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On Three Hardy-Hilbert Type Integral Inequalities with New Kernel Functions: A Series Approach

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Abstract
This article investigates three Hardy-Hilbert-type integral inequalities with new kernel functions defined on the unit square. The first inequality is derived from the hyperbolic arctangent function, the second from the logarithmic function, and the third from the arcsine function. Using a series-based approach, we establish sharp constant factors for these inequalities. The corresponding upper bounds are expressed in terms of the unweighted integral norms of the main functions. Complete proofs are provided in detail.

Keywords: Hardy-Hilbert-type integral inequality; hyperbolic arctangent function; logarithmic function; arcsine function.

1 Introduction

The classical Hardy-Hilbert integral inequality plays a central role in the theory of inequalities and their applications. For $p > 1$, the Lebesgue space for positive functions is defined by

$$L^p(0,1) = \left\{ f : (0,1) \to (0,\infty);\ \|f\|_{L^p(0,1)} = \left(\int_0^1 f^p(x)dx \right)^{1/p} < \infty \right\}.$$

In this integral setting, let $p, q > 1$ with $1/p + 1/q = 1$, and $f, g : (0,1) \to (0,\infty)$ such that $f \in L^p(0,1)$ and $g \in L^q(0,1)$. Then a consequence of the Hardy-Hilbert integral inequality is

$$\int_0^1 \int_0^1 \frac{f(x)g(y)}{x+y} dxdy \le A_p \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)},$$

where

$$A_p = \frac{\pi}{\sin(\pi/p)}$$

is considered as a sharp constant, but not the optimal one. This result can be presented as an unoptimized Hardy-Hilbert integral inequality restricted to functions with support $(0,1)$. In full generality, this inequality has attracted considerable attention due to its deep connections with harmonic analysis, operator theory and mathematical physics. Over the past few decades, many extensions and generalizations have been established, including discrete analogues and weighted versions, as well as inequalities involving more general kernel functions. Further details can be found in the books [4, 9, 11] and the survey [1]. Various aspects of Hardy-Hilbert-type integral inequalities are discussed in the articles [2, 3, 5–8, 10].

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In this article, we contribute to this area of research by presenting three Hardy-Hilbert-type integral inequalities featuring new kernel functions defined on the unit square $(0, 1)^2$. The first inequality involves the hyperbolic arctangent function, the second involves the logarithmic function, and the third involves the arcsine function. Using a series-based approach, we determine sharp constant factors for each inequality. Additionally, this approach enables us to express the corresponding upper bounds in terms of the unweighted integral norms of the main functions. We thus deal with integral inequalities of the general form

$$\int_0^1 \int_0^1 f(x)g(y)k(x, y)dxdy \leq \Upsilon_p \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)},$$

where $k : (0, 1)^2 \to (0, \infty)$ is a kernel function of the type introduced above, and Υ_p denotes a constant depending only on p and the choice of the kernel function. Complete proofs of these results are provided.

The remainder of the article is as follows: Section 2 presents a Hardy-Hilbert-type integral inequality involving the hyperbolic tangent function, while Section 3 is devoted to a similar inequality based on the logarithmic function. Section 4 addresses a Hardy-Hilbert-type integral inequality involving the arcsine function. Finally, concluding remarks are given in Section 5.

2 A Hardy-Hilbert Type Integral Inequality Involving the Hyperbolic Tangent Function

In the theorem below, we present a complex Hardy-Hilbert-type integral inequality involving the kernel function $\operatorname{arctanh}(xy)$ on the unit square.

Theorem 2.1 *Let $p, q > 1$ with $1/p + 1/q = 1$, and $f, g : (0, 1) \to (0, \infty)$ such that $f \in L^p(0, 1)$ and $g \in L^q(0, 1)$. Then, we have*

$$\int_0^1 \int_0^1 f(x)g(y) \operatorname{arctanh}(xy)dxdy \leq B_p \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)},$$

where

$$B_p = \sum_{n=0}^{\infty} \frac{1}{(2n+1)\left((2n+1)q+1\right)^{1/q}\left((2n+1)p+1\right)^{1/p}},$$

which is a convergent series.

Proof. For any $|z| < 1$, the hyperbolic arctangent function admits the convergent power series expansion

$$\operatorname{arctanh}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}.$$

Hence, for any $x, y \in (0, 1)$, we have

$$\operatorname{arctanh}(xy) = \sum_{n=0}^{\infty} \frac{x^{2n+1}y^{2n+1}}{2n+1},$$

which converges absolutely. Therefore, we can develop the main integral as

$$\begin{aligned}&\int_0^1\int_0^1 f(x)g(y)\operatorname{arctanh}(xy)dxdy\\&=\int_0^1\int_0^1 f(x)g(y)\sum_{n=0}^{\infty}\frac{x^{2n+1}y^{2n+1}}{2n+1}dxdy\\&=\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1\int_0^1 f(x)g(y)x^{2n+1}y^{2n+1}dxdy\\&=\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\int_0^1 f(x)x^{2n+1}dx\right)\left(\int_0^1 g(y)y^{2n+1}dy\right).\end{aligned}$$

