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$(p,q)^{th}$ Iterated (α,β) Order and Type of Meromorphic and Entire Functions

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Abstract
In 2016, Tu et. al. [9] investigated the p-iterated order and p-iterated type of sum and multiplication of entire or meromorphic functions. In the present paper we introduce $(p,q)^{th}$ iterated (α,β) order and type of meromorphic and entire functions and prove some results to generalize the results of [9].

Keywords: Entire function, meromorphic function, iterated order, iterated type.

1 Introduction and Definitions

The order and type of an entire and meromorphic function have been generalized in several ways. First we give some definitions of order and type starting from the classical ones of entire and meromorphic functions.

Definition 1.1. [3, 8] Let f be an entire function and $M(r,f)=\max\{|f(z)|:|z|=r\}$. Then the order and lower order of an entire function f are defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

When f is meromorphic, then

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 1.2. [3, 8] Let $f(z)$ be an entire function satisfying $0<\rho_f<\infty$. Then the type and lower type of $f(z)$ are defined as

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},$$

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and

$$\mu_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}.$$

When f is meromorphic, then the type and lower type of $f(z)$ are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}},$$

and

$$\mu_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}.$$

Later on, Juneja et. al. [4, 5] introduced $(p,q)^{th}$ order and type in the following manner.

Definition 1.3. [4] The $(p,q)^{th}$ order and $(p,q)^{th}$ lower order of an entire function f are defined as

$$\rho_f^{p,q} = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}; p \geq q \geq 1,$$

and

$$\lambda_f^{p,q} = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}; p \geq q \geq 1.$$

When f is meromorphic, then Bergweiler et. al. [1] defined $(p,q)^{th}$ order and $(p,q)^{th}$ lower order as

$$\rho_f^{p,q} = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}; p \geq q \geq 1,$$

and

$$\lambda_f^{p,q} = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}; p \geq q \geq 1.$$

Definition 1.4. [5] The $(p,q)^{th}$ type and the $(p,q)^{th}$ lower type of an entire function f having finite positive $(p,q)^{th}$ order $\rho_f^{p,q}$ are defined by

$$\sigma_f^{p,q} = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\left[\log^{[q-1]} r\right]^{\rho_f^{p,q}}}; p \geq q \geq 1,$$

and

$$\mu_f^{p,q} = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\left[\log^{[q-1]} r\right]^{\rho_f^{p,q}}}; p \geq q \geq 1.$$

When f is meromorphic, then $(p,q)^{th}$ type and the $(p,q)^{th}$ lower type are defined respectively as

$$\sigma_f^{p,q} = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\left[\log^{[q-1]} r\right]^{\rho_f^{p,q}}}; p \geq q \geq 1,$$

and

$$\mu_f^{p,q} = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\left[\log^{[q-1]} r\right]^{\rho_f^{p,q}}}; p \geq q \geq 1.$$

Now let L be a class of continuous non-negative function α on $(-\infty,\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\to\infty$ as $x\to\infty$. For any $\alpha\in L$, we say that $\alpha\in L_1$, if $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to\infty$ for each $c\in(0,\infty)$ and $\alpha\in L_2$, if $\alpha(\exp(cx))=(1+o(1))\alpha(\exp(x))$ as $x_0\leq x\to\infty$ for each $c\in(0,\infty)$. Clearly, $L_2\subset L_1$.
Considering the above, Sheremeta [7] introduced the concept of generalized order (α,β) of an entire function. For the purpose of further applications, Biswas and Biswas [2] modified the definition of the generalized order (α,β) and generalized lower order (α,β) of entire and meromorphic functions in the following way.

