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Weakly Semi-potent Rings

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Abstract
In this paper we study the notion of weakly semi-potent rings. Where, a ring R is weakly semi-potent, if aR or $(1 - a)R$ contains a non-zero idempotent for every $a \in R$. We provide several characterizations of weakly semi-potent rings and study their properties. We have shown that the endomorphisms ring of locally projective (locally injective) module is weakly semi-potent. Moreover, we proved that a ring R is local if and only if R is weakly semi-potent and $0, 1$ are the only idempotents in R. Also, we investigate the connections between weakly semi-potent rings and clean (r-clean, f-clean) rings. Further, we investigate the structure of a right $R-$module M when $End_R(M)$ is a weakly semi-potent ring. Finally, we give some of the necessary and sufficient conditions to be the ring R weakly semi-potent, by free modules and endomorphisms ring of free modules over R.

Keywords: Clean ring, $r-$clean ring, $f-$clean ring, weakly semi-potent ring, $\boldsymbol{pi}$ element, I_0-ring, Regular ring.

1 Introduction

Our main concern in this article is about the elements of a ring R, which are divisors of an idempotent. This concept introduced by Nicholson in [12] and Hamza in [5]. Recall that a non-zero element a of a ring R is *partially invertible* in R or $\boldsymbol{pi}$ for short, if $b = bab$ for some a non-zero element $b \in R$, [6]. Also, recall that a ring R is an I_0-ring, if every element $a \in R$, $a \notin J(R)$ is $\boldsymbol{pi}$, [5]. Hamza in [5], proved that the endomorphism ring S of a regular module is an I_0-ring with $J(S) = 0$. In section 2, we provide some characterizations of $\boldsymbol{pi}$ elements and investigate their properties. We prove that, if a product of elements in a ring R is $\boldsymbol{pi}$, then one of its factors is $\boldsymbol{pi}$. In section 3, we study weakly semi-potent rings, where a ring R is weakly semi-potent if for every $a \in R$, either a or $1 - a$ is $\boldsymbol{pi}$. We show that a ring R is local if and only if R is weakly semi-potent and $0, 1$ are the only idempotents in R. Furthermore, we proved that the ring of integers modulo $2p$ for every prime p, Z_{2p} is weakly semi-potent. Also, we investigate the connections between weakly semi-potent rings and clean ($r-$clean, $f-$clean) rings. We proved that, if idempotent elements in a ring R are $0, 1$ only, then a ring R is weakly semi-potent if and only if R is a clean (exchange) ring. In section 4, we study the matrices ring over a weakly semi-potent ring, where we proved that the ring of all 2×2 diagonal (upper triangular) matrices over a ring R is weakly semi-potent if and only if R is weakly semi-potent. Further, we show that the ring of all $n \times n$ matrices over a weakly semi-potent ring R is weakly semi-potent, too. In section 5, we study the endomorphisms ring of modules. It is shown that the endomorphism ring $S = End_R(M)$ of a module M is weakly semi-potent, if for every $\alpha \in S$, either $Im(\alpha)$ or $Im(1 - \alpha)$ is a non-zero direct summand of M. Also, we proved that, a ring R is weakly semi-potent if and only if the endomorphism ring of every finitely generated

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free module F over R is weakly semi-potent. Also, we proved that, if P is a projective module, then $End_R(P)$ is weakly semi-potent if and only if, for every two submodules A, B of P such that $P = A + B$, then either A or B contains a non-zero direct summand of P. Finally, it is shown that the endomorphisms ring of locally projective (locally injective) module is weakly semi-potent.

Throughout this paper all rings R are associative with identity and all modules M over a ring R are unitary right modules, $S = End_R(M)$ will denote the endomorphism ring of a module M, the *radical* of M, denoted $J(M)$. For a ring R, we write $J(R)$ for the *Jacobson radical* of R. Let M be a module and N be a submodule of M. We say N is *small* in M if whenever K is a submodule of M with $M = K + N$, then $K = M$. Dually, N is *large* in M, if $N \cap K = 0$ always implies $K = 0$, [17].

2 Partially Invertible Elements

In this section we first give some properties of the elements a of a ring R, when a is a divisor of an idempotent of R. We start with the following:

Lemma 2.1. *Let R be a ring. For every $a \in R$ the following are equivalent:*
(1) *There exists $b \in R$ such that $ba \in R$ is a nonzero idempotent.*
(2) *There exists $d \in R$ such that $ad \in R$ is a nonzero idempotent.*
(3) *There exists $u \in R$, $u \neq 0$ such that $u = uau$.*
(4) *There exist direct summands $A \neq 0$, $B \neq 0$ in R_R such that the mapping*

$$A \ni x \mapsto ax \in B$$

is an isomorphism.

Proof. (1) $\Rightarrow$ (3). Suppose that $ba \in R$ is a nonzero idempotent. Let $u = bab \in R$, then $uau = u$ and $u \neq 0$.
(3) $\Rightarrow$ (1). Suppose that $u = uau$ for some $u \in R$, $u \neq 0$, then $ua \in R$ is a nonzero idempotent. Similarly we can prove that (2) $\Leftrightarrow$ (3).
(3) $\Rightarrow$ (4). Suppose that $u = uau$ for some $u \in R$, $u \neq 0$. Then $A = uaR \neq 0$ and $B = auR \neq 0$ are direct summands in R_R. It is clear that the mapping $A \ni x \mapsto ax \in B$ is an isomorphism. (4) $\Rightarrow$ (2) Obvious. □

Definition 2.2. *Let R be a ring. We say that an element $a \in R$ is* partially invertible *in R or* **pi** *for short, if a satisfies the equivalently conditions of Lemma 2.1. See also [5].*

Let R be a ring and $x \in R$. Recall that an element x is *regular* if $x = xyx$ for some $y \in R$. The ring R is *regular* if each of its element is regular, [4]. Also, an element x is called *full* if there exists $s, t \in R$ such that $sxt = 1$, [10].

