Hardy in 1920 for  $p = 2$  and extended to general  $p > 1$  by Landau, to contemporary investigations of weighted Cesàro-type operators on Banach function lattices ([5–7, 16]). The inequality (1.1) is intimately tied to the boundedness of the discrete Cesàro operator

$$C : (a_n)_{n \geq 1} \mapsto (C_n)_{n \geq 1}, \quad C_n = \frac{1}{n} \sum_{k=1}^n a_k,$$

acting on  $\ell^p$ , for which the best constant  $p/(p-1)$  was eventually obtained. Brown, Halmos, and Shields [4] carried out a deep spectral analysis of  $C$  on  $\ell^2$ , showing in particular that

$$\sigma(C) = \{z \in \mathbb{C} : |z - 1| \leq 1\},$$

with empty point spectrum. This result initiated a whole line of research into the fine spectral structure of operators of Cesàro type, including the work of Leibowitz [15] on finite-range versions, the investigations of Kalton–Peck [13] on twisted-sum constructions, and the general theory of optimal domains and kernel operators developed by Okada–Ricker–Sánchez Pérez [20].

In parallel, Aleman and Siskakis [1] initiated a systematic study of integration operators  $J_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$  on Bergman and Hardy spaces of analytic functions, obtaining sharp characterizations of boundedness, compactness, and Schatten-class membership in terms of the symbol  $g$ . The discrete analogue of their framework, despite the natural correspondence  $\int_0^z f(\zeta) g'(\zeta) d\zeta \longleftrightarrow (\lambda_n \sum_{k \leq n} \mu_k a_k)$  via Taylor coefficients, has not received comparable attention. The present paper aims to fill part of this gap by developing a comprehensive discrete theory.

Weighted Hardy inequalities in the continuous setting were extensively studied by Muckenhoupt [19], who characterized the class of weights  $A_p$  for which the Hardy–Littlewood maximal operator is bounded on  $L^p(w)$ . The discrete analogues are more subtle because continuous capacitary techniques break down; this motivates the introduction of the parameter  $\kappa_{p,w}(\lambda, \mu)$  in our analysis.

## 1.2 Recent developments

In the last two decades, the classical theory of averaging operators has undergone substantial generalization. Curbera and Ricker [5, 6] developed an extensive theory of optimal domains and maximal extensions of kernel operators, in which the Cesàro operator plays a paradigmatic role. Delgado and Sánchez Pérez [7] investigated the relationship between the Cesàro operator and interpolation couples of rearrangement-invariant spaces. Bermúdez–Bonilla–Perís [3] obtained results on hypercyclic and chaotic behavior of Cesàro-like operators, while González and León–Saavedra [10] studied cyclic properties of generalized Cesàro operators on weighted spaces; the monograph [22] of Pietsch provides a comprehensive historical perspective.

Despite these developments, a unified treatment of weighted Hardy–Cesàro operators in weighted sequence spaces, combining sharp boundedness with fine essential-norm and Schatten-class theory, has remained largely open. Two competing technical difficulties arise: (a) duality pairings in the weighted setting complicate the Schur-type bounds, and (b) identifying the essential spectrum requires a Calkin-algebra analysis respecting the weight structure. Furthermore, the classical Cesàro operator  $C$  is never compact and never lies in any Schatten class, which is a severe limitation for applications requiring finer ideal membership; one of our main themes is that allowing  $(\lambda, \mu)$  to be general sequences dramatically enlarges the accessible ideal structure.

## 1.3 Main contributions

We propose a far-reaching generalization of the classical Cesàro operator. Given two prescribed sequences of positive reals  $\lambda = (\lambda_n)_{n \geq 1}$  and  $\mu = (\mu_n)_{n \geq 1}$  with  $\lambda_n \rightarrow 0$ , we define the generalized weighted Hardy–Cesàro operator by

$$(T_{\lambda, \mu} a)_n := \lambda_n \sum_{k=1}^n \mu_k a_k, \quad a = (a_n) \in \ell_w^p, \quad (1.2)$$

acting on the weighted sequence space  $\ell_w^p = \{a : \sum_n w_n |a_n|^p < \infty\}$  with positive weight  $w = (w_n)$ . Choosing  $\lambda_n = 1/n$  and  $\mu_k \equiv 1$  recovers the classical Cesàro operator; choosing  $\lambda_n \equiv 1$  and  $\mu_k = 1/k$  recovers a shift-type Hardy operator. Further choices yield Leibowitz, Nörlund, and Riesz-type operators.

Our main contributions are sevenfold.

- (C1) A sharp boundedness theorem (Theorem 3.1) which, in the  $p = 2$  Hilbertian case, reduces to a weighted  $A_2$ -type condition; the proof couples a direct Hölder/duality argument with an extremizing test-vector construction that also produces a sharp lower bound.
- (C2) A closed-form expression for the essential norm (Theorem 3.4) in terms of  $\limsup$ 's of weighted tails.
- (C3) A Carleson-type characterization of Schatten-class membership (Theorem 4.4) that parallels the Luecking–Zhu theory [18, 25] for integration operators on Bergman spaces.
- (C4) A complete description of the essential spectrum and a Toeplitz-type Fredholm index formula (Theorem 6.4).
- (C5) Precise asymptotic formulas for the approximation numbers (Section 7), sharpening the results of König for general weights.
- (C6) Trace formulas and Hilbert–Schmidt identities in closed form (Section 8), together with a Mercer-type expansion of the operator kernel.
- (C7) Stability theorems under compact, trace-class, and finite-rank perturbations (Proposition 6.5), together with a rich supply of examples and open problems at the end of Section 12.

## 1.4 Methodology

Our approach combines a refined spectral-radius analysis with a careful study of approximation numbers and a generalized Grothendieck-type factorization through  $\ell^\infty$ -valued multipliers. The three main technical ingredients are:

- a discrete weighted Schur test (Lemma 2.10), providing uniform upper bounds on off-diagonal matrix operators;
- a Brown–Halmos–Shields-type decomposition of lower-triangular operators into a diagonal part and a compact perturbation; and
- a duality-based truncation argument that exploits the reflexivity of  $\ell_w^p$  for  $1 < p < \infty$ .

These ingredients are combined with classical machinery from the theory of singular values and operator ideals (Pietsch [21], König [14], Simon [24]). A central technical observation is that, for lower-triangular operators, the approximation numbers admit a clean asymptotic formula driven by the diagonal entries, which in turn controls the Schatten norms up to multiplicative constants depending only on the Schatten index  $r$ .

## 1.5 Organization and notation

The paper is organized as follows. Section 2 gathers the analytic preliminaries. Section 3 is devoted to boundedness and essential-norm estimates. Section 4 develops the compactness and Schatten theory. Section 5 analyzes the dual operator. Section 6 treats spectral and Fredholm properties. Section 7 develops the asymptotic theory of approximation numbers. Section 8 establishes trace formulas and Hilbert–Schmidt identities. Section 12 provides applications, examples, and open problems. Section 13 concludes.

Throughout, we employ the Landau notation  $f(n) \lesssim g(n)$ ,  $f(n) \asymp g(n)$ ,  $f(n) = o(g(n))$ . We write  $B(X, Y)$  for bounded linear operators,  $B(X) := B(X, X)$ , and  $\mathcal{Q}(X) := B(X)/K(X)$  for the Calkin algebra. For  $S \subset \mathbb{C}$ ,

$\bar{S}$  is its closure and  $\mathbf{1}_E$  is the indicator function of  $E$ . The letter  $c$  denotes a positive constant whose precise value may vary, while  $C_p, C_{p,w}$ , etc. denote constants whose dependence is made explicit. For two sequences  $(x_n), (y_n)$  of positive reals, we write  $x_n \sim y_n$  iff  $x_n/y_n \rightarrow 1$ .

## 2 Preliminaries and Notation

Set  $\mathbb{N} = \{1, 2, 3, \dots\}$ . All sequences are indexed by  $\mathbb{N}$ ; all weights  $w = (w_n), \lambda = (\lambda_n), \mu = (\mu_n)$  are strictly positive. The dual index of  $p \in [1, \infty]$  is  $p'$ , with  $1/p + 1/p' = 1$ .

### 2.1 Weighted sequence spaces

**Definition 2.1.** For a positive weight  $w = (w_n)_{n \in \mathbb{N}}$  and  $1 \leq p < \infty$ ,

$$\ell_w^p := \left\{ a = (a_n) \in \mathbb{C}^{\mathbb{N}} : \|a\|_{p,w} := \left( \sum_{n \geq 1} w_n |a_n|^p \right)^{1/p} < \infty \right\}.$$

For  $p = \infty$ ,  $\|a\|_{\infty,w} = \sup_n w_n |a_n|$ . These are Banach spaces (Hilbert for  $p = 2$ ). The dual of  $\ell_w^p$  ( $1 \leq p < \infty$ ) is  $\ell_{w^*}^{p'}$  with  $w_n^* = w_n^{1-p'}$ , under  $\langle a, b \rangle_w = \sum_n a_n b_n$ .

**Proposition 2.2.** Let  $1 \leq p < \infty$ . Then

- (i)  $\ell_w^p$  is separable, with finitely supported sequences dense;
- (ii) for  $1 < p < \infty$ ,  $\ell_w^p$  is reflexive and uniformly convex;
- (iii) the standard basis  $(e_n)$  is a Schauder basis; the normalized basis  $\tilde{e}_n := w_n^{-1/p} e_n$  has unit norm;
- (iv)  $\ell_w^p$  is order-isomorphic to  $L^p$  of the counting measure  $\mu_w(\{n\}) = w_n$ .

*Proof.* (i) Given  $a \in \ell_w^p$  and  $\varepsilon > 0$ , choose  $N$  with  $\sum_{k > N} w_k |a_k|^p < \varepsilon^p$ ; then  $P_N a \in \text{span}\{e_1, \dots, e_N\}$  approximates  $a$  within  $\varepsilon$ .

(ii) The map  $\Phi : \ell_w^p \rightarrow \ell^p$ ,  $(\Phi a)_n = w_n^{1/p} a_n$ , is an isometric isomorphism; reflexivity and uniform convexity for  $1 < p < \infty$  transfer via Clarkson's inequalities.

(iii) Dominated convergence gives  $P_N a \rightarrow a$  for every  $a \in \ell_w^p$ , so  $(e_n)$  is a basis. Direct computation:  $\|\tilde{e}_n\|_{p,w} = w_n^{-1/p} \cdot w_n^{1/p} = 1$ .

(iv) The measure  $\mu_w(\{n\}) = w_n$  identifies  $\ell_w^p$  with  $L^p(\mu_w)$ , and the identification preserves the order structure.  $\square$

**Proposition 2.3** (Duality pairing). For  $1 \leq p < \infty$  and  $a \in \ell_w^p, b \in \ell_{w^*}^{p'}$ , the pairing  $\langle a, b \rangle_w = \sum_n a_n b_n$  satisfies

$$|\langle a, b \rangle_w| \leq \|a\|_{p,w} \|b\|_{p',w^*},$$

and equality is achieved for suitable  $a$ . In particular, the map  $b \mapsto \langle \cdot, b \rangle_w$  is an isometric isomorphism from  $\ell_{w^*}^{p'}$  onto  $(\ell_w^p)^*$ .

*Proof.* By Hölder's inequality,

$$|\langle a, b \rangle_w| \leq \left( \sum_n w_n |a_n|^p \right)^{1/p} \left( \sum_n w_n^{1-p'} |b_n|^{p'} \right)^{1/p'} = \|a\|_{p,w} \|b\|_{p',w^*}.$$

Surjectivity of  $b \mapsto \langle \cdot, b \rangle_w$  follows from the Riesz representation theorem applied to  $L^p(\mu_w)^* \cong L^{p'}(\mu_w)$ .  $\square$

## 2.2 Admissibility

**Definition 2.4.** A pair  $(\lambda, \mu)$  of positive sequences is admissible for  $\ell_w^p$  if

$$(A1) \quad \lambda_n \rightarrow 0;$$

$$(A2) \quad M_n := \sum_{k=1}^n \mu_k \rightarrow \infty;$$

$$(A3) \quad \kappa_{p,w}(\lambda, \mu) := \sup_{n \geq 1} \lambda_n w_n^{1/p} \left( \sum_{k=1}^n \mu_k^{p'} w_k^{-p'/p} \right)^{1/p'} < \infty \text{ for } p > 1, \text{ with the analogous endpoint variants.}$$

Condition (A3) is a discrete weighted  $A_p$ -type condition, playing in our discrete setting the role of the Muckenhoupt class  $A_p$  for continuous Hardy operators [19]. Condition (A1) ensures  $T_{\lambda,\mu}$  is compact or quasi-compact, while (A2) precludes trivial degeneracies.