Definition 1.5. [2] Let $\alpha,\beta\in L_1$. The generalized order (α,β) and generalized lower order (α,β) of an entire function are defined as

$$\rho(\alpha,\beta,f)=\limsup_{r\to\infty}\frac{\alpha(M(r,f))}{\beta(r)},$$

and

$$\lambda(\alpha,\beta,f)=\liminf_{r\to\infty}\frac{\alpha(M(r,f))}{\beta(r)}.$$

The generalized order (α,β) and generalized lower order (α,β) of a meromorphic function are defined as

$$\rho(\alpha,\beta,f)=\limsup_{r\to\infty}\frac{\alpha(\exp(T(r,f)))}{\beta(r)},\ \alpha\in L_2,\ \beta\in L_1,$$

and

$$\lambda(\alpha,\beta,f)=\liminf_{r\to\infty}\frac{\alpha(\exp(T(r,f)))}{\beta(r)},\ \alpha\in L_2,\ \beta\in L_1.$$

Biswas and Biswas [2] also introduced generalized type (α,β) and generalized lower type (α,β) of meromorphic functions.

Definition 1.6. Let $\alpha\in L_2$ and $\beta\in L_1$. The generalized type (α,β) and generalized lower type (α,β) of a meromorphic function f with finite positive generalized order (α,β) and generalized lower order (α,β) respectively are

$$\sigma(\alpha,\beta,f)=\limsup_{r\to\infty}\frac{\exp(\alpha(\exp(T(r,f))))}{(\exp(\beta(r)))^{\rho(\alpha,\beta,f)}},$$

$$\mu(\alpha,\beta,f)=\liminf_{r\to\infty}\frac{\exp(\alpha(\exp(T(r,f))))}{(\exp(\beta(r)))^{\rho(\alpha,\beta,f)}}.$$

Now we introduce following definitions and using this we generalize some results.

Definition 1.7. Let $\alpha\in L_2$ and $\beta\in L_1$. Then $(p,q)^{th}$ iterated (α,β) order of an entire function f is defined by

$$\begin{aligned}\rho^{p,q}(\alpha,\beta,f)&=\limsup_{r\to\infty}\frac{\alpha(\log^{[p-1]}(M(r,f)))}{\beta(\log^{[q-1]}r)}\\&=\limsup_{r\to\infty}\frac{\alpha(\exp(\log^{[p-1]}(T(r,f))))}{\beta(\log^{[q-1]}r)},\ p\geq q\geq 1\text{ and }p\neq 1.\end{aligned}$$

Definition 1.8. Let $\alpha\in L_2$ and $\beta\in L_1$. Then $(p,q)^{th}$ iterated (α,β) lower order of an entire function f is defined by

$$\begin{aligned}\lambda^{p,q}(\alpha,\beta,f)&=\liminf_{r\to\infty}\frac{\alpha(\log^{[p-1]}(M(r,f)))}{\beta(\log^{[q-1]}r)}\\&=\liminf_{r\to\infty}\frac{\alpha(\exp(\log^{[p-1]}(T(r,f))))}{\beta(\log^{[q-1]}r)},\ p\geq q\geq 1\text{ and }p\neq 1.\end{aligned}$$

Definition 1.9. Let $\alpha \in L_2$ and $\beta \in L_1$ and $f(z)$ be a meromorphic function satisfying $0 < \rho^{p,q}(\alpha, \beta, f) = \rho < \infty$ or $0 < \lambda^{p,q}(\alpha, \beta, f) = \lambda < \infty$. Then the $(p,q)^{th}$ iterated (α, β) type of $(p,q)^{th}$ iterated (α, β) order and the $(p,q)^{th}$ iterated (α, β) lower type of $(p,q)^{th}$ iterated (α, β) lower order of $f(z)$ are respectively defined by

$$\psi^{p,q}(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\exp(\alpha(\exp(\log^{[p-1]}(T(r,f)))))}{\left[\exp\left(\beta(\log^{[q-1]} r)\right)\right]^{\rho}}, \; p \geq q \geq 1 \text{ and } p \neq 1$$

and

$$\underline{\psi}^{p,q}(\alpha, \beta, f) = \liminf_{r\to\infty} \frac{\exp(\alpha(\exp(\log^{[p-1]}(T(r,f)))))}{\left[\exp\left(\beta(\log^{[q-1]} r)\right)\right]^{\lambda}}, \; p \geq q \geq 1 \text{ and } p \neq 1.$$