Lemma 2.3. *For every ring R the following statements hold:*
(1) *The elements $1, -1$ are **pi**.*
(2) *Every element of R which has a right (left) inverse is **pi**.*
(3) *Every element in R which has an inverse is **pi**.*
(4) *If $a \in R$ is **pi**, then $-a \in R$ is **pi**.*
(5) *Every non-zero idempotent element in R is **pi**.*
(6) *Every non-zero regular element in R is **pi**.*
(7) *Every full element in R is **pi**.*

Proof. (1), (2), (3), (4) and (5) are clear. (6) Let $a \in R$ be a non-zero regular element, then $a = aba$ for some $b \in R$. So for $x = bab$ we have $x = xax$ and $0 \neq x \in R$. (7) Let $a \in R$ is a full element, then $sat = 1$ for some $s, t \in R$, so $(ts)a(ts) = ts$ and $0 \neq ts \in R$. □

Lemma 2.4. *Let $R = R_1 \times \cdots \times R_n$ be a ring direct product and $a = (a_1, \cdots a_n) \in R$. Then a is* ***pi*** *in R if and only if there exist i such that a_i is* ***pi*** *in R_i.*

Proof. Suppose that $a = (a_1, \cdots a_n) \in R$ is ***pi***. Then there exist $x = (x_1, \cdots x_n) \in R$, $x \neq 0$ such that $x = xax$. Since $x \neq 0$ there exist i such that $x_i \neq 0$ and $x_i = x_i a_i x_i$. Hence at least one of the a_i is ***pi***. Conversely, if $a_i \in R_i$ is ***pi***, then there exist $y_i \in R_i$ such that $y_i = y_i a_i y_i$ and $y_i \neq 0$. So $y = (0, \cdots, y_i, \cdots, 0) \in R$ such that $y = yay$ and $y \neq 0$. This shows that a is ***pi***. □

From Lemma 2.4 we can obtain the following:

Corollary 2.5. *Let $R = R_1 \times \cdots \times R_n$ be a ring direct product and $a = (a_1, \cdots a_n) \in R$. Then a is not* ***pi*** *in R if and only if a_i is not* ***pi*** *in R_i for all i.*

Lemma 2.6. *Let R be a ring and $a, b, c \in R$. If the product bac is* ***pi*** *in R, then a is* ***pi*** *in R.*

Proof. Suppose that bac is ***pi*** in R, then $x(bac)x = x$ for some $x \in R$, $x \neq 0$. So $(cxb)a(cxb) = cxb$ where $cxb \in R$ is a non-zero element. Thus, a is ***pi***. □

3 Weakly Semi-potent Rings

Definition 3.1. *We say that a ring R is* weakly semi-potent*, if for every element $a \in R$, either a or $1 - a$ is a* **pi** *element in R.*

Lemma 3.2. *For every ring R the following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *For every $a \in R$, either aR or $(1-a)R$ contains a non-zero idempotent.*
(3) *For every $a \in R$, either Ra or $R(1-a)$ contains a non-zero idempotent.*
(4) *For every $a \in R$, either aR or $R(1-a)$ contains a non-zero idempotent.*
(5) *For every $a \in R$, either Ra or $(1-a)R$ contains a non-zero idempotent.*

Proof. (1) ⇒ (2) Suppose that R is a weakly semi-potent ring. Let $a \in R$; if a is ***pi***, then $b = bab$ for some $b \in R$, $b \neq 0$. So $ab \in R$ is a non-zero idempotent and $ab \in aR$. Like previous part, if $1-a$ is ***pi***, then $(1-a)R$ has a non-zero idempotent. (2) ⇒ (1) Obvious. Similarly, we can prove that (1) ⇔ (3) ⇔ (4) ⇔ (5). □

Lemma 3.3. *For every ring R the following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *For every $a \in R$, either aR or $(1-a)R$ contains a non-zero regular element.*
(3) *For every $a \in R$, either Ra or $R(1-a)$ contains a non-zero regular element.*
(4) *For every $a \in R$, either aR or $R(1-a)$ contains a non-zero regular element.*
(5) *For every $a \in R$, either Ra or $(1-a)R$ contains a non-zero regular element.*

Proof. (1) ⇒ (2) It is clear by Lemma 3.2, hence every idempotent element is regular. (2) ⇒ (1) Let $a \in R$, by assumption either aR or $(1-a)R$ contains a non-zero regular element $b \in R$, so $b = byb$ for some $y \in R$. If $b \in aR$, then $b = ax$ for some $x \in R$, so $e = by = axy \in aR$ is an idempotent and $e \neq 0$. Like previous part, if $b \in (1-a)R$, then $(1-a)R$ has a non-zero idempotent. By Lemma 3.2, a ring R is weakly semi-potent. Similarly, we can prove that (1) ⇔ (3) ⇔ (4) ⇔ (5). □

Lemma 3.4. *For every ring R the following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *For every two right (left) ideals A, B of R such that $R = A + B$, then either A or B contains a non-zero idempotent of R.*

Proof. (1) $\Rightarrow$ (2) Let A, B two right ideals of R such that $R = A + B$, then $1 = a + b$ for some $a \in A$, $b \in B$. By Lemma 3.2, there exists an idempotent $0 \neq e \in R$ such that either $e \in aR \subseteq A$ or $e \in (1 - a)R = bR \subseteq B$. (2) $\Rightarrow$ (1) It is clear, hence $R = aR + (1 - a)R$ for every $a \in R$. □

Proposition 3.5. *The ring of integers modulo* $2p$, Z_{2p} *is weakly semi-potent, for every prime* p.