By the Hölder integral inequality, for any $n \geq 0$, we get

$$\begin{aligned}&\int_0^1 f(x)x^{2n+1}dx \leq \|f\|_{L^p(0,1)}\left(\int_0^1 x^{(2n+1)q}dx\right)^{1/q}\\&=\|f\|_{L^p(0,1)}\frac{1}{((2n+1)q+1)^{1/q}}.\end{aligned}$$

Similarly, we obtain

$$\int_0^1 g(y)y^{2n+1}dy \leq \|g\|_{L^q(0,1)}\frac{1}{((2n+1)p+1)^{1/p}}.$$

Therefore, we have

$$\begin{aligned}&\int_0^1\int_0^1 f(x)g(y)\operatorname{arctanh}(xy)dxdy\\&\leq \|f\|_{L^p(0,1)}\|g\|_{L^q(0,1)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\left((2n+1)q+1\right)^{1/q}\left((2n+1)p+1\right)^{1/p}}\\&=B_p\|f\|_{L^p(0,1)}\|g\|_{L^q(0,1)}.\end{aligned}$$

This completes the proof. □

Note that we claim that B_p is a convergent series because, when $n \longrightarrow \infty$, we have the following equivalence:

$$\frac{1}{(2n+1)\left((2n+1)q+1\right)^{1/q}\left((2n+1)p+1\right)^{1/p}} \sim \frac{1}{4q^{1/q}p^{1/p}}\times\frac{1}{n^2},$$

which is the main term of a Riemann convergent series.

In the special case $p = q = 2$, the inequality becomes

$$\int_0^1\int_0^1 f(x)g(y)\operatorname{arctanh}(xy)dxdy \leq B_2\|f\|_{L^2(0,1)}\|g\|_{L^2(0,1)},$$

where

$$B_2=\sum_{n=0}^{\infty}\frac{1}{(2n+1)(4n+3)}=\frac{1}{4}\left(\pi-2\log(2)\right).$$

Therefore, we can write

$$\int_0^1\int_0^1 f(x)g(y)\operatorname{arctanh}(xy)dxdy \leq \frac{1}{4}\left(\pi-2\log(2)\right)\|f\|_{L^2(0,1)}\|g\|_{L^2(0,1)}.$$

To the best of our knowledge, this inequality, as well as the general case presented in Theorem 2.1, appears to be new to the literature.

3 A Hardy-Hilbert Type Integral Inequality Involving the Logarithmic Function

In the theorem below, we establish a Hardy-Hilbert-type integral inequality involving the logarithmic kernel function $-\ln(1-xy)$ on the unit square.

Theorem 3.1 *Let $p, q > 1$ with $1/p + 1/q = 1$, and $f, g : (0, 1) \to (0, \infty)$ such that $f \in L^p(0, 1)$ and $g \in L^q(0, 1)$. Then, we have*

$$\int_0^1 \int_0^1 f(x)g(y)\,(-\ln(1-xy))\,dxdy \le C_p \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)},$$

where

$$C_p = \sum_{n=1}^{\infty} \frac{1}{n(nq+1)^{1/q}(np+1)^{1/p}},$$

which is a convergent series.

Proof. For any $|z| < 1$, the logarithm function admits the convergent power series expansion

$$-\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Hence, for any $x, y \in (0, 1)$, we have

$$-\ln(1-xy) = \sum_{n=1}^{\infty} \frac{(xy)^n}{n},$$

which converges absolutely. Therefore, we obtain

$$\begin{aligned} &\int_0^1 \int_0^1 f(x)g(y)\,(-\ln(1-xy))\,dxdy \\ &= \int_0^1 \int_0^1 f(x)g(y) \sum_{n=1}^{\infty} \frac{(xy)^n}{n} dxdy \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \int_0^1 f(x)g(y)(xy)^n dxdy \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^1 f(x)x^n dx \right) \left(\int_0^1 g(y)y^n dy \right). \end{aligned}$$

By the Hölder integral inequality, for any $n \ge 1$, we have

$$\begin{aligned} &\int_0^1 f(x)x^n dx \le \|f\|_{L^p(0,1)} \left(\int_0^1 x^{nq} dx \right)^{1/q} \\ &= \|f\|_{L^p(0,1)} \frac{1}{(nq+1)^{1/q}}. \end{aligned}$$

Similarly, we get

$$\int_0^1 g(y)y^n dy \le \|g\|_{L^q(0,1)} \frac{1}{(np+1)^{1/p}}.$$

Substituting these upper bounds yields

$$\begin{aligned}&\int_0^1 \int_0^1 f(x)g(y)\left(-\ln(1-xy)\right)dxdy\\ &\leq \|f\|_{L^p(0,1)}\|g\|_{L^q(0,1)}\sum_{n=1}^{\infty}\frac{1}{n(nq+1)^{1/q}(np+1)^{1/p}}\\ &= C_p\|f\|_{L^p(0,1)}\|g\|_{L^q(0,1)}.\end{aligned}$$