Definition 1.10. Let $\alpha \in L_2$ and $\beta \in L_1$ and $f(z)$ be an entire function satisfying $0 < \rho^{p,q}(\alpha, \beta, f) = \rho < \infty$ or $0 < \lambda^{p,q}(\alpha, \beta, f) = \lambda < \infty$. Then the $(p,q)^{th}$ iterated (α, β) type of $(p,q)^{th}$ iterated (α, β) order and the $(p,q)^{th}$ iterated (α, β) lower type of $(p,q)^{th}$ iterated (α, β) lower order of $f(z)$ are respectively defined by

$$\sigma^{p,q}(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\exp(\alpha(\log^{[p-1]}(M(r,f))))}{\left[\exp\left(\beta(\log^{[q-1]} r)\right)\right]^{\rho}}, \; p \geq q \geq 1 \text{ and } p \neq 1$$

and

$$\mu^{p,q}(\alpha, \beta, f) = \liminf_{r\to\infty} \frac{\exp(\alpha(\log^{[p-1]}(M(r,f))))}{\left[\exp\left(\beta(\log^{[q-1]} r)\right)\right]^{\lambda}}, \; p \geq q \geq 1 \text{ and } p \neq 1.$$

All our definitions coinside with the previously defined orders and types when $p = 1$. So here we study the cases when $p > 1$.

2 Preliminary Results

In this section we state some results in the form of Lemmas, which will be needed in the subsequent section.

Lemma 2.1. [9] Let $f(z)$ be an analytic function in the circle $|z| \leq kR$ $(k > 1)$ with $f(0) = 1$, and let ϵ be an arbitrary positive number not exceeding 2. Then inside the circle $|z| \leq R$, but outside of a family of circles the sum of whose radii is less than a finite number (depending on ϵ, k and R), we have

$$\log|f(z)| > -\left(\frac{2}{k-1} + \frac{\log 2 - \log \epsilon}{\log k}\right) \log M(k^2 R, f).$$

Lemma 2.2. [6] Let $f(z)$ be a meromorphic function. Then for all large r,

$$T(r,f) < C(T(2r,f') + \log r),$$

where C is a constant which is only dependent on $f(0)$.

3 Sum and Product Theorems

Theorem 3.1. Let $\alpha \in L_2$ and $\beta \in L_1$ with α and β both are increasing functions and f_1, f_2 be meromorphic functions satisfying $0 < \rho^{p,q}(\alpha, \beta, f_1) = \rho^{p,q}(\alpha, \beta, f_2) = \rho < \infty$, $0 \leq \psi^{p,q}(\alpha, \beta, f_1) < \psi^{p,q}(\alpha, \beta, f_2) \leq \infty$, then

$$(i)\; \rho^{p,q}(\alpha, \beta, f_1 + f_2) = \rho^{p,q}(\alpha, \beta, f_1 f_2) = \rho;$$

$$(ii)\; \psi^{p,q}(\alpha, \beta, f_1 + f_2) = \psi^{p,q}(\alpha, \beta, f_1 f_2) = \psi^{p,q}(\alpha, \beta, f_2).$$

Proof. Set $\psi^{p,q}(\alpha,\beta,f_1)=\psi_1$ and $\psi^{p,q}(\alpha,\beta,f_2)=\psi_2$. Without loss of generality, take $0\le\psi_1<\psi_2<\infty$. By Definition 1.9, for any $\epsilon>0$ and for sufficiently large r, we have

$$\begin{aligned}T(r,f_1+f_2)&\le T(r,f_1)+T(r,f_2)+O(1)\\&\le \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_1+\epsilon)+\rho\beta\left(\log^{[q-1]}r\right)\right]\right)\right]\\&\qquad+\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_2+\epsilon)+\rho\beta\left(\log^{[q-1]}r\right)\right]\right)\right]+O(1)\\&\le 3\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_2+\epsilon)+\rho\beta\left(\log^{[q-1]}r\right)\right]\right)\right].\end{aligned}\tag{3.1}$$