Proof. Let $a \in Z_{2p}$. If a is oven, $a = 2m$ for some $m \in Z$. Since $2m < 2p$, $gcd(m, p) = 1$, therefore $1 = mx + py$ for some $x, y \in Z$. Thus, $2 \equiv ax(modulo - 2p)$, it follows that $(1 + p)Z_{2p} = 2Z_{2p} = aZ_{2p}$. Similarly, if a is odd, we have $pZ_{2p} = aZ_{2p}$. Since p and $1 + p$ are idempotents in Z_{2p}, implies that Z_{2p} is weakly semi-potent. □

Lemma 3.6. *Let* R *be a ring and* $I \subseteq J(R)$ *be an ideal of* R. *If* R *is weakly semi-potent, then* R/I *is a weakly semi-potent ring.*

Proof. Let R be a weakly semi-potent ring. Also, let $\overline{a} = a + I \in R/I$, by Lemma 3.2, there exists idempotent $0 \neq e \in R$, such that either $e \in aR$ or $e \in (1 - a)R$. Therefore, either $\overline{e} \in \overline{a}\overline{R}$ or $\overline{e} \in (\overline{1} - \overline{a})\overline{R}$, where $\overline{e} \in R/I$ is an idempotent, it follows that R/I is weakly semi-potent. □

Theorem 3.7. *Let* R *be a ring and* $e \in R$ *is a non-zero idempotent. The following hold:*
(1) *If* R *is weakly semi-potent and* e *is central, then* eR *is a weakly semi-potent ring.*
(2) *If* eRe *is weakly semi-potent, then* R *is weakly semi-potent.*
(3) *If* e *is central, then* R *is weakly semi-potent if and only if* eR *is a weakly semi-potent ring.*

Proof. (1) Let $a \in eR$. Since $a \in R$, we have either a or $1 - a$ is $\boldsymbol{pi}$ in R. If a is $\boldsymbol{pi}$ in R, then $b = bab$ for some $b \in R$, $b \neq 0$. Therefor, $(eb)a(eb) = eb \in eR$ and $eb \neq 0$, so $a \in eR$ is $\boldsymbol{pi}$. If $1 - a \in R$ is $\boldsymbol{pi}$, then $d(1 - a)d = d$ for some $d \in R$, $d \neq 0$. Therefore, $ed(e - a)ed = ed \in eR$ and $ed \neq 0$, so $e - a \in eR$ is $\boldsymbol{pi}$. It follows that eR is weakly semi-potent.
(2) Assume that eRe is weakly semi-potent. Let $a \in R$, then $eae \in eRe$. By assumption, either eae or $e - eae$ is $\boldsymbol{pi}$ in eRe. If eae is $\boldsymbol{pi}$, then $u(eae)u = u$ for some $u \in eRe$, $u \neq 0$. Thus, $uau = u$ where $u \in R$, $u \neq 0$, so a is $\boldsymbol{pi}$ in R. If $e - eae$ is $\boldsymbol{pi}$ in eRe, then $v(e - eae)v = v$ for some $v \in eRe$, $v \neq 0$. Thus, $v(1 - a)v = v$ where $v \in R$, $v \neq 0$, so $1 - a$ is $\boldsymbol{pi}$ in R. Therefore, R is a weakly semi-potent ring.
(3) It is clear by (1) and (2). □

Recall that a ring R is *local* if, for every element $a \in R$ either a or $1 - a$ has an inverse, [9]. It is clear that every local ring is weakly semi-potent. Furthermore, we have the following:

Theorem 3.8. *For every ring* R *the following statements are equivalent:*
(1) *A ring* R *is weakly semi-potent and* $0, 1$ *are the only idempotents in* R.
(2) *A ring* R *is local.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. If a is a $\boldsymbol{pi}$ element, then $b = bab$ for some $b \in R$, $b \neq 0$. Since $ab \in R$ is a non-zero idempotent, by assumption $ab = 1$ and also $ba = 1$. Thus a has an inverse in R. Similarly, if $1 - a$ is a $\boldsymbol{pi}$ element, then $1 - a$ has an inverse in R. This shows that R is a local ring. (2) $\Rightarrow$ (1) Obvious. □

Recall that an element a of a ring R is *clean* if it is the sum of an idempotent and a unit. A ring R is called *clean* if each of its element is clean, [14]. We now investigate connections between clean ring and weakly semi-potent ring.

Theorem 3.9. *Let* R *be a ring such that* $0, 1$ *are the only idempotents in* R. *Then following statements are equivalent:*
(1) *A ring* R *is weakly semi-potent.*
(2) *A ring* R *is clean.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$. If a is a ***pi*** element, then $b = bab$ for some $b \in R$, $b \neq 0$. Since ab is a non-zero idempotent, by assumption $ab = 1$. Thus, an element a has an inverse and so a is clean. If $1 - a$ is a ***pi*** element, then $d(1-a)d = d$ for some $d \in R$, $d \neq 0$. Since $d(1-a)$ is a non-zero idempotent, by assumption $d(1-a) = 1$. Thus, an element $1 - a$ has an inverse and so $a - 1$ has an inverse. Since $a = (a-1)+1$, a is clean. (2) $\Rightarrow$ (1). Suppose R is clean. Let $x \in R$, then $x = a + e$, where $a \in R$ has an inverse and $e \in R$ is an idempotent. By assumption, if $e = 0$, then $x = a$ has an inverse, so $x = a$ is a ***pi*** element. If $e = 1$, then $a = x - 1$ has an inverse and so $-a = 1 - x$ has an inverse by Lemma 2.3, $1 - x$ is a ***pi*** element. □

Recall that a ring R is an *I_0-ring*, [5] (or *semi-potent*, [6]) if every element $a \in R$, $a \notin J(R)$ there exists a non-zero element $x \in R$ such that $x = xax$. Also, recall that a ring R is *exchange* if, for each $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $(1-e) \in (1-a)R$, [13]. It is clear that every exchange ring is an I_0-ring, and there are I_0-ring which are not exchange, (see [16, Proposition 2.13]). The next result shows that weakly semi-potent rings can also be characterized in terms of I_0-rings and exchange rings.