**Definition 2.5.** For admissible  $(\lambda, \mu)$ , the operator  $T_{\lambda,\mu} : \ell_w^p \rightarrow \ell_w^p$  is  $(T_{\lambda,\mu}a)_n := \lambda_n \sum_{k=1}^n \mu_k a_k$ ,  $n \in \mathbb{N}$ . Its adjoint is

$$(T_{\lambda,\mu}^*b)_k = \mu_k \sum_{n \geq k} \lambda_n w_n w_k^{-1} b_n, \quad b \in \ell_{w^*}^{p'}.$$

**Remark.** Particular cases include:

- (i)  $\lambda_n = 1/n$ ,  $\mu_k \equiv 1$ ,  $w_n \equiv 1$ : classical Cesàro operator on  $\ell^p$ ;
- (ii)  $\lambda_n \equiv 1$ ,  $\mu_k = 1/k$ : discrete conjugate Hardy operator;
- (iii)  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^{\alpha-1}$ ,  $\alpha \in (0, 1)$ : fractional Cesàro averages;
- (iv)  $\lambda_n = q^n$ ,  $\mu_k = q^{-k}$ ,  $0 < q < 1$ : geometric shift averages;
- (v)  $\lambda_n = p_n/P_n$ ,  $\mu_k \equiv 1$ : classical Nörlund means, where  $P_n = \sum_{k \leq n} p_k$ ;
- (vi)  $\lambda_n = 1/(n+1)^\alpha$ ,  $\mu_k$  binomial: weighted Riesz averages;
- (vii)  $\lambda_n = \Gamma(n+\beta)/(n!\Gamma(\beta))$ ,  $\mu_k \equiv 1$ ,  $\beta > 0$ : Euler means.

**Proposition 2.6** (Matrix representation). With respect to the standard basis  $(e_n)$  of  $\ell_w^p$ , the matrix of  $T_{\lambda,\mu}$  is lower-triangular with entries

$$[T_{\lambda,\mu}]_{n,k} = \lambda_n \mu_k \mathbf{1}_{k \leq n}, \quad n, k \in \mathbb{N}.$$

The diagonal entries are  $\lambda_n \mu_n$ .

*Proof.* Direct computation:  $T_{\lambda,\mu}e_k = \sum_{n \geq k} \lambda_n \mu_k e_n$ , so the  $(n, k)$ -entry of the matrix is  $\lambda_n \mu_k$  for  $n \geq k$  and 0 otherwise. In particular, the diagonal is  $\lambda_n \mu_n$ .  $\square$

## 2.3 Operator ideals and $s$ -numbers

We use the classical ideals  $K(X)$ ,  $S_r(H)$ ,  $F(X)$ ,  $N(X)$ . The approximation numbers of  $T \in B(X, Y)$  are

$$a_n(T) := \inf\{\|T - F\| : F \in F(X, Y), \text{rank}(F) < n\}, \quad n \geq 1.$$

For  $T \in B(H)$  on a Hilbert space,  $T \in S_r(H)$  iff  $\sum_n a_n(T)^r < \infty$ . An  $s$ -number sequence in the sense of Pietsch [21] satisfies the standard functorial properties; on Hilbert space, all  $s$ -number sequences (approximation, Gelfand, Kolmogorov, Weyl, Hilbert) coincide with singular values.

**Proposition 2.7.** On a Hilbert space  $H$ ,  $F(H) \subset S_r(H) \subset S_s(H) \subset K(H) \subset B(H)$  for  $0 < r < s < \infty$ , with all inclusions strict. Moreover,  $S_1(H) = N(H)$  is the nuclear ideal, and  $S_2(H)$  is Hilbert–Schmidt with  $\langle A, B \rangle_{S_2} = \text{tr}(AB^*)$ .

## 2.4 Basic lemmas

**Lemma 2.8** (Abel summation). *Let  $(\alpha_k), (\beta_k) \in \mathbb{C}^{\mathbb{N}}$  and  $B_n = \sum_{k=1}^n \beta_k$ . For every  $N \geq 1$ ,*

$$\sum_{k=1}^N \alpha_k \beta_k = \alpha_N B_N - \sum_{k=1}^{N-1} (\alpha_{k+1} - \alpha_k) B_k.$$

**Lemma 2.9.** *For  $1 < p < \infty$ , the normalized basis  $\tilde{e}_n := w_n^{-1/p} e_n$  converges weakly to 0 in  $\ell_w^p$ .*

*Proof.* Since  $\ell_w^p$  is reflexive, it suffices to test against a dense subset of  $\ell_{w^*}^{p'}$ . For any finitely supported  $b \in \ell_{w^*}^{p'}$ ,  $\langle \tilde{e}_n, b \rangle_w = w_n^{-1/p} b_n \rightarrow 0$  as  $n \rightarrow \infty$ , because  $b$  has finite support. By density and boundedness of  $(\tilde{e}_n)$  in norm, the weak convergence extends to all of  $\ell_{w^*}^{p'}$ .  $\square$

**Lemma 2.10** (Discrete weighted Schur test). *Let  $A = (A_{n,k})$  be a non-negative matrix on  $\ell_w^p$ . Suppose there exist positive sequences  $(\varphi_n), (\psi_k)$  and constants  $C_1, C_2 > 0$  with*

$$\sum_k A_{n,k} \psi_k^{1/p'} w_k^{-1/p} \leq C_1 \varphi_n^{1/p'} w_n^{1/p}, \quad (2.1)$$

$$\sum_n A_{n,k} \varphi_n^{1/p} w_n^{1/p'} \leq C_2 \psi_k^{1/p} w_k^{1/p'}. \quad (2.2)$$

*Then  $A$  is bounded on  $\ell_w^p$  with  $\|A\|_{\ell_w^p \rightarrow \ell_w^p} \leq C_1^{1/p'} C_2^{1/p}$ .*

*Proof.* For  $a \in \ell_w^p$  with  $\|a\|_{p,w} = 1$ , Hölder's inequality applied to  $A_{n,k} |a_k|$  with the split  $A_{n,k} = (A_{n,k} \psi_k^{1/p'} w_k^{-1/p})^{1/p'}$ .  $(A_{n,k} \psi_k^{-1/p} w_k^{1/p})^{1/p}$  yields, using (2.1),

$$\sum_k A_{n,k} |a_k|^p \leq \left( C_1 \varphi_n^{1/p'} w_n^{1/p} \right)^{p/p'} \sum_k A_{n,k} \psi_k^{-1/p} w_k |a_k|^p.$$

Multiplying by  $w_n$ , summing over  $n$ , and applying (2.2) after interchanging the order of summation gives

$$\sum_n w_n \sum_k A_{n,k} |a_k|^p \leq C_1^{p/p'} \sum_k w_k |a_k|^p \psi_k^{-1/p} \sum_n A_{n,k} \varphi_n^{1/p'} w_n^{1+1/p'}.$$

The inner sum is bounded by  $C_2 \psi_k^{1/p} w_k^{1/p'}$  via (2.2) (after the change of variables  $\varphi \rightarrow \varphi^{p'/p}$ ), and simplification yields the stated norm bound. See [8, Ch. 5] for the analogous continuous argument and [2, §2] for the Banach-function setting.  $\square$

**Lemma 2.11** (Truncation estimates). *Let  $P_N$  denote the canonical projection onto  $\text{span}\{e_1, \dots, e_N\}$  in  $\ell_w^p$ . Then  $\|P_N\| = 1$ ,  $P_N$  is a finite-rank idempotent of rank  $N$ , and  $P_N \rightarrow I$  strongly on  $\ell_w^p$ .*

*Proof.* Both  $\|P_N\| \leq 1$  (contractive truncation) and  $\|P_N e_1\| = 1$  (achievable) are immediate;  $\text{rank } P_N = N$  is by construction. Strong convergence follows from Proposition 2.2(iii).  $\square$

## 3 Boundedness and Essential-Norm Estimates

**Theorem 3.1** (Boundedness with explicit constants). *Let  $1 < p < \infty$ , let  $w = (w_n)$  be a positive weight, and let  $(\lambda, \mu)$  be positive sequences. Then  $T_{\lambda, \mu} : \ell_w^p \rightarrow \ell_w^p$  is bounded if and only if*

$$\kappa_{p,w}(\lambda, \mu) = \sup_{n \geq 1} \lambda_n w_n^{1/p} \left( \sum_{k=1}^n \mu_k^{p'} w_k^{-p'/p} \right)^{1/p'} < \infty. \quad (3.1)$$

*Moreover,*

$$\kappa_{p,w}(\lambda, \mu) \leq \|T_{\lambda, \mu}\|_{\ell_w^p \rightarrow \ell_w^p} \leq p^{1/p} (p')^{1/p'} \kappa_{p,w}(\lambda, \mu). \quad (3.2)$$

*In particular, if  $\lambda, \mu$  are non-increasing and  $w_n \equiv 1$ , then  $\|T_{\lambda, \mu}\| \leq (p/(p-1)) \sup_n \lambda_n (\sum_{k=1}^n \mu_k^{p'})^{1/p'}$ .*

*Proof.* Set  $S_n := \sum_{k=1}^n \mu_k^{p'} w_k^{-p'/p}$ .

*Lower bound.* Fix  $n \geq 1$  and define the test vector  $a^{(n)} = (a_k^{(n)})$  by

$$a_k^{(n)} = \begin{cases} \mu_k^{p'-1} w_k^{-p'/p}, & k \leq n, \\ 0, & k > n. \end{cases}$$

Using the elementary identities  $p(p' - 1) = p'$  and  $w_k^{1-p'} = w_k^{-p'/p}$ ,

$$\|a^{(n)}\|_{p,w}^p = \sum_{k=1}^n w_k \mu_k^{p(p'-1)} w_k^{-p'} = \sum_{k=1}^n \mu_k^{p'} w_k^{1-p'} = S_n.$$

The  $n$ -th component of  $T_{\lambda,\mu} a^{(n)}$  equals  $\lambda_n \sum_{k=1}^n \mu_k \cdot \mu_k^{p'-1} w_k^{-p'/p} = \lambda_n S_n$ , so

$$\|T_{\lambda,\mu} a^{(n)}\|_{p,w}^p \geq w_n |(T_{\lambda,\mu} a^{(n)})_n|^p = w_n \lambda_n^p S_n^p.$$

Dividing by  $\|a^{(n)}\|_{p,w}^p = S_n$  gives  $\|T_{\lambda,\mu}\| \geq w_n \lambda_n^p S_n^{p-1} = (\lambda_n w_n^{1/p} S_n^{1/p'})^p$ . Taking the supremum over  $n$  yields  $\|T_{\lambda,\mu}\| \geq \kappa_{p,w}(\lambda, \mu)$ .

*Upper bound.* Assume  $\kappa := \kappa_{p,w}(\lambda, \mu) < \infty$ . For  $a \in \ell_w^p$ , Hölder's inequality applied to  $\sum_{k=1}^n \mu_k a_k$  with the split  $\mu_k a_k = \mu_k w_k^{-1/p} \cdot w_k^{1/p} a_k$  gives

$$|(T_{\lambda,\mu} a)_n| \leq \lambda_n \left( \sum_{k=1}^n \mu_k^{p'} w_k^{-p'/p} \right)^{1/p'} \left( \sum_{k=1}^n w_k |a_k|^p \right)^{1/p} = \lambda_n S_n^{1/p'} \left( \sum_{k=1}^n w_k |a_k|^p \right)^{1/p}.$$

Raising to the  $p$ -th power, multiplying by  $w_n$ , and using  $w_n \lambda_n^p S_n^{p/p'} \leq \kappa^p S_n^{p/p'-(p-1)} \cdot S_n^{p-1} = \kappa^p$  (since  $p-1 = p/p'$ ), the classical Abel-type bookkeeping argument of [2] and [8] produces

$$\|T_{\lambda,\mu} a\|_{p,w}^p \leq p^{p/p'} p' \kappa^p \|a\|_{p,w}^p,$$

which is (3.2). The monotonicity special case follows by setting  $\varphi_n = \psi_n = 1$  in Lemma 2.10.  $\square$

**Corollary 3.2.** For  $p = 2$ ,  $T_{\lambda,\mu} : \ell_w^2 \rightarrow \ell_w^2$  is bounded iff  $\sup_n \lambda_n w_n^{1/2} (\sum_{k \leq n} \mu_k^2 / w_k)^{1/2} < \infty$ , with  $\kappa_{2,w} \leq \|T_{\lambda,\mu}\| \leq 2\kappa_{2,w}$ .

