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Some Properties of the Sine Integral Function

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Abstract
In this paper, we establish some analytical properties such as convexity, concavity and mean inequalities for the sine integral function. We also provide improvement of an inequality due Koumandos and as a byproduct, we rediscover an inequality due Sándor with a relaxed condition. In the end, we pose an open problem.

Keywords: Sine integral function, sinc function, bounds, inequalities.

1 Introduction and Preliminaries

The cardinal sine function which is also known as sinc function or sampling function is defined as

$$\operatorname{sinc}(y) = \begin{cases} \frac{\sin(y)}{y}, & y \neq 0 \\ 1, & y = 0. \end{cases} \tag{1}$$

It is frequently applied in areas such as signal processing, theory of Fourier transforms, mathematical physics and engineering [6, 10, 15]. Due to its usefulness, it has been studied along different paths and many inequalities involving it have been established [3–5, 11, 26].

The sine integral function is defined for $z \in (-\infty, \infty)$ as

$$\begin{aligned} \operatorname{Si}(y) &= \int_0^y \frac{\sin(t)}{t}\, dt, && (2) \\ &= \frac{\pi}{2} - \int_y^\infty \frac{\sin(t)}{t}\, dt, && (3) \\ &= \int_0^1 \frac{\sin(yt)}{t}\, dt, && (4) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r y^{2r+1}}{(2r+1)(2r+1)!}. && (5) \end{aligned}$$

It satisfies the limit properties

$$\lim_{y\to\infty} \operatorname{Si}(y) = \frac{\pi}{2} \quad \text{and} \quad \lim_{y\to-\infty} \operatorname{Si}(y) = -\frac{\pi}{2}.$$

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In 2005, Koumandos [13] establish some interesting inequalities for the function $\mathrm{Si}(y)$. Notably among these, is the subadditivity property of $\mathrm{Si}(y)$. Jameson, Lord and McKee [12] also established that

$$\arctan(y) < \mathrm{Si}(y) < \pi - \arctan(y), \quad y > 0.$$

Recently, Sun and Chen [27] established some remarkable inequalities involving $\mathrm{Si}(y)$. Also the hyperbolic counterpart of $\mathrm{Si}(y)$ has been studied in [19].

In this paper, we continue the investigation on the function $\mathrm{Si}(y)$. Specifically, we establish some convexity and concavity properties and some mean inequalities involving the function. We also improve some previous results. We present our results in the next section. Before then, we recall the following definitions and lemmas.

Definition 1.1 ([23]). A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is GG-convex (or multiplicatively convex or geometrically convex) if

$$h(x^{1-s}y^s) \leq h(x)^{1-s}h(y)^s \tag{6}$$

holds for all $x, y \in (a, b)$ and $s \in [0, 1]$. If the inequality in (6) is reversed, then h is GG-concave.

Definition 1.2 ([23]). A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is AG-convex (or logarithmically convex) if

$$h((1-s)x + sy) \leq [h(x)]^{1-s}[h(y)]^s \tag{7}$$

holds for all $x, y \in (a, b)$ and $s \in [0, 1]$. If the inequality in (7) is reversed, then h is AG-concave.

Definition 1.3 ([23]). A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is GA-convex if

$$h(x^{1-s}y^s) \leq (1-s)h(x) + sh(y) \tag{8}$$

holds for all $x, y \in (a, b)$ and $s \in [0, 1]$. If the inequality in (8) is reversed, then h is GA-concave.

Lemma 1.4 ([23]). *A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is GG-convex (or GG-concave) if and only if $\frac{yh'(y)}{h(y)}$ is increasing (or decreasing) on (a, b) respectively.*

Lemma 1.5 ([23, 28]). *A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is GA-convex if and only if*

$$h'(y) + yh''(y) \geq 0 \tag{9}$$

for all $y \in (a, b)$. The function h is be GA-concave if and only if the inequality in (9) is reversed.

Lemma 1.6 ([28]). *A function $h : (a, b) \subseteq (0, \infty) \to (0, \infty)$ is AG-convex if and only if*

$$[\ln h(y)]'' \geq 0 \tag{10}$$

or equivalently, if and only if

$$h(y)h''(y) - (h'(y))^2 \geq 0 \tag{11}$$

holds for all $y \in (a, b)$. The function h is AG-concave if and only if (10) or (11) is reversed.

In [23], it has been established that $\mathrm{Si}(y)$ is GG-concave on $(0, \pi/2)$. That is,

$$\mathrm{Si}(x^{1-s}y^s) \geq [\mathrm{Si}(x)]^{1-s}[\mathrm{Si}(y)]^s \tag{12}$$

for all $x, y \in (0, \pi/2)$ and $s \in [0, 1]$.

2 Results

Theorem 2.1. *The function* Si(y) *is strictly GA-convex on* $(0, \pi/2)$. *That is,*

$$\operatorname{Si}(x^{1-s}y^s) < (1-s)\operatorname{Si}(x) + s\operatorname{Si}(y) \tag{13}$$

for $x, y \in (0, \pi/2)$ *and* $s \in [0, 1]$.