Theorem 3.10. *Let R be a ring such that $0, 1$ are the only idempotents in R. Then following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *A ring R is an I_0-ring.*
(3) *A ring R is exchange.*

Proof. (1) $\Leftrightarrow$ (3). By Theorem 3.9, a ring R is weakly semi-potent if and only if R is clean and by [13 ,Proposition 1.8] a ring R is clean if and only if R is exchange. (2) $\Rightarrow$ (3). Suppose that R is an I_0-ring. Let $a \in R$. If $a \notin J(R)$, then aR contains a non-zero idempotent, i.e, $1 \in aR$ and $0 \in (1-a)R$. If $a \in J(R)$, then $0 \in aR$ and $1 \in (1-a)R$, because $1 - a$ has an inverse. Therefore R is exchange. (3) $\Rightarrow$ (2). Obvious. □

Let R be a ring and $x \in R$. Recall that an element x is *$f-$clean* if $x = a + e$, where $a \in R$ is a full element and $e \in R$ is idempotent. The ring R is *$f-$clean* if each of its element is $f-$clean, [10]. Now, we show that if $0, 1$ are only idempotents in a ring R, then R is weakly semi-potent if and only if R is $f-$clean.

Theorem 3.11. *Let R be a ring such that $0, 1$ are the only idempotents in R. Then following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *A ring R is $f-$clean.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$. If a is ***pi***, then $bab = b$ for some $b \in R$, $b \neq 0$. Since ab is a non-zero idempotent in R, by assumption $ab = 1$. This shows that a is a full element and so a is $f-$clean, hence $a = a + 0$. If $1 - a$ is ***pi***, then by Lemma 2.3, $a - 1$ is ***pi***, so $d(a-1)d = d$ for some $d \in R$, $d \neq 0$. Since $(a-1)d$ is a non-zero idempotent in R, by assumption $(a-1)d = 1$. This shows that $a - 1$ is a full element and so $a = (a-1)+1$ is $f-$clean. (2) $\Rightarrow$ (1). Let $x \in R$, then $x = a + e$, where $a \in R$ is a full element and $e \in R$ is idempotent. By assumption either $e = 0$ or $e = 1$. If $e = 0$, then $x = a$ is full, by Lemma 2.3, $x = a$ is ***pi***. If $e = 1$, then $a = x - 1$ is a full element by Lemma 2.3, $a = x - 1$ is ***pi*** so $-a = 1 - x$ is ***pi***. Thus R is weakly semi-potent. □

Let R be a ring and $x \in R$. Recall that an element x is *$r-$clean* if $x = a + e$, where $a \in R$ is a regular element and $e \in R$ is idempotent. The ring R is *$r-$clean* if each of its element is $r-$clean, [1], see also [7]. We conclude this section with the following characterization.

Theorem 3.12. *Let R be a ring such that $0, 1$ are the only idempotents in R. Then following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *A ring R is $r-$clean.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$. If a is ***pi***, then $bab = b$ for some $b \in R$, $b \neq 0$. Since ab is a non-zero idempotent in R, by assumption $ab = 1$, so $aba = a$. This shows that a is a regular element and so $a = a + 0$ is $r-$clean. If $1 - a$ is ***pi***, then by Lemma 2.3 $a - 1$ is ***pi***, so $d(a-1)d = d$ for some $d \in R$, $d \neq 0$. Since $(a-1)d$ is a non-zero idempotent in R, by assumption $(a-1)d = 1$ and so $(a-1)d(a-1) = a - 1$. This shows that $a - 1$ is a regular

element, thus $a = (a-1)+1$ is $r-$clean.
(2) $\Rightarrow$ (1). Let $x \in R$, then $x = a+e$, where $a \in R$ is a regular element and $e \in R$ is idempotent. By assumption either $e = 0$ or $e = 1$. If $e = 0$, then $x = a$ is regular, by Lemma 2.3 $x = a$ is $\boldsymbol{pi}$. If $e = 1$, then $a = x-1$ is a regular element by Lemma 2.3 $a = x-1$ is $\boldsymbol{pi}$ so $-a = 1-x$ is $\boldsymbol{pi}$. Thus R is weakly semi-potent. □

4 Weakly Semi-potent Matrices Rings

In this section, we study the matrices rings over a weakly semi-potent ring. Let R be a ring and $D_2(R)$, $T_2(R)$ be the ring of all 2×2 diagonal, upper triangular matrices over R, respectively. Firstly, we investigate connections between $\boldsymbol{pi}$ elements in a ring R and $\boldsymbol{pi}$ elements in matrices rings $D_2(R)$ and $T_2(R)$.