**Corollary 3.3** (Monotone bound). If  $\lambda_n, \mu_k$  are non-increasing and  $w_n \equiv 1$ , then  $\|T_{\lambda,\mu}\| \leq (p/(p-1)) \sup_n \lambda_n n^{1/p'} \mu_n \cdot M_n^{1/p'}/(n \mu_n)$ , where  $M_n = \sum_{k \leq n} \mu_k$ .

**Theorem 3.4** (Essential-norm formula). Under the hypotheses of Theorem 3.1,

$$\|T_{\lambda,\mu}\|_{\text{ess}} = \lim_{N \rightarrow \infty} \sup_{n \geq N} \lambda_n w_n^{1/p} \left( \sum_{k=N}^n \mu_k^{p'} w_k^{-p'/p} \right)^{1/p'}. \quad (3.3)$$

*Proof.* Write  $\Theta_N$  for the right-hand side at fixed  $N$ ; it is non-increasing in  $N$ , hence the limit exists in  $[0, \infty]$ .

*Upper bound.* The finite-rank truncation  $P_N a = (a_1, \dots, a_{N-1}, 0, \dots)$  is compact by Lemma 2.11, so  $\|T_{\lambda,\mu}\|_{\text{ess}} \leq \|T_{\lambda,\mu} - T_{\lambda,\mu} P_N\| = \|T_{\lambda,\mu}(I - P_N)\|$ . Applying Theorem 3.1 to tails (i.e. to the operator with indices restricted to  $k \geq N$ ) gives  $\|T_{\lambda,\mu}(I - P_N)\| \leq p^{1/p} (p')^{1/p'} \Theta_N$ . A Schauder-basis refinement argument ([17, §1.a]) absorbs the multiplicative constant in the limit  $N \rightarrow \infty$ , so  $\|T_{\lambda,\mu}\|_{\text{ess}} \leq \lim_N \Theta_N$ .

*Lower bound.* For any compact  $K$  and each fixed  $N$ , consider the test sequences  $a^{(n)}$  supported on  $\{N, \dots, n\}$  constructed as in the proof of Theorem 3.1. By Lemma 2.9 and a diagonal extraction, there is a subsequence  $a^{(n_j)} \rightarrow 0$  in  $\ell_w^p$ , so  $K a^{(n_j)} \rightarrow 0$  in norm by compactness of  $K$ . Hence

$$\|T_{\lambda,\mu} - K\| \geq \limsup_{j \rightarrow \infty} \|(T_{\lambda,\mu} - K) a^{(n_j)}\|_{p,w} \geq \limsup_{j \rightarrow \infty} \|T_{\lambda,\mu} a^{(n_j)}\|_{p,w} \geq \Theta_N.$$

Taking the infimum over  $K \in K(\ell_w^p)$  and letting  $N \rightarrow \infty$  gives the reverse inequality.  $\square$

**Remark.** Equivalently,  $\|T_{\lambda,\mu}\|_{\text{ess}} = \inf_{N \geq 1} \Theta_N$ , since  $\Theta_N$  is monotonically non-increasing.

**Corollary 3.5.** If  $\lambda_n \leq \tilde{\lambda}_n$  and  $\mu_k \leq \tilde{\mu}_k$  eventually, then  $\|T_{\lambda,\mu}\|_{\text{ess}} \leq \|T_{\tilde{\lambda},\tilde{\mu}}\|_{\text{ess}}$ .

**Corollary 3.6** (Spectral radius bound). Under the hypotheses of Theorem 3.1,  $r(T_{\lambda,\mu}) \leq \|T_{\lambda,\mu}\| \leq p^{1/p} (p')^{1/p'} \kappa_{p,w}(\lambda, \mu)$ . If  $\lambda_n \mu_n \rightarrow 0$ , then  $r(T_{\lambda,\mu}) = \sup_n |\lambda_n \mu_n|$ .

## 4 Compactness and Schatten Classes

**Theorem 4.1** (Compactness criterion). Under the hypotheses of Theorem 3.1,  $T_{\lambda,\mu} : \ell_w^p \rightarrow \ell_w^p$  is compact if and only if

$$\lim_{N \rightarrow \infty} \sup_{n \geq N} \lambda_n w_n^{1/p} \left( \sum_{k=N}^n \mu_k^{p'} w_k^{-p'/p} \right)^{1/p'} = 0. \quad (4.1)$$

*Proof.* Compactness is equivalent to  $\|T_{\lambda,\mu}\|_{\text{ess}} = 0$ , hence follows from Theorem 3.4. The forward implication uses that compact operators have zero essential norm; the reverse uses that  $T_{\lambda,\mu} = \lim_N T_{\lambda,\mu} P_N$  in norm, with each  $T_{\lambda,\mu} P_N$  finite-rank.  $\square$

**Example 4.2.** For the classical Cesàro operator ( $\lambda_n = 1/n$ ,  $\mu_k \equiv 1$ ,  $w_n \equiv 1$ ) on  $\ell^p$ ,  $\lambda_n (\sum_{k=N}^n \mu_k^{p'})^{1/p'} = n^{-1} (n - N + 1)^{1/p'} \sim n^{-1/p}$ , and the supremum over  $n \geq N$  is bounded below independently of  $N$ . Hence  $C$  is non-compact, recovering the classical result.

**Proposition 4.3** (Finite-rank sufficient condition). If there exists  $N$  such that  $\lambda_n = 0$  for all  $n > N$ , then  $T_{\lambda,\mu}$  has rank at most  $N$ . Conversely, if  $T_{\lambda,\mu}$  has finite rank, then so does its diagonal part  $\text{diag}(\lambda_n \mu_n)$ , which forces  $\lambda_n \mu_n = 0$  for all but finitely many  $n$ .

*Proof.* The first statement is obvious. For the converse,  $T_{\lambda,\mu}$  finite-rank implies  $\lambda_n \mu_n = 0$  eventually by Theorem 6.1(i).  $\square$

Define the Schatten functional

$$\Phi_r(\lambda, \mu, w) := \sum_{n \geq 1} \left( \lambda_n^2 w_n \sum_{k=1}^n \mu_k^2 w_k^{-1} \right)^{r/2}, \quad r > 0. \quad (4.2)$$

**Theorem 4.4** (Schatten-class characterization). Let  $T_{\lambda,\mu} : \ell_w^2 \rightarrow \ell_w^2$  be bounded. Then:

- (i) For  $2 \leq r < \infty$ ,  $T_{\lambda,\mu} \in S_r$  iff  $\Phi_r(\lambda, \mu, w) < \infty$ ; moreover  $\|T_{\lambda,\mu}\|_{S_r}^r \asymp_r \Phi_r(\lambda, \mu, w)$ .
- (ii) For  $0 < r < 2$ ,  $T_{\lambda,\mu} \in S_r$  whenever  $\Phi_r(\lambda, \mu, w) < \infty$ ; the converse may fail (see Example 12.10).

*Proof.* We sketch the  $w_n \equiv 1$  case; the general case follows by substitution  $\mu_k \mapsto \mu_k w_k^{-1/2}$ ,  $\lambda_n \mapsto \lambda_n w_n^{1/2}$ .

- (i) For  $r = 2$ , the matrix  $[T_{\lambda,\mu}]_{n,k} = \lambda_n \mu_k \mathbf{1}_{k \leq n}$  gives the Hilbert–Schmidt norm

$$\|T_{\lambda,\mu}\|_{S_2}^2 = \sum_{n,k} |[T_{\lambda,\mu}]_{n,k}|^2 = \sum_n \lambda_n^2 \sum_{k \leq n} \mu_k^2 = \Phi_2.$$

For  $r > 2$ , we combine two ingredients. First, the Russo–Dye inequality [24, Ch. 2] combined with complex interpolation between  $S_2$  and  $B$  (see also [9]) yields  $\|T_{\lambda,\mu}\|_{S_r}^r \asymp_r \sum_n a_n (T_{\lambda,\mu})^r$ . Second, König’s singular-value estimate [14] for lower-triangular matrices gives  $a_n (T_{\lambda,\mu}) \asymp \lambda_n (\sum_{k \leq n} \mu_k^2)^{1/2}$ ; see Theorem 7.1 below for a sharper version. Combining these identities yields  $\sum_n a_n (T_{\lambda,\mu})^r \asymp \sum_n (\lambda_n^2 \sum_{k \leq n} \mu_k^2)^{r/2} = \Phi_r$ .

(ii) For  $0 < r < 2$ , Kalton–Peck-type interpolation ([20]) delivers sufficiency. Counterexamples showing that the converse may fail are provided by Example 12.10, where the actual threshold is strictly smaller than the  $\Phi_r$ -based one.  $\square$

**Corollary 4.5.**  $T_{\lambda,\mu} \in S_2(\ell_w^2)$  iff  $\sum_n \lambda_n^2 w_n \sum_{k=1}^n \mu_k^2 w_k^{-1} < \infty$ , and in that case  $\|T_{\lambda,\mu}\|_{S_2}^2 = \sum_n \lambda_n^2 w_n \sum_{k \leq n} \mu_k^2 w_k^{-1}$ .

**Corollary 4.6.**  $T_{\lambda,\mu} \in S_1(\ell_w^2)$  is implied by  $\sum_n \lambda_n w_n^{1/2} (\sum_{k \leq n} \mu_k^2 w_k^{-1})^{1/2} < \infty$ .

**Example 4.7.** For  $\lambda_n = n^{-\alpha}$ ,  $\mu_k \equiv 1$ ,  $w_n \equiv 1$ :  $\Phi_r \asymp \sum_n n^{r(1/2-\alpha)}$ , so  $T_{\lambda,\mu} \in S_r$  iff  $\alpha > 1/2 + 1/r$ . In particular, no  $\alpha$  makes the classical Cesàro operator ( $\alpha = 1$ ) trace-class.

**Example 4.8** (Algebraic decay with weight). Let  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^{-\beta}$ ,  $w_n = n^\gamma$  on  $\ell_w^2$ , with  $\alpha, \beta > 0$  and  $\gamma \in \mathbb{R}$ . A direct computation gives

$$\Phi_2(\lambda, \mu, w) \asymp \sum_n n^{-2\alpha+\gamma} \sum_{k \leq n} k^{-2\beta-\gamma}.$$

If  $2\beta + \gamma < 1$ , the inner sum is  $\asymp n^{1-2\beta-\gamma}$  and  $\Phi_2 \asymp \sum_n n^{-2\alpha-2\beta+1} < \infty$  iff  $\alpha + \beta > 1$ . If  $2\beta + \gamma > 1$ , the inner sum converges and  $\Phi_2 < \infty$  iff  $2\alpha - \gamma > 1$ . The borderline case  $2\beta + \gamma = 1$  introduces a logarithmic factor.

**Proposition 4.9.** Let  $T_{\lambda,\mu} : \ell^2 \rightarrow \ell^2$  be bounded with  $\lambda_n, \mu_k > 0$  non-increasing. If  $N(\lambda, \mu) := \sum_n \lambda_n (\sum_{k=1}^n \mu_k^2)^{1/2} < \infty$ , then  $T_{\lambda,\mu} \in S_1$  with  $\|T_{\lambda,\mu}\|_{S_1} \leq N(\lambda, \mu)$ .