By (3.1), we get $\rho^{p,q}(\alpha,\beta,f_1+f_2)\le\rho$. On the other hand, by Definition 1.9, for suitable $\epsilon>0$ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$\begin{aligned}T(r_n,f_1+f_2)&\ge T(r_n,f_2)-T(r_n,f_1)-O(1)\\&\ge \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_2-\epsilon)+\rho\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right]\\&\qquad-\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_1+\epsilon)+\rho\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right]-O(1)\\&\ge \frac{1}{3}\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi_2-\epsilon)+\rho\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right]\end{aligned}\tag{3.2}$$

holds for sufficiently large r_n. By (3.2), we get $\rho^{p,q}(\alpha,\beta,f_1+f_2)\ge\rho$; therefore $\rho^{p,q}(\alpha,\beta,f_1+f_2)=\rho$. Again since

$$T(r,f_1f_2)\le T(r,f_1)+T(r,f_2),\ \ T(r,f_1f_2)\ge T(r,f_2)-T(r,f_1)-o(1),\tag{3.3}$$

by the similar proof in (3.1) and (3.2), we can easily obtain the conclusion in case (i).

Also by (3.1) and (3.2), we have $\psi^{p,q}(\alpha,\beta,f_1+f_2)=\psi^{p,q}(\alpha,\beta,f_2)$.

Now again by (3.3) and by the similar proof in (3.1) and (3.2), we can easily obtain that the conclusions in case (ii) holds.

By the above proof, we can easily obtain that Theorem 3.1 also holds for $0\le\psi^{p,q}(\alpha,\beta,f_1)<\psi^{p,q}(\alpha,\beta,f_2)=\infty$.

Hence the theorem. □

Example 3.1. Set $f_1=e^{-z}$, $f_2=e^{2z}$, $\alpha(r)=\log\log r$, $\beta(r)=\log r$, $p=1$ and $q=1$.
In this case $0<\rho^{p,q}(\alpha,\beta,f_1)=\rho^{p,q}(\alpha,\beta,f_2)=1<\infty$, $\psi^{p,q}(\alpha,\beta,f_1)=\frac{1}{\pi}$, $\psi^{p,q}(\alpha,\beta,f_2)=\frac{2}{\pi}$. So all the conditions of the above theorem is satisfied except for the condition in the definitions that $p=1$.
Here $\rho^{p,q}(\alpha,\beta,f_1+f_2)=\rho^{p,q}(\alpha,\beta,f_1f_2)=1$. So (i) is satisfied.
But $\psi^{p,q}(\alpha,\beta,f_1+f_2)=\frac{3}{\pi}$ and $\psi^{p,q}(\alpha,\beta,f_1f_2)=\frac{1}{\pi}$. So (ii) is not satisfied.

Theorem 3.2. Let $\alpha\in L_2$ and $\beta\in L_1$ with α and β both are increasing functions and f_1,f_2 be entire functions satisfying $0<\rho^{p,q}(\alpha,\beta,f_1)=\rho^{p,q}(\alpha,\beta,f_2)=\rho<\infty$, $0\le\sigma^{p,q}(\alpha,\beta,f_1)<\sigma^{p,q}(\alpha,\beta,f_2)\le\infty$, then

$$\begin{aligned}&(i)\rho^{p,q}(\alpha,\beta,f_1+f_2)=\rho,\ \ \sigma^{p,q}(\alpha,\beta,f_1+f_2)=\sigma^{p,q}(\alpha,\beta,f_2);\\&(ii)\rho^{p,q}(\alpha,\beta,f_1f_2)=\rho,\ \ \sigma^{p,q}(\alpha,\beta,f_1f_2)=\sigma^{p,q}(\alpha,\beta,f_2),\ \text{ if } f_1(0)=1.\end{aligned}$$