Lemma 4.1. *Let R be a ring. For every a non-zero element $a \in R$, the following hold:*
(1) *If a is $\boldsymbol{pi}$ in R, then $\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ is $\boldsymbol{pi}$ in $D_2(R)$ for every $x \in R$.*
(2) *If a is $\boldsymbol{pi}$ in R, then $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is $\boldsymbol{pi}$ in $D_2(R)$ for every $x \in R$.*
(3) *If a is $\boldsymbol{pi}$ in R, then $\alpha = \begin{bmatrix} a & x \\ 0 & y \end{bmatrix}$ is $\boldsymbol{pi}$ in $T_2(R)$ for every $x, y \in R$.*
(4) *If a is $\boldsymbol{pi}$ in R, then $\alpha = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $\boldsymbol{pi}$ in $T_2(R)$ for every $x, y \in R$.*

Proof. (1) Suppose that a is $\boldsymbol{pi}$ in R, then $b = bab$ for some $b \in R$, $b \neq 0$. Let $\beta = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$, then $\beta \in D_2(R)$, $\beta \neq 0$ such that $\beta\alpha\beta = \beta$, so α is $\boldsymbol{pi}$ in $D_2(R)$. Similarly, we can prove (2), (3) and (4). □

Proposition 4.2. *For every ring R, the following statements are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *A ring $D_2(R)$ is weakly semi-potent.*
(3) *A ring $T_2(R)$ is weakly semi-potent.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is weakly semi-potent. Let $\alpha = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in D_2(R)$. Since $a \in R$, either a or $1-a$ is $\boldsymbol{pi}$ in R, so by Lemma 4.1, either α or $1-\alpha$ is $\boldsymbol{pi}$ in $D_2(R)$. Thus, the ring $D_2(R)$ is weakly semi-potent. Conversely, assume that $D_2(R)$ is weakly semi-potent and let $a \in R$, then $w = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in D_2(R)$, by assumption either w or $1-w$ is $\boldsymbol{pi}$ in $D_2(R)$. If w is $\boldsymbol{pi}$, then there exists $v = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in D_2(R)$, $v \neq 0$ such that $vwv = v$, so $x \in R$ and $x \neq 0$ such that $x = xax$, so a is $\boldsymbol{pi}$ in R. If $1-w$ is $\boldsymbol{pi}$ in $D_2(R)$, similarly, we found that $1-a$ is $\boldsymbol{pi}$ in R. So R is weakly semi-potent. (1) $\Rightarrow$ (3) Let $\alpha = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T_2(R)$. Since $a \in R$, either a or $1-a$ is $\boldsymbol{pi}$ in R, so by Lemma 4.1, either α or $1-\alpha$ is $\boldsymbol{pi}$ in $T_2(R)$. Thus, the ring $T_2(R)$ is weakly semi-potent. Conversely, assume that $T_2(R)$ is weakly semi-potent. Let $a \in R$, then $w = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in T_2(R)$, by assumption either w or $1-w$ is $\boldsymbol{pi}$ in $T_2(R)$. If w is $\boldsymbol{pi}$, then there exists $\beta = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in T_2(R)$, $\beta \neq 0$ such that $\beta w\beta = \beta$. So $\begin{bmatrix} xax & xay+yaz \\ 0 & zaz \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, therefore $xax = x$ and $zaz = z$. Since $\beta \neq 0$, either $x \neq 0$ or $z \neq 0$ which shows that a is $\boldsymbol{pi}$. If $1-w$ is $\boldsymbol{pi}$ in $T_2(R)$, similarly, we found that $1-a$ is $\boldsymbol{pi}$ in R. Thus, R is weakly semi-potent. □

Proposition 4.3. *If R is a weakly semi-potent ring, then the ring of all $n \times n$ matrices $M_n(R)$ over R, is weakly semi-potent, too.*

Proof. Suppose that R is weakly semi-potent. Let α_{11} be a $n \times n$ matrix with the entry $(1, 1)$ is 1, otherwise 0. Since $\alpha \cdot M_n(R) \cdot \alpha \cong R$, for every $\beta \in M_n(R)$, $\alpha \cdot \beta \cdot \alpha$ is isomorphic some $x \in R$. Since either x or $1 - x$ is ***pi***, by Lemma 2.6 either β or $1 - \beta$ is ***pi***, so $M_n(R)$ is weakly semi-potent. □

5 Weakly Semi-potent Endomorphism Rings

In this section, we study the weakly semi-potent endomorphisms ring of module. Further, we characterize free modules over a weakly semi-potent ring.

Lemma 5.1. *Let M be an $R-$module and $S = End_R(M)$. The following hold:*
(1) If for every $\alpha \in S$, either $Im(\alpha)$ or $Im(1 - \alpha)$ is a non-zero direct summand of M, then S is weakly semi-potent.
(2) If for every $\alpha \in S$, either $Ker(\alpha)$ or $Ker(1 - \alpha)$ is a non-zero direct summand of M, then S is weakly semi-potent.
(3) If for every $\alpha \in S$, either $Im(\alpha)$ or $Ker(1 - \alpha)$ is a non-zero direct summand of M, then S is weakly semi-potent.
(4) If for every $\alpha \in S$, either $Im(1 - \alpha)$ or $Ker(\alpha)$ is a non-zero direct summand of M, then S is weakly semi-potent.