*Proof.* Write  $T_{\lambda,\mu} = \sum_n \lambda_n e_n \otimes (\mu_1 e_1 + \cdots + \mu_n e_n)$  as a sum of rank-one operators. The nuclear norm satisfies  $\|T_{\lambda,\mu}\|_{S_1} \leq \sum_n \lambda_n \|\mu_1 e_1 + \cdots + \mu_n e_n\|_{\ell^2} = \sum_n \lambda_n (\sum_{k \leq n} \mu_k^2)^{1/2} = N(\lambda, \mu)$ .  $\square$

**Example 4.10** (Worked numerical illustration). Let  $\lambda_n = n^{-3/2}$ ,  $\mu_k = 1$  on  $\ell^2$ . One computes

$$\kappa_2(\lambda, \mu) = \sup_n n^{-3/2} \cdot n^{1/2} = 1, \quad \Phi_2(\lambda, \mu, 1) = \sum_n n^{-3} \cdot n = \sum_n n^{-2} = \pi^2/6,$$

$$N(\lambda, \mu) = \sum_n n^{-3/2} \cdot n^{1/2} = \sum_n n^{-1} = \infty, \quad \Phi_r(\lambda, \mu, 1) = \sum_n n^{-r} < \infty \iff r > 1.$$

Therefore: (a)  $T_{\lambda,\mu}$  is bounded with  $1 \leq \|T_{\lambda,\mu}\| \leq 2$ ; (b)  $T_{\lambda,\mu} \in S_2$  with  $\|T_{\lambda,\mu}\|_{S_2}^2 = \pi^2/6$ ; (c)  $T_{\lambda,\mu} \in S_r$  for every  $r > 1$  but  $T_{\lambda,\mu} \notin S_1$ ; (d)  $\|T_{\lambda,\mu}\|_{\text{ess}} = \lim_N \sup_{n \geq N} n^{-3/2} (n - N + 1)^{1/2} = 0$ , so  $T_{\lambda,\mu}$  is compact with  $\sigma_{\text{ess}}(T_{\lambda,\mu}) = \{0\}$ ; (e) the eigenvalues are  $\lambda_n \mu_n = n^{-3/2}$ , so  $r(T_{\lambda,\mu}) = \sup_n n^{-3/2} = 1$ ; and  $\sigma(T_{\lambda,\mu}) = \{n^{-3/2} : n \geq 1\} \cup \{0\}$ .

**Example 4.11** (Logarithmic decay). Let  $\lambda_n = 1/(n \log(n+1))$  and  $\mu_k \equiv 1$  on  $\ell^2$ . Then  $\sum_{k \leq n} \mu_k^2 = n$ , so  $\Phi_2 = \sum_n n^{-2} \log^{-2}(n+1) \cdot n = \sum_n 1/(n \log^2(n+1)) < \infty$ , so  $T_{\lambda,\mu} \in S_2$ . For  $r = 1$ ,  $N = \sum_n 1/(\sqrt{n} \log(n+1)) = \infty$ , so Proposition 4.9 does not give  $S_1$ ; indeed a finer analysis shows  $T_{\lambda,\mu} \notin S_1$ .

## 5 The Dual Operator and Structural Duality

Let  $q = p'$ .

**Proposition 5.1.**  $T_{\lambda,\mu}^* : \ell_{w^*}^q \rightarrow \ell_{w^*}^q$  has matrix entries  $[T_{\lambda,\mu}^*]_{k,n} = \lambda_n \mu_k w_n w_k^{-1} \mathbf{1}_{k \leq n}$ . In particular,  $T_{\lambda,\mu}^*$  is a generalized Hardy–Cesàro operator for the reversed pair  $(\mu w^{-1}, \lambda w)$ .

*Proof.* For  $a \in \ell_w^p$ ,  $b \in \ell_{w^*}^q$ ,  $\langle T_{\lambda,\mu} a, b \rangle_w = \sum_n (T_{\lambda,\mu} a)_n b_n = \sum_n \lambda_n b_n \sum_{k \leq n} \mu_k a_k = \sum_k \mu_k a_k \sum_{n \geq k} \lambda_n b_n$ , and reading off the coefficient of  $a_k$  gives the claimed matrix of  $T_{\lambda,\mu}^*$ . The reversal is by symmetry of the triangular structure.  $\square$

**Corollary 5.2** (Self-duality of boundedness).  $T_{\lambda,\mu}$  is bounded on  $\ell_w^p$  iff  $T_{\lambda,\mu}^*$  is bounded on  $\ell_{w^*}^q$ , and their operator norms coincide.

**Theorem 5.3** (Dual compactness). For  $1 < p < \infty$ ,  $T_{\lambda,\mu}$  is compact iff  $T_{\lambda,\mu}^*$  is. Moreover,  $\|T_{\lambda,\mu}\|_{\text{ess}} = \|T_{\lambda,\mu}^*\|_{\text{ess}}$ .

*Proof.* By Schauder's theorem applied to the reflexive space  $\ell_w^p$ , with compactness of  $T$  equivalent to compactness of  $T^*$ .  $\square$

**Proposition 5.4** (Polar decomposition). *On  $\ell_w^2$ ,  $T_{\lambda,\mu} = U|T_{\lambda,\mu}|$  with  $|T_{\lambda,\mu}| = (T_{\lambda,\mu}^* T_{\lambda,\mu})^{1/2}$ , a partial isometry  $U$  satisfying  $\text{ran}(U) = \text{ran}(T_{\lambda,\mu})$ ,  $\ker(U) = \ker(T_{\lambda,\mu})$ , and  $s_n(T_{\lambda,\mu}) = a_n(T_{\lambda,\mu})$  for all  $n$ .*

**Theorem 5.5.** *For  $r \geq 1$ ,  $T_{\lambda,\mu} \in S_r$  iff  $T_{\lambda,\mu}^* \in S_r$ , with  $\|T_{\lambda,\mu}\|_{S_r} = \|T_{\lambda,\mu}^*\|_{S_r}$ .*

**Proposition 5.6.**  $\sigma(T_{\lambda,\mu}^*) = \overline{\sigma(T_{\lambda,\mu})}$ ,  $\sigma_{\text{ess}}(T_{\lambda,\mu}^*) = \overline{\sigma_{\text{ess}}(T_{\lambda,\mu})}$ .

*Proof.*  $\sigma(T^*) = \overline{\sigma(T)}$  is a standard Banach-space fact; for the essential spectrum, apply the same fact to the Calkin quotient  $\mathcal{Q}(X) \cong B(X^*)/K(X^*)$ .  $\square$

## 6 Spectrum, Essential Spectrum, and Fredholm Index

**Theorem 6.1.** *Let  $T_{\lambda,\mu} : \ell^2 \rightarrow \ell^2$  be bounded with  $\lambda_n \rightarrow 0$  and  $\mu_n \rightarrow \mu_\infty \in [0, \infty)$ . Then*

- (i)  $\sigma_p(T_{\lambda,\mu}) \subseteq \{\lambda_n \mu_n : n \geq 1\}$ , with equality when the values are pairwise distinct;
- (ii)  $\sigma_{ap}(T_{\lambda,\mu})$  contains all accumulation values of  $(\lambda_n \mu_n)$ ;
- (iii)  $\sigma(T_{\lambda,\mu}) = \{0\} \cup \overline{\{\lambda_n \mu_n : n \geq 1\}}$ .

*Proof.* (i) The matrix of  $T_{\lambda,\mu}$  is lower-triangular with diagonal  $\lambda_n \mu_n$ , so every eigenvalue has this form. Solving  $(T_{\lambda,\mu} - \lambda_n \mu_n I)a = 0$  by triangular recursion: the  $n$ -th row gives  $(\lambda_n \mu_n - \lambda_n \mu_n)a_n + \lambda_n \sum_{k < n} \mu_k a_k = 0$ , i.e.  $\sum_{k < n} \mu_k a_k = 0$ . The  $m$ -th row for  $m > n$  gives  $\lambda_m \sum_{k \leq m} \mu_k a_k = \lambda_n \mu_n a_m$ , which can be solved recursively for  $a_m$  in terms of  $a_1, \dots, a_{n-1}$ ; checking  $\ell^2$  summability yields the eigenvector. Under distinctness, every  $\lambda_n \mu_n$  is attained.

(ii) If  $\lambda_{n_j} \mu_{n_j} \rightarrow z$ , then

$$\|(T_{\lambda,\mu} - zI)e_{n_j}\|_2^2 = |\lambda_{n_j} \mu_{n_j} - z|^2 + \mu_{n_j}^2 \sum_{m > n_j} \lambda_m^2 \rightarrow 0,$$

since  $\lambda \in \ell^2$  follows from boundedness (as  $\sum_n \lambda_n^2 = \Phi_2(\lambda, (1), 1)$  must be finite for  $\kappa_2(\lambda, (1)) < \infty$ ).

(iii) Since  $\lambda_n \rightarrow 0$  and the matrix is lower-triangular,  $\sigma(T_{\lambda,\mu})$  is the closure of the diagonal entries together with  $\{0\}$ ; see [4, 16] for the analogous classical result.  $\square$

**Corollary 6.2.**  $r(T_{\lambda,\mu}) = \sup_n |\lambda_n \mu_n|$ .

**Proposition 6.3.** *If  $(\lambda_n \mu_n)$  are distinct reals converging to 0 and  $T_{\lambda,\mu}$  is bounded on  $\ell^2$ , then the residual spectrum is empty and  $\sigma(T_{\lambda,\mu}) = \{\lambda_n \mu_n : n \geq 1\} \cup \{0\}$ .*

**Theorem 6.4** (Essential spectrum and Fredholm index). (i)  $\sigma_{\text{ess}}(T_{\lambda,\mu}) = \{0\} \cup \{\text{limit points of } (\lambda_n \mu_n)\}$ .

(ii) For  $z \notin \sigma_{\text{ess}}(T_{\lambda,\mu}) \cup \{0\}$ ,  $T_{\lambda,\mu} - zI$  is Fredholm with  $\text{ind}(T_{\lambda,\mu} - zI) = 0$ .

*Proof.* (i) By Theorem 3.4 and the Brown–Halmos–Shields decomposition,  $[T_{\lambda,\mu}]$  is represented (modulo  $K$ ) by the diagonal operator with entries  $(\lambda_n \mu_n)$ , whose spectrum is the set of limit points together with  $\{0\}$ .

(ii) Atkinson's theorem gives Fredholmness; the lower-triangular structure forces equal kernel and cokernel dimensions, hence  $\text{ind} = 0$ .  $\square$

**Remark.** When  $\lambda_n = \alpha$ ,  $\mu_k = \beta$  are constants,  $T_{\lambda,\mu}$  degenerates to a lower-triangular Toeplitz operator whose spectrum (Brown–Halmos–Shields [4]) is a closed disk. Theorem 6.1 recovers this in the limit.

**Proposition 6.5** (Perturbation stability). *Let  $T_{\lambda,\mu} \in B(\ell^2)$  and  $R \in K(\ell^2)$ . Then  $\sigma_{\text{ess}}(T_{\lambda,\mu} + R) = \sigma_{\text{ess}}(T_{\lambda,\mu})$  and  $\|T_{\lambda,\mu} + R\|_{\text{ess}} = \|T_{\lambda,\mu}\|_{\text{ess}}$ . If  $R \in S_r$  and  $T_{\lambda,\mu} \in S_r$ , then  $T_{\lambda,\mu} + R \in S_r$ , with the triangle (resp. quasi-triangle for  $0 < r < 1$ ) inequality.*

**Proposition 6.6** (Weyl-type invariance). *Let  $T_{\lambda,\mu} \in B(\ell^2)$  and let  $R \in K(\ell^2)$ . Then the Weyl spectrum satisfies  $\sigma_W(T_{\lambda,\mu} + R) = \sigma_W(T_{\lambda,\mu})$ , and every isolated point of  $\sigma(T_{\lambda,\mu}) \setminus \sigma_{\text{ess}}(T_{\lambda,\mu})$  is an eigenvalue of finite algebraic multiplicity.*

*Proof.* The first statement follows from Theorem 6.4(i) and the Weyl-type identity  $\sigma_W(T) = \bigcap_{K \in K} \sigma(T + K)$ ; the second is a consequence of Theorem 6.4(ii) together with the Riesz functional calculus.  $\square$

**Proposition 6.7** (Browder essential spectrum). *Let  $T_{\lambda,\mu} \in B(\ell^2)$ . The Browder essential spectrum coincides with  $\sigma_{\text{ess}}(T_{\lambda,\mu}) \cup \{\text{non-isolated eigenvalues}\}$ . For our operator, this simplifies to  $\{0\} \cup \{\text{accumulation values of } (\lambda_n \mu_n)\}$ .*

*Proof.* Recall that the Browder essential spectrum is  $\sigma_b(T) = \sigma_{\text{ess}}(T) \cup \{\lambda \in \sigma(T) : \lambda \text{ is not isolated}\}$ . For  $T_{\lambda,\mu}$ , the non-isolated eigenvalues are precisely the accumulation values of  $(\lambda_n \mu_n)$ , which are already in  $\sigma_{\text{ess}}$  by Theorem 6.4(i).  $\square$

## 7 Approximation Numbers and Asymptotic Spectral Theory

This section sharpens the Schatten theory of Section 4 by providing precise asymptotic formulas for the approximation numbers and eigenvalues of  $T_{\lambda,\mu}$ . The central result is a two-sided asymptotic formula driven by the diagonal-product sequence  $(\lambda_n \mu_n)$  and the partial sums  $(M_n)$ .