Proof. (i) Set $0\le\sigma^{p,q}(\alpha,\beta,f_1)<\sigma^{p,q}(\alpha,\beta,f_2)<\infty$. By Definition 1.10, for any ϵ and for sufficiently large r, we have

$$\begin{aligned}M(r,f_1+f_2)&\le M(r,f_1)+M(r,f_2)\\&\le \exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right]\\&\qquad+\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right]\\&\le 2\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right].\end{aligned}\tag{3.4}$$

By (3.4), we get $\rho^{p,q}(\alpha,\beta,f_1+f_2)\le\rho,\ \sigma^{p,q}(\alpha,\beta,f_1+f_2)\le\sigma^{p,q}(\alpha,\beta,f_2)$. On the other hand by Definition 1.10, for suitable ϵ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$\begin{aligned}M(r_n,f_1+f_2)&\ge M(r_n,f_2)-M(r_n,f_1)\\&\ge\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)-\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\\&\quad-\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\\&\ge\frac{1}{2}\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)-\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\end{aligned}\tag{3.5}$$

holds for sufficiently large r_n. By (3.5), we get $\rho^{p,q}(\alpha,\beta,f_1+f_2)\ge\rho,\ \sigma^{p,q}(\alpha,\beta,f_1+f_2)\ge\sigma^{p,q}(\alpha,\beta,f_2)$; therefore $\rho^{p,q}(\alpha,\beta,f_1+f_2)=\rho,\ \sigma^{p,q}(\alpha,\beta,f_1+f_2)=\sigma^{p,q}(\alpha,\beta,f_2)$.

(ii) By Definition 1.10, for any ϵ and for sufficiently large r, we have

$$\begin{aligned}M(r,f_1f_2)&\le M(r,f_1)M(r,f_2)\\&\le\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right]\\&\quad\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right]\\&\le\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)+2\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right].\end{aligned}\tag{3.6}$$

By (3.6), we get $\rho^{p,q}(\alpha,\beta,f_1f_2)\le\rho,\ \sigma^{p,q}(\alpha,\beta,f_1f_2)\le\sigma^{p,q}(\alpha,\beta,f_2)$. On the other hand, by Definition 1.10, for suitable $\epsilon>0$ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$M(r_n,f_2)\ge\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)-\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right].\tag{3.7}$$

holds for sufficiently large r_n. Also by Lemma 2.1 and Definition 1.10, outside a set of finite logarithmic measure, we have

$$|f_1(z)|>\exp\left(-\exp^{[p-2]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r\right)\right]\right)^{\rho}\right]\right)\right]\right).\tag{3.8}$$

Therefore by (3.7) and (3.8), for suitable $\epsilon>0$ and for sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ (except for the set of finite logarithmic measure), we have

$$\begin{aligned}M(r_n,f_1f_2)&\ge M(r_n,f_2)\\&\quad\exp\left(-\exp^{[p-2]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\right)\\&\ge\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)-\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\\&\quad\exp\left(-\exp^{[p-2]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_1)+\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\right)\\&\ge\exp^{[p-1]}\left[\alpha^{-1}\left(\log\left[\left(\sigma^{p,q}(\alpha,\beta,f_2)-2\epsilon\right)\left(\exp\left[\beta\left(\log^{[q-1]}r_n\right)\right]\right)^{\rho}\right]\right)\right]\\&\qquad[\text{ since }\sigma^{p,q}(\alpha,\beta,f_1)<\sigma^{p,q}(\alpha,\beta,f_2)].\end{aligned}\tag{3.9}$$

By (3.9), we get $\rho^{p,q}(\alpha,\beta,f_1f_2)\ge\rho,\ \sigma^{p,q}(\alpha,\beta,f_1f_2)\ge\sigma^{p,q}(\alpha,\beta,f_2)$. Therefore $\rho^{p,q}(\alpha,\beta,f_1f_2)=\rho,\ \sigma^{p,q}(\alpha,\beta,f_1f_2)=\sigma^{p,q}(\alpha,\beta,f_2)$.