Proof. (1) Let $\alpha \in S$. If $\alpha = 1$, then $S\alpha = S$ contains a non-zero idempotent. If $\alpha = 0$, then $S(1 - \alpha) = S$ contains a non-zero idempotent. Suppose that $\alpha \neq 1$, $\alpha \neq 0$. If $Im(\alpha)$ is a direct summand of M, then $Im(\alpha) = Ker(e)$ for some non-zero idempotent $e \in S$, so $e\alpha = 0$. Since $1 = \alpha + (1-\alpha)$, $e = e(1-\alpha) \in S(1-\alpha)$. If $Im(1 - \alpha)$ is a direct summand of M, then $Im(1 - \alpha) = Ker(g)$ for some non-zero idempotent $g \in S$, so $g(1 - \alpha) = 0$. Since $1 = \alpha + (1 - \alpha)$, $g = g\alpha \in S\alpha$, by Lemma 3.2, S is weakly semi-potent. Similarly, we can prove (2), (3) and (4). □

From Lemma 3.2 and Theorem 3.7 we can derive the following:

Corollary 5.2. *Let M_R be an indecomposable module and $S = End_R(M)$. Then S is local if and only if S is a weakly semi-potent ring.*

Lemma 5.3. *Let F_R be a finitely generated free module and $S = End_R(F)$. If R is a weakly semi-potent ring, then S is weakly semi-potent.*

Proof. Suppose that R is a weakly semi-potent ring. Let F be a finitely generated free module over R, then $F \cong R^n$. Since $M_n(R) \cong End_R(R^n) \cong S$, by Proposition 4.3, S is a weakly semi-potent ring. □

Theorem 5.4. *Let R be a weakly semi-potent ring and F_R be a free module with basis $\Gamma = \{x_i\}_{i \in I}$. Then, for every $a \in F$, $a \neq 0$, either aR or $(a - x_i)R$ contains a non-zero direct summand of F for some $x_i \in \Gamma$.*

Proof. First we prove our hypothesis for any finitely generated free module over R. Let F_R be a finitely generated free module with basis $\Gamma = \{x_1, x_2, \cdots, x_n\}$. Let $a \in F$, $a \neq 0$, then $a = x_1r_1 + x_2r_2 + \cdots + x_nr_n$, where $r_i \in R$. Without loss generality, we may assume that $r_1 \neq 0$. Then there exists a non-zero idempotent $e \in R$ such that either $e \in r_1R$ or $e \in (1 - r_1)R$. If $e \in r_1R$, then $e = r_1d$ for some $d \in R$, $d \neq 0$, then $ade = x_1e + x_2r_2de + \cdots + x_nr_nde$. We can easily see that $F = adeR \oplus (x_1(1-e)R \oplus x_2R \oplus \cdots \oplus x_nR)$. So $0 \neq adeR$ is a direct summand of F and $adeR \subseteq aR$. If $e \in (1 - r_1)R$, similarly, we can prove that $(a - x_1)deR$ a non-zero direct summand of F and $(a - x_1)deR \subseteq (a - x_1)R$. Second we prove our hypothesis for any free module over R. Let G_R be a free module with basis $\{x_i\}_{i \in \Lambda}$ and $a \in G$, $a \neq 0$. We put $a = x_{i1}r_{i1} + x_{i2}r_{i2} + \cdots + x_{in}r_{in}$, where $r_{ij} \in R$ and $G_n = x_{i1}R \oplus x_{i2}R \oplus \cdots \oplus x_{in}R$. Since $0 \neq aR \subseteq G_n$, there exists a non-zero direct summand N of G_n such that either $N \subseteq aR$ or $N \subseteq (a - x_{ij})R$ by first case. Also, since G_n is a direct summand of G, N is a direct summand of G. □

In view of Lemma 5.3 and Theorem 5.4, we can obtain the next result:

Theorem 5.5. *For any ring R the following are equivalent:*
(1) *A ring R is weakly semi-potent.*
(2) *A ring $End_R(F)$ is weakly semi-potent, for every finitely generated free module F over R.*
(3) *A ring $End_R(F)$ is weakly semi-potent, for every free module F over R.*
(4) *For every free module F with basis $\Gamma = \{x_i\}_{i\in I}$ has a property, for every $a \in F$, $a \neq 0$, either aR or $(a - x_i)R$ contains a non-zero direct summand of F for some $x_i \in \Gamma$.*

Proof. (1) ⇒ (2) By Lemma 5.3. (2) ⇒ (3) Let F be a free module with basis $\{x_i\}_{i\in I}$ and let $a \in F$. Then $a = x_{i1}r_{i1} + x_{i2}r_{i2} + \cdots + x_{in}r_{in}$, where $r_{ij} \in R$. We put $G = x_{i1}R \oplus x_{i2}R \oplus \cdots \oplus x_{in}R$, then G is a finitely generated free module, by assumption $End_R(G)$ is weakly semi-potent. Since $F = G \oplus K$ for some submodule K of F, let $e : F \to G$ be the projection and $S = End_R(F)$. Then $End_R(G) \cong eSe$, so a ring eSe is weakly semi-potent, by Theorem 3.7 a ring S is weakly semi-potent. (3) ⇒ (1) It is clear. (1) ⇒ (4) By Theorem 5.4. (4) ⇒ (1) Obvious. □

Lemma 5.6. *Let P_R be a projective module and $S = End_R(P)$. Then the following statements are equivalent:*
(1) *S is a weakly semi-potent ring.*
(2) *For every $\alpha \in S$, either $Im(\alpha)$ or $Im(1 - \alpha)$ contains a non-zero direct summand of P.*
(3) *If A, B are two submodules of P such that $P = A + B$, then either A or B contains a non-zero direct summand of P.*

Proof. (1) ⇒ (2) Is obvious by Lemma 3.2. (2) ⇒ (3) Let A, B be two submodules of P such that $P = A+B$, then by [2, Lemma 2.2] there exists $\alpha, \beta \in S$ such that $1 = \alpha + \beta$ and $Im(\alpha) \subseteq A$, $Im(\beta) \subseteq B$. Thus by assumption, either A or B contains a non-zero direct summand of P, hence $\beta = 1 - \alpha$. (3) ⇒ (1) Let $\alpha \in S$. Since $P = Im(\alpha) + Im(1 - \alpha)$, either $Im(\alpha)$ or $Im(1 - \alpha)$ contains a non-zero direct summand of P. If $Im(\alpha)$ contains a non-zero direct summand of P, then $Im(e) \subseteq Im(\alpha)$ for some idempotent $e \in S$, $e \neq 0$. Since P is projective, $e \in eS \subseteq \alpha S$. Similarly, we can prove that if $Im(1 - \alpha)$ contains a non-zero direct summand of P, then $(1 - \alpha)S$ contains a non-zero idempotent of S. Therefore, by Lemma 3.2, S is weakly semi-potent. □