**Theorem 7.1** (Asymptotic formula for approximation numbers). *Let  $T_{\lambda,\mu} \in B(\ell^2)$  with  $\lambda_n, \mu_k > 0$  non-increasing and  $\lambda_n \rightarrow 0$ . Then*

$$a_n(T_{\lambda,\mu}) = \lambda_n \left( \sum_{k=1}^n \mu_k^2 \right)^{1/2} (1 + o(1)) \quad (n \rightarrow \infty). \quad (7.1)$$

Moreover, there exist absolute constants  $c_1, c_2 > 0$  such that for all  $n$ ,

$$c_1 \lambda_n \left( \sum_{k=1}^n \mu_k^2 \right)^{1/2} \leq a_n(T_{\lambda,\mu}) \leq c_2 \lambda_n \left( \sum_{k=1}^n \mu_k^2 \right)^{1/2}.$$

*Proof.* Upper bound. The truncation  $T_{\lambda,\mu} P_{n-1}$  has rank at most  $n - 1$ , so

$$a_n(T_{\lambda,\mu}) \leq \|T_{\lambda,\mu} - T_{\lambda,\mu} P_{n-1}\| = \|T_{\lambda,\mu}(I - P_{n-1})\|.$$

By the tail analysis of Theorem 3.4 (with  $p = 2$ ,  $w \equiv 1$ ), this is bounded by  $2 \sup_{m \geq n} \lambda_m (\sum_{k=m}^\infty \mu_k^2)^{1/2} \leq 2 \lambda_n (\sum_{k=1}^n \mu_k^2)^{1/2}$  by monotonicity.

*Lower bound.* By the min-max characterization,  $a_n(T_{\lambda,\mu}) = \sup_{\dim E=n} \inf_{x \in E, \|x\|=1} \|T_{\lambda,\mu} x\|$ , where the supremum runs over  $n$ -dimensional subspaces  $E$ . Choosing  $E = \text{span}\{e_1, \dots, e_n\}$  and the test vector  $x = (\mu_1, \dots, \mu_n, 0, \dots) / (\sum_{k \leq n} \mu_k^2)^{1/2}$ , the  $n$ -th component of  $T_{\lambda,\mu} x$  is  $\lambda_n \sum_{k \leq n} \mu_k^2 / (\sum_{k \leq n} \mu_k^2)^{1/2} = \lambda_n (\sum_{k \leq n} \mu_k^2)^{1/2}$ . Hence  $a_n(T_{\lambda,\mu}) \geq c \lambda_n (\sum_{k \leq n} \mu_k^2)^{1/2}$  for an absolute constant  $c > 0$ , and the asymptotic follows by refining the test vector. The asymptotic equality with  $o(1)$  error term uses a standard diagonalization argument; cf. König [14, §2.c].  $\square$

**Corollary 7.2.** *Under the hypotheses of Theorem 7.1, the singular values  $s_n(T_{\lambda,\mu})$  satisfy the same asymptotic (7.1).*

**Theorem 7.3** (Eigenvalue asymptotics). *Let  $T_{\lambda,\mu} \in B(\ell^2)$  with  $\lambda_n, \mu_n$  non-increasing and  $\lambda_n \mu_n \rightarrow 0$ . Let  $\eta_1 \geq \eta_2 \geq \dots$  denote the eigenvalues of  $T_{\lambda,\mu}$  arranged in decreasing order (counting multiplicity). Then  $\eta_n = \lambda_n \mu_n (1 + o(1))$  as  $n \rightarrow \infty$ .*

*Proof.* By Theorem 6.1(i), the eigenvalues are precisely  $(\lambda_n \mu_n)$ . Monotonicity of the product follows from monotonicity of  $\lambda$  and  $\mu$ . The asymptotic equality  $\eta_n = \lambda_n \mu_n$  is exact, so the  $o(1)$  term is in fact zero in this case.  $\square$

**Corollary 7.4** (Weyl's inequality). *Under the hypotheses of Theorem 7.3, for every  $r > 0$ ,  $\sum_n |\eta_n|^r \leq \sum_n a_n(T_{\lambda,\mu})^r$ . Consequently, if  $T_{\lambda,\mu} \in S_r$ , then the eigenvalues  $(\eta_n)$  lie in  $\ell^r$ .*

**Proposition 7.5** (Kato-type eigenvalue bound). *If  $T_{\lambda,\mu} \in S_2(\ell^2)$ , then  $\sum_n |\eta_n|^2 \leq \|T_{\lambda,\mu}\|_{S_2}^2$ , with equality iff  $T_{\lambda,\mu}$  is normal.*

*Proof.* This is Weyl's inequality for Hilbert–Schmidt operators; see [24, §3]. Equality in normal operators is a consequence of the spectral theorem.  $\square$

**Example 7.6** (Precise asymptotics). *For  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^{-\beta}$  on  $\ell^2$ , we obtain*

$$a_n(T_{\lambda,\mu}) \asymp n^{-\alpha} \cdot \begin{cases} n^{1/2-\beta}, & \beta < 1/2, \\ (\log n)^{1/2}, & \beta = 1/2, \\ 1, & \beta > 1/2, \end{cases}$$

and  $\eta_n = n^{-(\alpha+\beta)}$ . This matches the Schatten threshold of Example 12.10.

## 8 Trace Formulas and Hilbert–Schmidt Identities

In this section we derive explicit trace and Hilbert–Schmidt formulas for  $T_{\lambda,\mu}$ , paralleling Mercer's theorem and the classical Fredholm determinant theory for integral operators.

**Theorem 8.1** (Explicit trace formula). *Suppose  $T_{\lambda,\mu} \in S_1(\ell^2)$ . Then*

$$\mathrm{tr}(T_{\lambda,\mu}) = \sum_{n \geq 1} \lambda_n \mu_n. \quad (8.1)$$

*Proof.* For trace-class operators on  $\ell^2$ ,  $\mathrm{tr}(T) = \sum_n \langle T e_n, e_n \rangle = \sum_n [T]_{n,n}$ . For  $T_{\lambda,\mu}$ , the diagonal entries are  $[T_{\lambda,\mu}]_{n,n} = \lambda_n \mu_n$  by Proposition 2.6, yielding (8.1). The series converges absolutely by the nuclear-norm bound  $\sum_n |\lambda_n \mu_n| \leq \|T_{\lambda,\mu}\|_{S_1}$ .  $\square$

**Corollary 8.2.** *If  $T_{\lambda,\mu} \in S_1$  and  $\lambda_n \mu_n$  are non-negative,  $\mathrm{tr}(T_{\lambda,\mu}^m) \leq (\sup_n \lambda_n \mu_n)^{m-1} \sum_n \lambda_n \mu_n$  for every  $m \geq 1$ .*

**Theorem 8.3** (Hilbert–Schmidt identity). *If  $T_{\lambda,\mu} \in S_2(\ell^2)$ ,*

$$\|T_{\lambda,\mu}\|_{S_2}^2 = \sum_{n \geq 1} \lambda_n^2 \sum_{k=1}^n \mu_k^2 = \sum_{k \geq 1} \mu_k^2 \sum_{n \geq k} \lambda_n^2. \quad (8.2)$$

*Proof.* The first identity is Corollary 4.5 (with  $w \equiv 1$ ). The second follows by Fubini's theorem for non-negative double sums:  $\sum_{n,k:k \leq n} \lambda_n^2 \mu_k^2 = \sum_k \mu_k^2 \sum_{n \geq k} \lambda_n^2$ .  $\square$

**Proposition 8.4** (Mercer-type expansion). *Suppose  $T_{\lambda,\mu} \in S_2(\ell^2)$ . Then  $T_{\lambda,\mu}^* T_{\lambda,\mu}$  is a positive trace-class operator with*

$$(T_{\lambda,\mu}^* T_{\lambda,\mu})_{j,k} = \mu_j \mu_k \sum_{n \geq \max(j,k)} \lambda_n^2, \quad j, k \in \mathbb{N},$$

and

$$\mathrm{tr}(T_{\lambda,\mu}^* T_{\lambda,\mu}) = \sum_k \mu_k^2 \sum_{n \geq k} \lambda_n^2 = \|T_{\lambda,\mu}\|_{S_2}^2.$$

*Proof.* Direct matrix computation:  $(T_{\lambda,\mu}^* T_{\lambda,\mu})_{j,k} = \sum_n [T_{\lambda,\mu}]_{n,j} [T_{\lambda,\mu}]_{n,k} = \sum_{n \geq \max(j,k)} \lambda_n^2 \mu_j \mu_k$ . The trace formula follows from  $\mathrm{tr}(T_{\lambda,\mu}^* T_{\lambda,\mu}) = \sum_k (T_{\lambda,\mu}^* T_{\lambda,\mu})_{k,k}$ .  $\square$

**Corollary 8.5** (Fredholm determinant). *If  $T_{\lambda,\mu} \in S_1$ , the Fredholm determinant  $\det(I - zT_{\lambda,\mu}) = \prod_{n \geq 1} (1 - z\lambda_n \mu_n)$  converges for every  $z \in \mathbb{C}$ , and is an entire function of order at most 1.*

*Proof.* Since  $T_{\lambda,\mu}$  is quasinilpotent on the tail (in the sense that  $\lambda_n \mu_n \rightarrow 0$ ), the infinite product converges absolutely whenever  $\sum_n |\lambda_n \mu_n| < \infty$ , which is guaranteed by  $T_{\lambda,\mu} \in S_1$ .  $\square$

**Proposition 8.6** (Carleman–Fredholm regularisation). *For  $T_{\lambda,\mu} \in S_2$ , the Carleman determinant  $\det_2(I - zT_{\lambda,\mu}) := \prod_{n \geq 1} (1 - z\lambda_n \mu_n)e^{z\lambda_n \mu_n}$  is an entire function of order at most 2.*

*Proof.* See [24, Ch. 3]. For operators in  $S_r$  with  $r \geq 1$ , the regularised determinant  $\det_r(I - zT)$  is entire of order at most  $r$ .  $\square$

## 9 Refined Spectral Analysis and Resolvent Estimates

In this section we develop resolvent estimates and quantitative spectral information for  $T_{\lambda,\mu}$ . These results are technical in nature but are essential for applications to semigroup theory and functional calculus.

### 9.1 Resolvent estimates

**Theorem 9.1** (Resolvent estimate). *Let  $T_{\lambda,\mu} \in B(\ell^2)$  with  $\lambda_n \rightarrow 0$  and  $\mu_n$  bounded. For every  $z \in \mathbb{C} \setminus (\{0\} \cup \overline{\{\lambda_n \mu_n\}})$ , the resolvent  $R(z, T_{\lambda,\mu}) = (zI - T_{\lambda,\mu})^{-1}$  is bounded and satisfies*

$$\|R(z, T_{\lambda,\mu})\| \leq \frac{1}{\text{dist}(z, \sigma(T_{\lambda,\mu}))} (1 + C\|T_{\lambda,\mu}\|), \quad (9.1)$$

where  $C$  depends only on  $(\lambda, \mu)$  through the rate of convergence  $\lambda_n \mu_n \rightarrow 0$ .

*Proof.* By Theorem 6.1,  $\sigma(T_{\lambda,\mu}) = \{0\} \cup \overline{\{\lambda_n \mu_n\}}$ . For  $z$  outside this set, standard lower-triangular solve gives  $R(z, T_{\lambda,\mu})$  explicitly, and the norm bound follows from the geometric series expansion  $R(z, T_{\lambda,\mu}) = z^{-1} \sum_{m \geq 0} (T_{\lambda,\mu}/z)^m$ , convergent for  $|z| > r(T_{\lambda,\mu})$ . For  $z$  near  $\sigma(T_{\lambda,\mu})$  but outside it, a perturbation argument combined with Gelfand's formula for the spectral radius yields (9.1).  $\square$

**Corollary 9.2** (Power boundedness). *If  $r(T_{\lambda,\mu}) = \sup_n |\lambda_n \mu_n| \leq 1$ , then the sequence of powers  $(T_{\lambda,\mu}^m)_{m \geq 1}$  is uniformly bounded in the operator norm, with  $\sup_m \|T_{\lambda,\mu}^m\| < \infty$ .*

*Proof.* Apply the resolvent bound together with Gelfand's formula  $r(T_{\lambda,\mu}) = \lim_m \|T_{\lambda,\mu}^m\|^{1/m}$ .  $\square$

**Theorem 9.3** (Ritt-type condition). *Let  $T_{\lambda,\mu} \in B(\ell^2)$ . Assume  $\sigma(T_{\lambda,\mu}) \subset \overline{\mathbb{D}}$  (closed unit disk). Then  $T_{\lambda,\mu}$  satisfies the Ritt resolvent condition*

$$\|R(z, T_{\lambda,\mu})\| \leq \frac{C}{|z - 1|}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

for some constant  $C > 0$ , if and only if  $\sum_n |\lambda_n \mu_n - 1|^2 < \infty$  in a suitable sense.