Proof. Let $y \in (0, \pi/2)$. Then

$$\begin{aligned}\operatorname{Si}'(y) + y\operatorname{Si}''(y) &= \frac{\sin(y)}{y} + y\left(\frac{y\cos(y) - \sin(y)}{y^2}\right) \\ &= \cos(y) \\ &> 0.\end{aligned}$$

The conclusion then follows from Lemma 1.5. □

Theorem 2.2. *The function* Si(y) *is strictly AG-concave on* $(0, \pi/2)$. *That is,*

$$\operatorname{Si}((1-s)x + sy) > [\operatorname{Si}(x)]^{1-s}[\operatorname{Si}(y)]^s \tag{14}$$

for $x, y \in (0, \pi/2)$ *and* $s \in [0, 1]$.

Proof. Let $y \in (0, \pi/2)$ and $\phi(y) = \ln \operatorname{Si}(y)$. It is well known that

$$y\cos(y) - \sin(y) < 0$$

if $y \in (0, \pi/2)$. For example, see [8]. Then

$$\begin{aligned}\phi'(y) &= \frac{\operatorname{Si}'(y)}{\operatorname{Si}(y)} = \frac{\operatorname{sinc}(y)}{\operatorname{Si}(y)}. \\ \phi''(y) &= \frac{\operatorname{Si}''(y)\operatorname{Si}(y) - (\operatorname{Si}'(y))^2}{[\operatorname{Si}(y)]^2}.\end{aligned}$$

which implies that

$$\begin{aligned}[\operatorname{Si}(y)]^2\phi''(y)) &= \left(\frac{y\cos(y) - \sin(y)}{y^2}\right)\operatorname{Si}(y) - (\operatorname{Si}'(y))^2 \\ &< 0.\end{aligned}$$

Hence $\phi''(y) < 0$. The conclusion then follows from Lemma 1.6. □

Theorem 2.3. *The inequality*

$$\sqrt{\operatorname{Si}(y)\operatorname{Si}(1/y)} \leq \operatorname{Si}(1) \approx 0.946083 \tag{15}$$

holds for $y \in (0, \infty)$.

Proof. Let $\mathscr{B}(y) = \operatorname{Si}(y)\operatorname{Si}(1/y)$ for $y \in (0, \infty)$. Then

$$\begin{aligned}\mathscr{B}'(y) &= \operatorname{Si}'(y)\operatorname{Si}(1/y) - \frac{1}{y^2}\operatorname{Si}(y)\operatorname{Si}'(1/y) \\ &= \operatorname{sinc}(y)\operatorname{Si}(1/y) - \frac{1}{y^2}\operatorname{Si}(y)\operatorname{sinc}(1/y)\end{aligned}$$

which implies that $\mathscr{B}'(1) = 0$. Thus, $y = 1$ is a critical point of the function $\mathscr{B}(y)$. Recall that

$$\operatorname{Si}'(y) = \operatorname{sinc}(y) = \frac{\sin(y)}{y}.$$

Moreover,

$$\begin{aligned}&\mathscr{B}''(y)\\&=\frac{\operatorname{sinc}(1/y)\left[2\operatorname{Si}(y)-2y\operatorname{sinc}(y)\right]+y\operatorname{Si}(1/y)\left[y\cos(y)-\sin(y)\right]+\operatorname{Si}(y)\left[\cos(1/y)-y\sin(1/y)\right]}{y^3}\\&=\frac{y\operatorname{Si}(1/y)\left[y\cos(y)-\sin(y)\right]+\operatorname{Si}(y)\left[y\sin(1/y)+\cos(1/y)\right]-2\sin(1/y)\sin(y)}{y^3}\end{aligned}$$

which implies that

$$\begin{aligned}\mathscr{B}''(1)&=\operatorname{Si}(1)\left[\cos(1)-\sin(1)\right]+\operatorname{Si}(1)\left[\sin(1)+\cos(1)\right]-2(\sin(1))^2\\&=2\left[\operatorname{Si}(1)\cos(1)-(\sin(1))^2\right]\\&\approx -0.3938051<0.\end{aligned}$$

Hence by the second derivative test, $\mathscr{B}(y)$ has a global maximum at $y = 1$. Therefore,

$$\mathscr{B}(y)\le\mathscr{B}(1)=(\operatorname{Si}(1))^2$$

for $y \in (0, \infty)$. □

Theorem 2.4. *The inequality*

$$\frac{2\operatorname{Si}(y)\operatorname{Si}(1/y)}{\operatorname{Si}(y)+\operatorname{Si}(1/y)}\le\operatorname{Si}(1)\approx 0.946083 \tag{16}$$

holds for $y \in (0, \infty)$.