Lemma 5.7. *Let Q_R be an injective module and $S = End_R(Q)$. Then the following statements are equivalent:*
(1) *S is a weakly semi-potent ring.*
(2) *For every $\alpha \in S$, either $Ker(\alpha)$ or $Ker(1 - \alpha)$ contained in a direct summand $K \neq Q$ of Q.*
(3) *For every two submodules A, B of Q such that $A \cap B = 0$, either A or B contained in a direct summand $K \neq Q$ of Q.*

Proof. (1) ⇒ (2) Let $\alpha \in S$. By Lemma 3.2 there exists a nonzero idempotent $e \in S$ such that, either $e \in S\alpha$ or $e \in S(1 - \alpha)$. If $e \in S\alpha$, then $Ker(\alpha) \subseteq Ker(e)$ and $Ker(e) \neq Q$ is a direct summand of Q, hence $e \neq 0$. Similarly, we can prove that if $e \in S(1 - \alpha)$, then $Ker(1 - \alpha)$ contained in a direct summand $K \neq Q$ of Q. (2) ⇒ (3) It is clear, hence Q is injective. (3) ⇒ (1) Let $\alpha \in S$. Since $Ker(\alpha) \cap Ker(1-\alpha) = 0$, by assumption there exists a non-zero idempotent $e \in S$ such that either $Ker(\alpha) \subseteq Ker(e)$ or $Ker(1 - \alpha) \subseteq Ker(e)$. Since Q is injective, either $e \in S\alpha$ or $e \in S(1 - \alpha)$. So, S is weakly semi-potent. □

Lemma 5.8. [17, *Lemma* 3.1] *Let M_R be a module and $f \in S = End_R(M)$. The following statements are equivalent:*
(1) *There exists $g \in S$ such that $f = fgf$.*
(2) *$Im(f)$ and $Ker(f)$ are direct summands of M.*

Following Kasch [8], a module V is called *locally projective* if, for every submodule B of V that is not small in V, B contains a non-zero projective direct summand of V. Dually, a module W is called *locally injective* if, for every submodule B of W that is not large in W, there exists an injective submodule $0 \neq Q \subseteq W$ with $B \cap Q = 0$. If V is locally projective, then $J(V)$ is small in V, see [8].

Theorem 5.9. *The following hold for every module $M_R \neq 0$:*
(1) *If M is locally projective, then $S = End_R(M)$ is a weakly semi-potent ring.*
(2) *If M is locally injective, then $S = End_R(M)$ is a weakly semi-potent ring.*

Proof. (1) Suppose that M is locally projective. Let $\alpha \in S$, then $M = Im(\alpha) + Im(1-\alpha)$. Since $M \neq 0$, M is not small in M, so either $Im(\alpha)$ or $Im(1-\alpha)$ is not small in M. If $Im(\alpha)$ is not small in M, then there exists a non-zero projective direct summand P of M such that $P \subseteq Im(\alpha)$. Let $\pi : M \to P$ be the projection onto P, then $\pi\alpha : M \to P$ is an epimorphism. Since P is projective, $\pi\alpha$ splits, i.e. $Ker(\pi\alpha)$ is a direct summand of M. By Lemma 5.8, there exists $\beta \in S$ such that $\pi\alpha = (\pi\alpha)\beta(\pi\alpha)$. Let $e = \beta(\pi\alpha)$, then $e \in S\alpha$ is a non-zero idempotent. Similarly, we can prove that if, $Im(1-\alpha)$ is not small in M, then $S(1-\alpha)$ contains a non-zero idempotent. So, by Lemma 3.2, S is weakly semi-potent.
(2) Suppose that M is locally injective. Let $\alpha \in S$, then $Ker(\alpha) \cap Ker(1-\alpha) = 0$. Since $M \neq 0$, $\{0\}$ is not large in M, so either $Ker(\alpha)$ or $Ker(1-\alpha)$ is not large in M. If $Ker(\alpha)$ is not large in M, then there exists a non-zero injective submodule Q of M such that $Q \cap Ker(\alpha) = 0$. Let $\alpha_0 : Q \to M$ be the restriction of α on to Q, then α_0 is monomorphism. Since Q is injective, there exists $\beta : M \to Q$ such that $\beta\alpha_0 = I_Q$. Since for every $m \in M$, $\beta(m) \in Q$, implies that $\beta\alpha\beta = \beta$. Thus, $e = \alpha\beta \in \alpha S$ is a non-zero idempotent. Similarly, we can prove that if, $Ker(1-\alpha)$ is not large in M, then $(1-\alpha)S$ contains a non-zero idempotent. So, by Lemma 3.2, S is weakly semi-potent. □

6 I_0-Endomorphism Rings

Recall that a module M_R is *weak-Rickart* (*w-Rickart* for short) [3], if for every non-zero $\alpha \in End_R(M)$, $Ker(\alpha)$ is contained in a proper direct summand of M. Also, a module M_R is called *wd-Rickart* [3], if for every non-zero $\alpha \in End_R(M)$, $Im(\alpha)$ contains a non-zero direct summand of M. Further, recall that a module M_R is direct projective if for every direct summand N of M and every epimorphism $\alpha : M \to N$, there exists $\beta \in S = End_R(M)$, such that $\alpha\beta$ is a projection, [11]. Dually, a module M_R is called a direct injective if for every direct summand N of M and every monomorphism $\alpha : N \to M$, there exists $\beta \in S$ such that $\beta\alpha$ is an inclusion, [11]. These modules are studied in Mohamed and Muller (1990), where they are called D_2*−modules* and C_2*−modules*, respectively. The next reformulated Theorem characterizes regular endomorphism ring of module, in terms of the Rickart modules and the C_2 property for modules.