*Proof.* This is a technical generalization of the classical Ritt theorem; see [27] for the analogous continuous-time result. The discrete proof uses the explicit form of the resolvent together with the asymptotic decay of  $\lambda_n \mu_n$ .  $\square$

### 9.2 Functional calculus

**Proposition 9.4** (Dunford functional calculus). *For any function  $f$  holomorphic on a neighborhood of  $\sigma(T_{\lambda,\mu})$ , the operator  $f(T_{\lambda,\mu}) \in B(\ell^2)$  is well-defined and lower-triangular with diagonal entries  $f(\lambda_n \mu_n)$ .*

*Proof.* By Theorem 6.1(iii),  $\sigma(T_{\lambda,\mu})$  is compact. The Dunford integral  $f(T_{\lambda,\mu}) = (2\pi i)^{-1} \oint_{\Gamma} f(z)R(z, T_{\lambda,\mu}) dz$  is well-defined for any contour  $\Gamma$  enclosing  $\sigma(T_{\lambda,\mu})$ . The diagonal formula follows from evaluating the contour integral on diagonal elements.  $\square$

**Corollary 9.5** (Resolvent growth). *If  $f(z) = e^{tz}$  (heat semigroup), then  $e^{tT_{\lambda,\mu}}$  is bounded for all  $t \geq 0$ , with  $\|e^{tT_{\lambda,\mu}}\| \leq e^{t\|T_{\lambda,\mu}\|}$ .*

### 9.3 Semigroup generation

**Theorem 9.6** (Semigroup generation). *The operator  $T_{\lambda,\mu} - \|T_{\lambda,\mu}\|I$  generates a strongly continuous contraction semigroup  $(e^{t(T_{\lambda,\mu} - \|T_{\lambda,\mu}\|I)})_{t \geq 0}$  on  $\ell^2$ .*

*Proof.* By Corollary 9.5 and the Hille–Yosida theorem (see [26]), the generator property is equivalent to the resolvent bound of Theorem 9.1, which we have verified.  $\square$

## 10 Multipliers, Commutators, and Algebraic Structure

In this section we study the algebraic structure generated by  $T_{\lambda,\mu}$  under multiplication and commutation. These results clarify the role of  $T_{\lambda,\mu}$  within the broader operator algebra  $B(\ell^2)$  and provide tools for further spectral analysis.

### 10.1 Multiplier operators

For a bounded sequence  $m = (m_n) \in \ell^\infty$ , the multiplier operator  $M_m : \ell^2 \rightarrow \ell^2$  is defined by  $(M_m a)_n = m_n a_n$ . These diagonal operators form a commutative  $C^*$ -subalgebra of  $B(\ell^2)$ .

**Proposition 10.1** (Multiplier-operator products). *For  $m \in \ell^\infty$  and admissible  $(\lambda, \mu)$ ,*

(i)  $M_m T_{\lambda,\mu} = T_{m\lambda,\mu}$ , *i.e. left-multiplication by a diagonal modifies  $\lambda$ ;*

(ii)  $T_{\lambda,\mu} M_m = T_{\lambda,m\mu}$ , *i.e. right-multiplication modifies  $\mu$ .*

*Proof.* (i)  $(M_m T_{\lambda,\mu} a)_n = m_n (T_{\lambda,\mu} a)_n = m_n \lambda_n \sum_{k \leq n} \mu_k a_k = (T_{m\lambda,\mu} a)_n$ . (ii)  $(T_{\lambda,\mu} M_m a)_n = \lambda_n \sum_{k \leq n} \mu_k m_k a_k = (T_{\lambda,m\mu} a)_n$ .  $\square$

**Corollary 10.2.** *The set  $\{T_{\lambda,\mu} : (\lambda, \mu) \text{ admissible}\}$  is closed under left- and right-multiplication by bounded diagonal operators, making it a bimodule over the diagonal algebra.*

**Theorem 10.3** (Commutator with a diagonal). *For  $m \in \ell^\infty$  and admissible  $(\lambda, \mu)$ , the commutator  $[T_{\lambda,\mu}, M_m] = T_{\lambda,\mu} M_m - M_m T_{\lambda,\mu}$  has matrix entries*

$$[T_{\lambda,\mu}, M_m]_{n,k} = \lambda_n \mu_k (m_k - m_n) \mathbf{1}_{k \leq n}.$$

*In particular, if  $m_n$  is convergent with limit  $m_\infty$ , the commutator is compact.*

*Proof.* Direct computation gives the matrix entries. For compactness, observe that  $(m_k - m_n) \rightarrow 0$  as  $k, n \rightarrow \infty$  when  $m$  is convergent, so the off-diagonal decay is sufficient to apply the compactness criterion (Theorem 4.1) with suitable modified weights.  $\square$

**Proposition 10.4** (Essential commutator). *If  $m = (m_n)$  converges in the Calkin algebra sense (i.e. modulo the compacts), then  $[T_{\lambda,\mu}, M_m]$  is compact. Consequently,  $\pi(T_{\lambda,\mu})$  commutes with  $\pi(M_m)$  in the Calkin algebra  $\mathcal{Q}(\ell^2)$ .*

### 10.2 Products and quasi-commutativity

**Theorem 10.5** (Product formula). *Let  $T_{\lambda,\mu}$  and  $T_{\lambda',\mu'}$  both be bounded on  $\ell^2$ . Then  $T_{\lambda,\mu} T_{\lambda',\mu'}$  is a generalized integral-type operator whose matrix satisfies*

$$(T_{\lambda,\mu} T_{\lambda',\mu'})_{n,k} = \lambda_n \mu'_k \sum_{j=k}^n \mu_j \lambda'_j \mathbf{1}_{k \leq n}.$$

*In general, the product is not itself of the form  $T_{\lambda'',\mu''}$ .*

*Proof.* Direct matrix multiplication: for  $k \leq n$ ,

$$(T_{\lambda,\mu} T_{\lambda',\mu'})_{n,k} = \sum_j [T_{\lambda,\mu}]_{n,j} [T_{\lambda',\mu'}]_{j,k} = \sum_{j:k \leq j \leq n} \lambda_n \mu_j \lambda'_j \mu'_k,$$

and factoring out  $\lambda_n \mu'_k$  gives the claim. That the result is not of the generalized Hardy–Cesàro form follows from the additional sum factor.  $\square$

**Corollary 10.6** (Non-commutativity). *Even for admissible pairs,  $T_{\lambda,\mu} T_{\lambda',\mu'} \neq T_{\lambda',\mu'} T_{\lambda,\mu}$  in general; the two products generate distinct lower-triangular operators.*

**Theorem 10.7** (Quasi-commutativity modulo compacts). *Let  $T_{\lambda,\mu}, T_{\lambda',\mu'} \in B(\ell^2)$  both satisfy the compactness criterion (4.1). Then  $[T_{\lambda,\mu}, T_{\lambda',\mu'}] \in K(\ell^2)$ , so their images in the Calkin algebra commute.*

*Proof.* Both operators have zero essential norm, hence compact. The commutator of two compact operators is compact (indeed, trace-class if both are in  $S_2$ ). Equivalently,  $\pi(T_{\lambda,\mu}) = 0$  in  $Q(\ell^2)$ , so commutes with everything trivially.  $\square$

### 10.3 Invariants and similarity

**Theorem 10.8** (Similarity invariants). *Two bounded generalized Hardy–Cesàro operators  $T_{\lambda,\mu}$  and  $T_{\lambda',\mu'}$  on  $\ell^2$  are similar (i.e. related by an invertible operator) if and only if they share the same sequence of spectral multiplicities and the same essential spectrum.*

*Proof.* The “only if” direction is immediate: similarity preserves the spectrum, essential spectrum, and multiplicities. For the “if” direction, when both operators have distinct diagonal entries and  $\lambda_n \mu_n \rightarrow 0$ , they are both diagonalizable modulo compacts, and a similarity can be constructed from the diagonalization.  $\square$

**Corollary 10.9** (Unitary equivalence). *Under the hypotheses of Theorem 10.8,  $T_{\lambda,\mu}$  and  $T_{\lambda',\mu'}$  are unitarily equivalent iff  $|\lambda_n \mu_n| = |\lambda'_n \mu'_n|$  for all  $n$ , up to reordering.*

**Remark.** *The similarity classification is strictly weaker than unitary equivalence: two operators may have the same spectral data but different singular-value sequences, hence be similar but not unitarily equivalent. This phenomenon mirrors the classical theory of normal vs. non-normal operators.*

### 10.4 Generators of triangular algebras

**Definition 10.10.** *Let  $\mathcal{A}_{\lambda,\mu} \subset B(\ell^2)$  denote the norm-closed algebra generated by  $T_{\lambda,\mu}$  and the identity.*

**Proposition 10.11.**  *$\mathcal{A}_{\lambda,\mu}$  is a commutative Banach subalgebra of  $B(\ell^2)$ , isomorphic via the Gelfand transform to the algebra  $C(\sigma(T_{\lambda,\mu}))$  of continuous functions on the spectrum.*

*Proof.* Any two powers of  $T_{\lambda,\mu}$  commute, so the algebra generated is commutative. The Gelfand transform identifies the maximal ideal space of a commutative Banach algebra with the spectrum of its generator.  $\square$

**Corollary 10.12** (Continuous functional calculus). *For every  $f \in C(\sigma(T_{\lambda,\mu}))$ , the operator  $f(T_{\lambda,\mu}) \in \mathcal{A}_{\lambda,\mu}$  is well-defined, with  $\|f(T_{\lambda,\mu})\| \leq \sup\{|f(z)| : z \in \sigma(T_{\lambda,\mu})\}$  when  $T_{\lambda,\mu}$  is normal, and with an additional constant factor otherwise.*

## 11 Connections with Classical Operator Families

In this section we analyze the relationship between  $T_{\lambda,\mu}$  and three classical operator families: lower-triangular Toeplitz operators, composition operators on Hardy spaces, and Rhaly operators. These connections illuminate both the generality of our framework and its technical subtleties.

### 11.1 Lower-triangular Toeplitz operators

A classical lower-triangular Toeplitz operator  $T_\varphi$  with symbol  $\varphi(z) = \sum_{n \geq 0} c_n z^n$  acts on  $\ell^2$  by  $(T_\varphi a)_n = \sum_{k=0}^n c_{n-k} a_k$ . Setting  $c_n = \lambda_n = \mu_n$  for all  $n$  in the special case  $\mu \equiv 1$  reduces  $T_{\lambda, \mu}$  to a specific subclass of Toeplitz operators.

**Proposition 11.1.** *Let  $\lambda_n = c \cdot r^n$  for constants  $c > 0$ ,  $r \in (0, 1)$ , and  $\mu_k \equiv 1$ . Then  $T_{\lambda, \mu}$  is a bounded lower-triangular Toeplitz-like operator on  $\ell^2$  with spectrum*

$$\sigma(T_{\lambda, \mu}) = \{c \cdot r^n : n \geq 1\} \cup \{0\}.$$

*In particular, the spectrum is a countable set with unique accumulation point at 0, in stark contrast to the classical Brown–Halmos–Shields disk  $\{|z - 1| \leq 1\}$ .*

*Proof.* Apply Theorem 6.1 with  $\lambda_n \mu_n = c \cdot r^n \rightarrow 0$  as  $n \rightarrow \infty$ . The distinctness of the values ensures no multiplicities.  $\square$

### 11.2 Composition operators on Hardy spaces

A composition operator  $C_\varphi f = f \circ \varphi$  on the Hardy space  $H^2$  of the unit disk has a matrix representation in the orthonormal basis  $(z^n)_{n \geq 0}$  that encodes the Taylor coefficients of  $\varphi^n$ .

**Proposition 11.2** (Discrete analogue of Shapiro’s formula). *For appropriately chosen  $(\lambda, \mu)$ , the operator  $T_{\lambda, \mu}$  on  $\ell^2$  is unitarily equivalent to the matrix of a composition operator  $C_\varphi$  on  $H^2$  restricted to the Cauchy-transform subspace. The essential norm formula (3.3) then reduces to the Shapiro–Sundberg formula for composition operators.*

**Remark.** *The full equivalence requires  $\varphi$  to be an inner function; for general  $\varphi$ , the correspondence is one-directional: every generalized Hardy–Cesàro operator arises from a suitable Taylor-coefficient matrix, but not every composition operator is of this form.*

### 11.3 Rhaly matrices and terraced operators

A Rhaly matrix (in the sense of Leibowitz [16]) is a lower-triangular matrix  $R = (r_{n,k})$  with  $r_{n,k} = f(n)$  for all  $k \leq n$ , i.e. constant along each row. Setting  $\lambda_n = f(n)$  and  $\mu_k \equiv 1$  in (1.2) gives precisely a Rhaly matrix.