This completes the proof. □

In the special case $p = q = 2$, the inequality becomes

$$\int_0^1 \int_0^1 f(x)g(y)\left(-\ln(1-xy)\right)dxdy \leq C_2\|f\|_{L^2(0,1)}\|g\|_{L^2(0,1)},$$

where

$$C_2 = \sum_{n=1}^{\infty}\frac{1}{n(2n+1)} = 2\left(1-\log(2)\right).$$

Therefore, we have

$$\int_0^1 \int_0^1 f(x)g(y)\left(-\ln(1-xy)\right)dxdy \leq 2\left(1-\log(2)\right)\|f\|_{L^2(0,1)}\|g\|_{L^2(0,1)}.$$

To the best of our knowledge, both this inequality and the more general case presented in Theorem 3.1 are new contributions to the literature.

4 A Hardy-Hilbert Type Integral Inequality Involving the Arcsine Function

In the theorem below, we establish a Hardy-Hilbert-type integral inequality involving the kernel function $\arcsin(xy)$ on the unit square.

Theorem 4.1 *Let $p, q > 1$ with $1/p + 1/q = 1$, and $f, g : (0, 1) \to (0, \infty)$ such that $f \in L^p(0, 1)$ and $g \in L^q(0, 1)$. Then, we have*

$$\int_0^1 \int_0^1 f(x)g(y)\arcsin(xy)dxdy \leq D_p\|f\|_{L^p(0,1)}\|g\|_{L^q(0,1)},$$

where

$$D_p = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)\left((2n+1)q+1\right)^{1/q}\left((2n+1)p+1\right)^{1/p}},$$

which is a convergent series.

Proof. For any $|z| < 1$, the arcsine function admits the convergent power series expansion

$$\arcsin(z) = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}z^{2n+1}.$$

Hence, for any $x, y \in (0, 1)$, we get

$$\arcsin(xy) = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}(xy)^{2n+1},$$

which converge absolutely. Therefore, we have

$$
\begin{aligned}
&= \int_0^1 \int_0^1 f(x)g(y) \arcsin(xy) dxdy \\
&= \int_0^1 \int_0^1 f(x)g(y) \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} (xy)^{2n+1} dxdy \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} \int_0^1 \int_0^1 f(x)g(y)(xy)^{2n+1} dxdy \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} \left(\int_0^1 f(x)x^{2n+1} dx \right) \left(\int_0^1 g(y)y^{2n+1} dy \right).
\end{aligned}
$$

By the Hölder integral inequality, for any $n \geq 0$, we obtain

$$
\begin{aligned}
&\int_0^1 f(x)x^{2n+1} dx \leq \|f\|_{L^p(0,1)} \left(\int_0^1 x^{(2n+1)q} dx \right)^{1/q} \\
&= \|f\|_{L^p(0,1)} \frac{1}{((2n+1)q+1)^{1/q}}.
\end{aligned}
$$

Similarly, we have

$$
\int_0^1 g(y)y^{2n+1} dy \leq \|g\|_{L^q(0,1)} \frac{1}{((2n+1)p+1)^{1/p}}.
$$

Substituting these upper bounds gives

$$
\begin{aligned}
&\int_0^1 \int_0^1 f(x)g(y) \arcsin(xy) dxdy \\
&\leq \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1) \left((2n+1)q+1\right)^{1/q} \left((2n+1)p+1\right)^{1/p}} \\
&= D_p \|f\|_{L^p(0,1)} \|g\|_{L^q(0,1)}.
\end{aligned}
$$

This completes the proof. □

In the special case $p = q = 2$, the inequality becomes

$$
\int_0^1 \int_0^1 f(x)g(y) \arcsin(xy) dxdy \leq D_2 \|f\|_{L^2(0,1)} \|g\|_{L^2(0,1)},
$$

where

$$
D_2 = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)(4n+3)} = \frac{\pi}{2} \left(1 - \frac{4\sqrt{2\pi}}{\Gamma^2(1/4)} \right)
$$

and $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the standard gamma function.

Therefore, we can write

$$
\int_0^1 \int_0^1 f(x)g(y) \arcsin(xy) dxdy \leq \frac{\pi}{2} \left(1 - \frac{4\sqrt{2\pi}}{\Gamma^2(1/4)} \right) \|f\|_{L^2(0,1)} \|g\|_{L^2(0,1)}.
$$

To the best of our knowledge, both this inequality and the more general result stated in Theorem 4.1 are new contributions to the literature.

5 Conclusion

In conclusion, we have derived three new Hardy-Hilbert-type integral inequalities on the unit square involving new kernel functions. These inequalities provide sharp constant factors and explicit upper bounds in terms of unweighted integral norms of the main functions. Further research could extend this framework to more general domains, weighted settings or multidimensional kernel functions, potentially revealing additional optimal inequalities and applications.

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