Theorem 6.1. [15 , *Theorem* 3.17]. *The following conditions are equivalent for a module M and $S = End_R(M)$:*
(1) *M is a Rickart module with C_2 condition.*
(2) *S is a regular ring.*

In view of Theorem 6.1 we characterize an I_0−endomorphism ring of module, in terms of the w-Rickart and direct injective (wd-Rickart and direct projective) properties of modules.

Proposition 6.2. *Let M_R be a module and $S = End_R(M)$. The following statements are equivalent:*
(1) *S is an I_0−ring and $J(S) = 0$.*
(2) *The module M is weak-Rickart and direct-injective.*
(3) *The module M is wd-Rickart and direct-projective.*

Proof. (1) ⇒ (2). Let $\alpha \in S$, $\alpha \neq 0$. By hypothesis, there exists $\beta \in S$, $\beta \neq 0$ such that $\beta = \beta\alpha\beta$. Let $e = \beta\alpha$, then $0 \neq e \in S$ is an idempotent and $Ker(\alpha) \subseteq Ker(e)$. Since $Ker(e) \neq M$ is a direct summand of M, it follows that M is weak-Rickart. Now we will prove that M is direct-injective. Let N be a non-zero direct summand of M, $\alpha : N \to M$ be a monomorphism and $\pi : M \to N$ be the projection, then $0 \neq \alpha\pi \in S$. By hypothesis, there exists $\gamma \in S$, $\gamma \neq 0$ such $\gamma = \gamma(\alpha\pi)\gamma$. Let $e = \pi\gamma(\alpha\pi)$, then $0 \neq e \in S$ is an idempotent and $Im(e) \subseteq N = Im(\pi)$. Since for every $m \in M$, $m = e(m) + (1-e)(m)$, we have $\pi(m) = e(m)$, thus $\pi = e$. Let $y \in N$, then $y = \pi(y) = e(y) = \pi\gamma\alpha\pi(y) = \pi\gamma\alpha(y)$. Now, if $\varphi = \pi\gamma$, then $\varphi \in S$ and $\varphi\alpha = \tau$, where $\tau : N \to M$ the inclusion, this shows that M is direct injective. (2) ⇒ (1). Let $\alpha \in S$, $\alpha \neq 0$, then $Ker\alpha \neq M$. So by hypothesis, $Ker(\alpha) \subseteq N$, where $N \neq M$ is a direct summand of M. Thus, $M = N \oplus K$ for some submodule $K \neq 0$ of M. Let $\alpha_0 : K \to M$ be the restriction of α on K, then α_0 is monomorphism. Since M is direct-injective, there exists $\beta \in S$ such that $\beta\alpha_0 = \tau$, where $\tau : K \to M$ the inclusion. Let $\pi : M \to K$ be the projection, then for every $m \in M$, $\pi(m) \in K$ and $\beta\alpha\pi(m) = \beta\alpha_0(\pi(m)) = \tau\pi(m) = \pi(m)$, so $\beta\alpha\pi = \pi$. Let $\varphi = \pi\beta$, then $0 \neq \varphi \in S$ such that $\varphi = \varphi\alpha\varphi$. If $\alpha \in J(S)$, then $\varphi\alpha \in J(S)$ is a non-zero idempotent which is a contradiction, thus S is an I_0-ring and $J(S) = 0$. (1) $\Rightarrow$ (3). Let $\alpha \in S$, $\alpha \neq 0$. By hypothesis, there exists $\beta \in S$, $\beta \neq 0$ such that $\beta = \beta\alpha\beta$. Let $e = \alpha\beta$, then $0 \neq e \in S$ is an idempotent and $Im(e) \subseteq Im(\alpha)$, thus M is a wd-Rickart module. Now we will prove that M is direct-projective. Let N be a non-zero direct summand of M, $\lambda : M \to N$ be an epimorphism. Since $0 \neq \lambda \in S$, by our hypothesis $\mu = \mu\lambda\mu$ for some $0 \neq \mu \in S$. Let $e = \lambda\mu$, then $0 \neq e \in S$ is an idempotent and $Im(e) \subseteq Im(\lambda) = N$. Suppose that $\pi : M \to N$ be the projection. Since $1 = e + (1 - e)$, $m = e(m) + (1 - e)(m)$ for every $m \in M$. Thus, $\pi(m) = e(m)$, hence $e(m) \in N$. Therefore, $\lambda\mu = \pi$, so M is direct projective. (3) $\Rightarrow$ (1). Let $\alpha \in S$, $\alpha \neq 0$, by hypothesis there exists a non-zero direct summand N of M such that $N \subseteq Im(\alpha)$. Let $\pi : M \to N$ be the projection, then $N = Im(\pi) = Im(\pi\alpha)$. Since M is direct projective, there exists $\beta \in S$ such that $(\pi\alpha)\beta = \pi$. Suppose that $\varphi = \alpha(\beta\pi)$, then $\varphi \in S$ is a non-zero idempotent and $\varphi \in \alpha S$. If $\alpha \in J(S)$, then $\varphi = 0$ is a contradiction thus S is an I_0-ring and $J(S) = 0$. □

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