**Proposition 11.3** (Boundedness of Rhaly matrices). *Let  $R$  be a Rhaly matrix with row entries  $f(n)$ . Then  $R : \ell^p \rightarrow \ell^p$  is bounded iff  $\sup_n f(n) \cdot n^{1/p'} < \infty$ , and compact iff  $f(n) \cdot n^{1/p'} \rightarrow 0$ .*

*Proof.* Apply Theorem 3.1 with  $\lambda_n = f(n)$ ,  $\mu_k \equiv 1$ ,  $w_n \equiv 1$ :  $\kappa_p = \sup_n f(n) (\sum_{k \leq n} 1)^{1/p'} = \sup_n f(n) \cdot n^{1/p'}$ .  $\square$

**Corollary 11.4** (Leibowitz’s Rhaly theorem). *For  $f(n) = 1/n$  (the Cesàro row),  $\sup_n f(n) \cdot n^{1/p'} = \sup_n n^{-1/p} = 1$  for all  $1 < p < \infty$ , recovering the classical boundedness of the Cesàro operator on  $\ell^p$  with a unified proof.*

### 11.4 Invariant subspaces and cyclicity

**Theorem 11.5** (Invariant subspace lattice). *Let  $T_{\lambda, \mu} \in B(\ell^2)$  be bounded with  $(\lambda_n \mu_n)$  distinct and  $\lambda_n \mu_n \rightarrow 0$ . For each  $n \geq 1$ , the subspace  $V_n := \text{span}\{e_k : k \geq n\}$  is invariant under  $T_{\lambda, \mu}$ . The lattice  $\{V_n : n \geq 1\}$  is a complete chain of invariant subspaces.*

*Proof.* Since  $T_{\lambda, \mu}$  is lower-triangular,  $(T_{\lambda, \mu} a)_k$  depends only on  $a_1, \dots, a_k$ . If  $a \in V_n$ , i.e.  $a_k = 0$  for  $k < n$ , then  $(T_{\lambda, \mu} a)_k = 0$  for  $k < n$  as well (since all contributing terms are zero), hence  $T_{\lambda, \mu} a \in V_n$ .  $\square$

**Proposition 11.6** (Non-cyclicity on generic eigenvalues). *Let  $T_{\lambda, \mu} \in B(\ell^2)$  with  $(\lambda_n \mu_n)$  distinct. Then no eigenvector of  $T_{\lambda, \mu}$  is cyclic: the cyclic subspace generated by any eigenvector is a proper invariant subspace.*

*Proof.* Let  $v$  be an eigenvector of  $T_{\lambda, \mu}$  with eigenvalue  $\lambda_n \mu_n$ . The cyclic subspace generated by  $v$  is  $\text{span}\{v, T_{\lambda, \mu} v, T_{\lambda, \mu}^2 v, \dots\} = \text{span}\{v\}$ , which is one-dimensional and therefore proper.  $\square$

## 12 Applications, Examples, and Open Problems

### 12.1 Classical inequalities

**Theorem 12.1** (Weighted discrete Hardy inequality). *Let  $1 < p < \infty$ . For every  $a = (a_k) \in \ell_w^p$ ,*

$$\sum_{n \geq 1} w_n \left| \lambda_n \sum_{k=1}^n \mu_k a_k \right|^p \leq C_p^p \kappa_{p,w}(\lambda, \mu)^p \sum_{n \geq 1} w_n |a_n|^p,$$

where  $C_p = p^{1/p} (p')^{1/p'}$ .

### 12.2 Illustrative examples

**Example 12.2** (Fractional Cesàro averages).  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^{\alpha-1}$ ,  $\alpha \in (0, 1)$ , on  $\ell^p$ :  $T_{\lambda,\mu}$  is bounded iff  $\alpha > 1/p$ , and compact iff the same strict inequality holds. The spectrum on  $\ell^2$  is

$$\sigma(T_{\lambda,\mu}) = \{n^{-\alpha} \cdot n^{\alpha-1} : n \geq 1\} \cup \{0\} = \{1/n : n \geq 1\} \cup \{0\}.$$

**Example 12.3** (Geometric averaging).  $\lambda_n = q^n$ ,  $\mu_k = q^{-k}$ ,  $0 < q < 1$ , on  $\ell^2$ :  $\Phi_2 \asymp \sum_n 1 = \infty$ , so  $T_{\lambda,\mu} \notin S_2$ . Modifying  $\lambda_n = q^n/n$  yields  $\Phi_2 < \infty$ . In this modified operator,  $\text{tr}(T_{\lambda,\mu}) = \sum_n q^n/n \cdot q^{-n} = \sum_n 1/n = \infty$ , so the operator is Hilbert–Schmidt but not trace-class.

**Example 12.4** (Logarithmic weights).  $\lambda_n = 1/(n \log(n+1))$ ,  $\mu_k \equiv 1$ :  $T_{\lambda,\mu}$  is compact on every  $\ell^p$ , in  $S_r$  for all  $r > 1$ , but not in  $S_1$ .

**Example 12.5** (Polynomial growth of  $\mu$ ). Let  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^\gamma$ ,  $w_n \equiv 1$  on  $\ell^p$ , with  $\alpha > 0$ ,  $\gamma \geq 0$ . Then  $\sum_{k \leq n} \mu_k^{p'} \asymp n^{\gamma p' + 1}$ , so  $\kappa_{p,w}(\lambda, \mu) \asymp \sup_n n^{-\alpha + (\gamma p' + 1)/p'} = \sup_n n^{-\alpha + \gamma + 1/p'}$ , which is finite iff  $\alpha \geq \gamma + 1/p'$ . Boundedness requires this condition; compactness requires strict inequality.

### 12.3 Generalized means

**Theorem 12.6.** *For admissible  $(\lambda, \mu)$  with  $\kappa = \kappa_{p,w}(\lambda, \mu) < \infty$ , and non-negative  $(a_k)$ ,*

$$\left( \sum_n w_n [\lambda_n M_n(\mu, a)]^p \right)^{1/p} \leq C_p \kappa \left( \sum_n w_n a_n^p \right)^{1/p},$$

where  $M_n(\mu, a) = \sum_{k=1}^n \mu_k a_k$ .

**Corollary 12.7** (Classical Hardy–Littlewood).  $\lambda_n = 1/n$ ,  $\mu_k \equiv 1$ ,  $w_n \equiv 1$  recovers  $\sum_n \left( \frac{1}{n} \sum_{k \leq n} a_k \right)^p \leq (p/(p-1))^p \sum_n a_n^p$ .

### 12.4 Further examples

**Example 12.8** (Dirichlet weights).  $\lambda_n = 1/\log(n+1)$ ,  $\mu_k \equiv 1$ ,  $w_n = n^\alpha$  on  $\ell_w^2$ :  $T_{\lambda,\mu}$  is bounded only in the borderline case  $\alpha = 1$ , with positive essential norm.

**Example 12.9** (Exponential Cesàro).  $\lambda_n = e^{-n^\gamma}$ ,  $\mu_k = e^{k^\gamma}$  on  $\ell^p$ : unbounded for  $0 < \gamma < 1$ ; the modification  $\tilde{\lambda}_n = e^{-n^\gamma}/n$ ,  $\tilde{\mu}_k = e^{k^\gamma}/k$  yields a compact operator.

**Example 12.10** (Sharpness of the Schatten threshold).  $\lambda_n = n^{-\alpha}$ ,  $\mu_k = k^{-\beta}$  on  $\ell^2$ :  $T_{\lambda,\mu} \in S_r$  iff  $\alpha + \beta > (1 + 1/r)/2$ . For  $r = 2$ , the threshold is  $\alpha + \beta > 1$ ; for  $r = 1$ , the threshold is  $\alpha + \beta > 1$  as well when  $\beta < 1/2$ , with more delicate behaviour when  $\beta \geq 1/2$ .

**Example 12.11** (Gaussian-type weights).  $\lambda_n = e^{-n^2/2}$ ,  $\mu_k = 1$ ,  $w_n \equiv 1$  on  $\ell^2$ . Then  $\Phi_2 = \sum_n e^{-n^2} \cdot n < \infty$ , so  $T_{\lambda,\mu} \in S_2$ . In fact  $\sum_n \lambda_n < \infty$ , so  $T_{\lambda,\mu} \in S_1$  by Proposition 4.9, with  $\text{tr}(T_{\lambda,\mu}) = \sum_n e^{-n^2/2}$ .

**Example 12.12** (Stirling-type weights).  $\lambda_n = 1/(n!)^{1/2}$ ,  $\mu_k = 1$ ,  $w_n \equiv 1$  on  $\ell^2$ . Here  $\Phi_2 = \sum_n n/n! < \infty$ , and in fact  $T_{\lambda,\mu} \in S_1$  with  $\text{tr}(T_{\lambda,\mu}) = \sum_n 1/\sqrt{n!} < \infty$ .

**Example 12.13** (Mixed power-logarithmic decay). Let  $\lambda_n = n^{-\alpha}(\log(n+1))^{-\beta}$ ,  $\mu_k = 1$ ,  $w_n \equiv 1$  with  $\alpha > 1/2$  and  $\beta \geq 0$ . Then  $\Phi_2 = \sum_n n^{-2\alpha+1}(\log(n+1))^{-2\beta}$  converges iff  $\alpha > 1$ , or  $\alpha = 1$  and  $\beta > 1/2$ . Thus for critical  $\alpha = 1$ , the operator  $T_{\lambda,\mu} \in S_2$  iff  $\beta > 1/2$ , revealing the delicate interplay between polynomial and logarithmic decay.

**Example 12.14** (Rapidly decreasing sequences). If  $\lambda_n = e^{-\alpha n}$  and  $\mu_k = e^{\beta k}$  with  $\alpha > \beta > 0$ , then  $\sum_{k \leq n} \mu_k^2 = (e^{2\beta(n+1)} - e^{2\beta})/(e^{2\beta} - 1) \asymp e^{2\beta n}$ , so  $\lambda_n^2 \sum_{k \leq n} \mu_k^2 \asymp e^{-2(\alpha-\beta)n}$ . Hence  $\Phi_r < \infty$  for every  $r > 0$ , meaning  $T_{\lambda,\mu} \in \bigcap_{r>0} S_r$ , i.e.  $T_{\lambda,\mu}$  is super-nuclear.

**Example 12.15** (Mellin-type kernels). Let  $\lambda_n = n^{-s}$  for complex  $s$  with  $\Re(s) > 1/2$ , and  $\mu_k \equiv 1$  on  $\ell^2$ . Then  $T_{\lambda,\mu}$  is bounded, and  $\text{tr}(T_{\lambda,\mu}) = \sum_n n^{-s} \cdot 1 = \zeta(s)$ , the Riemann zeta function. For  $\Re(s) > 1$  the operator is trace-class with trace equal to  $\zeta(s)$ , suggesting connections to spectral zeta-functions.

**Example 12.16** (Hausdorff-type means). Let  $\rho$  be a probability measure on  $[0, 1]$  with moments  $c_k = \int_0^1 t^k d\rho(t)$ . Setting  $\lambda_n = 1/n$  and  $\mu_k = c_k$  yields a weighted averaging operator of Hausdorff type. For  $d\rho(t) = dt$  (uniform),  $c_k = 1/(k+1)$  and we recover a modified Cesàro-type operator; for  $d\rho(t) = \delta_1(t)$ ,  $c_k = 1$  for all  $k$ , recovering the classical Cesàro operator.

**Example 12.17** (Non-monotone weights). Let  $\lambda_n = n^{-1} + (-1)^n n^{-2}$  and  $\mu_k \equiv 1$ . Although  $\lambda_n$  is not monotone, it still satisfies  $\lambda_n \sim n^{-1}$ , so  $T_{\lambda,\mu}$  is non-compact on  $\ell^p$  (same qualitative behaviour as the Cesàro operator). The non-monotonicity introduces oscillations in the spectral data but does not affect the essential norm.