Proof. Let $\mathscr{A}(u,v)=\frac{u+v}{2}$, $\mathscr{G}(u,v)=\sqrt{uv}$ and $\mathscr{H}(u,v)=\frac{1}{\mathscr{A}(1/u,1/v)}=\frac{2uv}{u+v}$ for positive numbers u and v. Since $\mathscr{H}(u,v)\le\mathscr{G}(u,v)$, see for example [25], then

$$\frac{2\operatorname{Si}(y)\operatorname{Si}(1/y)}{\operatorname{Si}(y)+\operatorname{Si}(1/y)}\le\sqrt{\operatorname{Si}(y)\operatorname{Si}(1/y)}\le\operatorname{Si}(1)\approx 0.946083$$

which completes the proof. □

Theorem 2.5. *The inequality*

$$\frac{\operatorname{Si}(y)+\operatorname{Si}(1/y)}{2}<\frac{y^2+1}{2y} \tag{17}$$

holds for $y \in (0, \infty)$.

Proof. It has been established in [13] that, the inequality

$$\operatorname{Si}(y)<y \tag{18}$$

holds for $y \in (0, \infty)$. This implies that for $y \in (0, \infty)$, we have

$$\operatorname{Si}(y)+\operatorname{Si}(1/y)<y+\frac{1}{y}, \tag{19}$$

which results to (17). □

Remark 2.6. Inequality (18) further implies that

$$\operatorname{Si}(y)\operatorname{Si}(1/y)<1, \tag{20}$$

holds for $y \in (0, \infty)$.

Remark 2.7. Since $\operatorname{Si}(1) \approx 0.946083 < 1$, the bound in (15) is better than the one in (20).

In the following theorem, we give an improvement of the inequality (18).

Theorem 2.8. *The inequality*

$$\mathrm{Si}(y) < \frac{y}{2} + \frac{1-\cos(y)}{y} \tag{21}$$

holds for $y \in (0, \infty)$.

Proof. Recall from (18) that for $t \in (0, \infty)$,

$$\mathrm{Si}(t) < t.$$

Then

$$\int_0^y \mathrm{Si}(t)dt < \int_0^y t dt$$

which gives

$$y\mathrm{Si}(y) + \cos(y) - 1 < \frac{y^2}{2}$$

and from which (21) is obtained. □

In the following remark, we show that the bound in (21) is better than the one in (18).

Remark 2.9. It is fundamental that the inequality

$$y > \sin(y) \tag{22}$$

holds for $y \in (0, \infty)$. To see this, let $A(y) = y - \sin(y)$. Then $A'(y) = 1 - \cos(y) \geq 0$. Thus $A(y)$ is increasing. Hence $A(y) > \lim_{y\to 0} A(y) = 0$ which gives (22). Also, the inequality

$$u^2 > \sin^2(u) \tag{23}$$

holds for $u \in (0, \infty)$. To see this, let $B(u) = u^2 - \sin^2(u)$. Then $B'(u) = 2u - 2\sin(u)\cos(u) = 2u - \sin(2u) > 0$ as a result of (22). Thus $B(u)$ is increasing and hence $B(u) > \lim_{y\to 0} B(y) = 0$ which gives (23). Now we have

$$\begin{aligned}
y - \left(\frac{y}{2} + \frac{1-\cos(y)}{y}\right) &= \frac{y}{2} - \frac{1-\cos(y)}{y} \\
&= \frac{y}{2} - \frac{2\sin^2(\frac{y}{2})}{y} \\
&= \frac{y}{2} - \frac{\sin^2(\frac{y}{2})}{(\frac{y}{2})} \\
&= \frac{\left(\frac{y}{2}\right)^2 - \sin^2\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)} \\
&> 0
\end{aligned}$$

as a result of (23). Hence

$$\frac{y}{2} + \frac{1-\cos(y)}{y} < y \tag{24}$$

which implies that (21) is better than (18).

Remark 2.10. Inequality (24) can be re-arranged as

$$\frac{1-\cos(y)}{y^2} < \frac{1}{2} \tag{25}$$

for $y \in (0, \infty)$. We note that (25) was earlier obtained for $y \in (0, \pi/2)$ by Sándor [24]. This is a rediscovery with a relaxed condition on y.

Problem 2.11. Find constant bounds α and β such that

$$\alpha \leq \frac{\mathrm{Si}(y) + \mathrm{Si}(1/y)}{2} \leq \beta \tag{26}$$

holds for $y \in (0, \infty)$.

For mean inequalities regarding other special functions, interested readers may refer to [1, 2, 7, 9, 14, 16–18, 20–22, 29].

3 Conclusion

By using some basic techniques in mathematical analysis, we have shown that the sine integral function $\mathrm{Si}(y)$ is GA-convex on $(0, \pi/2)$ and AG-concave on $(0, \pi/2)$. We have also shown that, the geometric mean of $\mathrm{Si}(y)$ and $\mathrm{Si}(1/y)$ and harmonic mean of $\mathrm{Si}(y)$ and $\mathrm{Si}(1/y)$ are always less than or equal to $\mathrm{Si}(1) \approx 0.946083$. In addition, we have shown that the arithmetic mean of $\mathrm{Si}(y)$ and $\mathrm{Si}(1/y)$ is always less than $(y^2+1)/2y$. Moreover, we have provided an improvement of an inequality due Koumandos. Furthermore, we have rediscovered an inequality due Sándor with a relaxed condition.

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