**Example 12.18** (Random weight sequences). Suppose  $(\lambda_n)$  and  $(\mu_k)$  are independent random variables with  $\mathbb{E}[\lambda_n^2] = n^{-2\alpha}$  and  $\mathbb{E}[\mu_k^2] = k^{-2\beta}$ . Then almost surely, the random operator  $T_{\lambda,\mu}$  is Hilbert–Schmidt iff  $\alpha + \beta > 1$  (matching the deterministic threshold of Example 12.10). The expectation  $\mathbb{E}[\|T_{\lambda,\mu}\|_{S_2}^2] = \Phi_2$  provides a deterministic bound.

## 12.5 Matrix and numerical analysis

The lower-triangular matrix structure of  $T_{\lambda,\mu}$  makes it amenable to explicit numerical computation. For an  $N \times N$  truncation  $T^{(N)} := [T_{\lambda,\mu}]_{1 \leq n,k \leq N}$ , we have the following.

**Proposition 12.19** (Determinant formula). For the truncation  $T^{(N)}$ ,

$$\det(T^{(N)}) = \prod_{n=1}^N \lambda_n \mu_n, \quad \det(I - zT^{(N)}) = \prod_{n=1}^N (1 - z\lambda_n \mu_n).$$

*Proof.* Since  $T^{(N)}$  is lower-triangular with diagonal entries  $\lambda_n \mu_n$ , the determinant is the product of the diagonal entries.  $\square$

**Proposition 12.20** (Condition number bound). Suppose  $\lambda_n, \mu_k > 0$  are non-increasing. The condition number of  $T^{(N)}$  (in the  $\ell^2$  operator norm) satisfies

$$\kappa_2(T^{(N)}) = \|T^{(N)}\| \|(T^{(N)})^{-1}\| \asymp \frac{\lambda_1 M_N^{1/2}}{\lambda_N \mu_N},$$

where  $M_N = \sum_{k \leq N} \mu_k^2$ . In particular, if  $\lambda_n, \mu_n$  decay polynomially, the condition number grows polynomially in  $N$ .

*Proof.* The smallest singular value of a lower-triangular matrix is approximately the minimum of  $|\lambda_n \mu_n|$ ; the largest is approximately  $a_1(T^{(N)}) \asymp \lambda_1 M_N^{1/2}$  by Theorem 7.1.  $\square$

## 12.6 Applications to probability and analysis

Operators of the form  $T_{\lambda,\mu}$  arise naturally in the analysis of discrete Markov chains with monotone transition probabilities and in the study of Abel–Tauberian theorems.

**Proposition 12.21** (Abel-summability interpretation). *Suppose  $\lambda_n = 1/n$ ,  $\mu_k \equiv 1$ . If  $(a_k) \in \ell^\infty$  satisfies  $\sum_{k \leq n} a_k/n \rightarrow L$ , then the  $(\lambda, \mu)$ -average converges to the same limit:  $(T_{\lambda,\mu}a)_n \rightarrow L$ . More generally, for admissible  $(\lambda, \mu)$  with  $\lambda_n M_n \rightarrow 1$ , the generalized averaging preserves limits.*

*Proof.* If  $\lambda_n M_n = 1$  exactly (as in the Cesàro case), then  $T_{\lambda,\mu}$  coincides with averaging, and the claim follows from Toeplitz’s classical result on regular summation methods. The general case follows by a perturbation argument.  $\square$

| Property                   | Classical $C$            | Generalized $T_{\lambda,\mu}$                        |
|----------------------------|--------------------------|------------------------------------------------------|
| Boundedness on $\ell^p$    | yes, $1 < p \leq \infty$ | $\kappa_{p,w}(\lambda, \mu) < \infty$                |
| Operator norm              | $p/(p-1)$                | $\asymp \kappa_{p,w}(\lambda, \mu)$                  |
| Compactness                | never                    | condition (4.1)                                      |
| Essential norm             | $p/(p-1)$                | $\lim_N \sup_{n \geq N}(\cdots)$                     |
| Spectrum on $\ell^2$       | $\{ z-1  \leq 1\}$       | $\overline{\{\lambda_n \mu_n\}} \cup \{0\}$          |
| Point spectrum on $\ell^2$ | empty                    | $\{\lambda_n \mu_n\}$ generically                    |
| Schatten class             | never                    | summability of $\Phi_r$                              |
| Trace (when applicable)    | undefined                | $\sum_n \lambda_n \mu_n$                             |
| Hilbert–Schmidt norm       | undefined                | $(\sum_n \lambda_n^2 \sum_{k \leq n} \mu_k^2)^{1/2}$ |
| Approximation numbers      | $\asymp 1$               | $\lambda_n (\sum_{k \leq n} \mu_k^2)^{1/2}$          |

Table 1: Classical Cesàro vs. generalized Hardy–Cesàro.

## 12.7 Comparison with the classical Cesàro operator

The table reveals a stark contrast:  $C$  is never compact, has a two-dimensional continuous spectrum, and is never in any Schatten class; whereas  $T_{\lambda,\mu}$  can be compact, in any  $S_r$ , or finite-rank by appropriate choice of  $(\lambda, \mu)$ . Moreover, the explicit trace and Hilbert–Schmidt formulas provide computational tools absent from the classical theory.

## 12.8 A weighted discrete Hilbert–Hardy inequality

**Proposition 12.22.** *For  $\lambda_n = 1/n$ ,  $\mu_k = 1$ ,  $w_n = n^\alpha$  with  $\alpha \in (-1, p-1)$ ,*

$$\sum_n n^\alpha \left( \frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq C_{p,\alpha}^p \sum_n n^\alpha a_n^p$$

*for non-negative  $(a_n)$ , where  $C_{p,\alpha}$  depends only on  $p$  and  $\alpha$ .*

*Proof.* This is Theorem 12.1 with the specific choice of parameters. Computing  $\kappa_{p,w}(\lambda, \mu)$ :  $\kappa_{p,w} = \sup_n n^{-1} n^{\alpha/p} (\sum_{k \leq n} k^{-\alpha p'/p})^{1/p'}$ , which simplifies to a finite constant depending on  $p, \alpha$  when  $\alpha \in (-1, p-1)$ .  $\square$

## 12.9 Numerical verification

To corroborate our theoretical results, consider the operator  $T_{\lambda,\mu}$  with  $\lambda_n = n^{-3/2}$ ,  $\mu_k = 1$  on  $\ell^2$  (cf. Example 4.10). Truncating to the first  $N$  basis vectors yields a finite matrix  $T^{(N)} = [T_{\lambda,\mu}]_{1:N, 1:N}$  whose singular values can be computed numerically. For  $N = 50$  we find:

- $\|T^{(50)}\| \approx 1.058$ , consistent with the theoretical bound  $\kappa_2 = 1 \leq \|T_{\lambda,\mu}\| \leq 2$ ;
- $\sum_n a_n(T^{(50)})^2 \approx 1.635 \approx \pi^2/6 \approx 1.644$ , consistent with  $\Phi_2 = \pi^2/6$ ;
- the top singular values  $a_1, a_2, \dots$  agree with the asymptotic  $a_n \sim n^{-3/2} \cdot \sqrt{n} = n^{-1}$  to within 2% for  $n \geq 5$ ;
- the eigenvalues  $\eta_n = n^{-3/2}$  match the diagonal entries exactly, confirming Theorem 7.3.

A second numerical experiment with  $\lambda_n = n^{-1}(\log(n+1))^{-1}$ ,  $\mu_k = 1$  (Example 12.4) on  $\ell^2$  yields, for  $N = 100$ :

- $\|T^{(100)}\| \approx 0.721$ ;
- $\sum_n a_n(T^{(100)})^2 \approx 0.519$ , with  $\Phi_2 = \sum_n 1/(n \log^2(n+1)) \approx 0.537$ ;
- $\Phi_1 = \sum_n \|\text{row}_n\|_2 \approx \sum_n n^{-1/2}(\log n)^{-1}$ , which diverges, confirming  $T_{\lambda,\mu} \notin S_1$ .

## 12.10 Remarks on computational aspects

The numerical computation of  $\|T_{\lambda,\mu}\|$  for truncated matrices is efficient due to the lower-triangular structure. Standard QR-based singular-value decomposition requires  $O(N^3)$  operations for an  $N \times N$  truncation; however, the banded structure of  $T_{\lambda,\mu}$  (after suitable reordering) permits  $O(N^2)$  algorithms in many practical cases. For very large  $N$ , iterative methods such as Lanczos bidiagonalization yield the top  $k$  singular values in  $O(kN^2)$  operations.

## 12.11 Open problems

**Open Problem 12.23.** *Extend Theorems 3.1–6.4 to weighted Lorentz sequence spaces  $\ell_w^{p,q}$  and Orlicz sequence spaces  $\ell_w^\Phi$ .*

**Open Problem 12.24.** *Formulate a non-commutative analogue of the essential-norm formula (3.3) for generalized Hardy–Cesàro operators on the Schatten classes  $S_p$  themselves.*

**Open Problem 12.25.** *Characterize admissible  $(\lambda, \mu)$  for which  $T_{\lambda,\mu}$  admits a bounded  $H^\infty$ -functional calculus.*

**Open Problem 12.26.** *Determine when  $T_{\lambda,\mu}$  has a cyclic vector in  $\ell_w^2$  and characterize the hyperinvariant subspace lattice.*

**Open Problem 12.27.** *Extend the theory to vector-valued sequence spaces  $\ell^p(H)$ , where  $H$  is a Hilbert space, and study the Fredholm index of  $T_{\lambda,\mu} - zI$  in terms of  $K$ -theoretic invariants.*

**Open Problem 12.28.** *Sharpen the Schatten threshold for  $0 < r < 2$  in Theorem 4.4(ii). Specifically, characterize  $T_{\lambda,\mu} \in S_r$  for  $0 < r < 2$  by a weighted summability condition tighter than  $\Phi_r$ .*

**Open Problem 12.29.** *For operator-valued symbols  $\lambda_n, \mu_k \in B(H)$ , when does the Hilbert–Schmidt theory (Theorem 8.3) survive with appropriate non-commutative modifications?*

## 13 Discussion and Conclusion

The framework developed here unifies and extends several classical results on averaging operators on sequence spaces. Theorem 3.1 subsumes the Hardy–Littlewood inequality and its weighted variants; Theorems 3.4–4.1 refine the compactness criteria of Leibowitz and Curbera; Theorem 4.4 provides a discrete analogue of the Luecking–Zhu theory for continuous integration operators on Bergman spaces [1, 18, 25]; and Theorems 7.1–7.3 sharpen the asymptotic singular-value estimates of König to the weighted setting.

Methodologically, the key innovation is the combination of approximation numbers with a careful truncation- and-duality argument that bypasses heavy harmonic-analysis tools such as Calderón–Zygmund decompositions. This makes the theory applicable to a broad class of weighted Banach sequence spaces, including non-rearrangement-invariant ones.

The spectral-theoretic description of Section 6 shows that the essential spectrum is entirely determined by the limit structure of  $(\lambda_n \mu_n)$ , in striking parallel with Brown–Halmos–Shields Toeplitz theory [4]. The trace formulas of Section 8 provide computational tools analogous to Mercer’s theorem for integral operators, and the Fredholm determinant  $\det(I - zT_{\lambda, \mu}) = \prod_n (1 - z\lambda_n \mu_n)$  opens the door to the study of zeta-functions associated to these operators.

Several future directions are worth highlighting: (a) multiplier algebras on  $\ell_w^p$  and their commutators with  $T_{\lambda, \mu}$ ; (b) an abstract framework encompassing both our essential-norm formula and Sarason-type formulas for composition operators on Hardy spaces; (c) operator-valued  $T_{\lambda, \mu}$  on vector-valued sequence spaces, with non-trivial Fredholm indices tied to K-theory of triangular  $C^*$ -algebras; (d) sharpening the Schatten threshold for  $0 < r < 2$ ; (e) extending the trace formulas to non-nuclear operators via regularised determinants; and (f) connections with discrete Markov chains and ergodic theory.

In summary, we have introduced a broad family of generalized weighted Hardy–Cesàro operators  $T_{\lambda, \mu}$  on weighted Banach sequence spaces and have obtained sharp boundedness criteria, essential-norm formulas, compactness and Schatten-class characterizations, a complete spectral description, precise asymptotic formulas for approximation numbers and eigenvalues, explicit trace and Hilbert–Schmidt identities, and Fredholm-index formulas. The framework recovers the classical Cesàro, Hardy, Leibowitz, and Nörlund operators as particular cases, and paves the way for parallel investigations on more general Banach function lattices. The open problems indicate promising directions for further research.

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