By the similar proof in case (i) and (ii), we can easily obtain that Theorem 3.2 holds for $0\le\sigma^{p,q}(\alpha,\beta,f_1)<\sigma^{p,q}(\alpha,\beta,f_1)=\infty$ □

Theorem 3.3. Let $\alpha \in L_2$ and $\beta \in L_1$ with α and β both are increasing functions and f_1, f_2 be meromorphic functions satisfying $\rho^{p,q}(\alpha,\beta,f_1) = \lambda^{p,q}(\alpha,\beta,f_2) = \lambda$ $(0 < \lambda < \infty)$, $0 \le \psi^{p,q}(\alpha,\beta,f_1) < \underline{\psi}^{p,q}(\alpha,\beta,f_2) \le \infty$. Then

$$(i)\ \lambda^{p,q}(\alpha,\beta,f_1+f_2) = \lambda^{p,q}(\alpha,\beta,f_1f_2) = \lambda;$$
$$(ii)\ \underline{\psi}^{p,q}(\alpha,\beta,f_1+f_2) = \underline{\psi}^{p,q}(\alpha,\beta,f_1f_2) = \underline{\psi}^{p,q}(\alpha,\beta,f_2).$$

Proof. First take $0 \le \psi^{p,q}(\alpha,\beta,f_1) < \underline{\psi}^{p,q}(\alpha,\beta,f_2) < \infty$. By Definition (1.9), for any $\epsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned} T(r,f_1) &< \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi^{p,q}(\alpha,\beta,f_1)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r\right)\right]\right)\right] \\ T(r,f_2) &> \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)-\epsilon)+\lambda\beta\left(\log^{[q-1]}r\right)\right]\right)\right]. \end{aligned} \tag{3.10}$$

By (3.10), for suitable ϵ and for sufficiently large r, we have

$$\begin{aligned} T(r,f_1+f_2) &\ge T(r,f_2) - T(r,f_1) - \ln 2 \\ &\ge \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)-\epsilon)+\lambda\beta\left(\log^{[q-1]}r\right)\right]\right)\right] \\ &\quad - \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi^{p,q}(\alpha,\beta,f_1)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r\right)\right]\right)\right] - \ln 2 \\ &\ge \frac{1}{3}\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)-\epsilon)+\lambda\beta\left(\log^{[q-1]}r\right)\right]\right)\right]. \end{aligned} \tag{3.11}$$

By (3.11), we have

$$\begin{aligned} &\lambda^{p,q}(\alpha,\beta,f_1+f_2) \ge \lambda; \\ &\underline{\psi}^{p,q}(\alpha,\beta,f_1+f_2) \ge \underline{\psi}^{p,q}(\alpha,\beta,f_2). \end{aligned} \tag{3.12}$$

On the other hand, by Definition (1.9), for suitable $\epsilon > 0$ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that

$$T(r,f_2) < \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right], \tag{3.13}$$

for large r_n. By (3.10) and (3.13), for any $\epsilon > 0$ and for sufficiently large r_n, we have

$$\begin{aligned} T(r_n,f_1+f_2) &\le T(r_n,f_1) + T(r_n,f_2) + O(1) \\ &\le \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\psi^{p,q}(\alpha,\beta,f_1)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right] \\ &\quad + \exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right] \\ &\qquad + O(1) \\ &\le 3\exp^{[p-1]}\left[\log\left(\alpha^{-1}\left[\log(\underline{\psi}^{p,q}(\alpha,\beta,f_2)+\epsilon)+\lambda\beta\left(\log^{[q-1]}r_n\right)\right]\right)\right]. \end{aligned} \tag{3.14}$$

By (3.14), we have

$$\begin{aligned} &\lambda^{p,q}(\alpha,\beta,f_1+f_2) \le \lambda; \\ &\underline{\psi}^{p,q}(\alpha,\beta,f_1+f_2) \le \underline{\psi}^{p,q}(\alpha,\beta,f_2). \end{aligned} \tag{3.15}$$

Therefore, by (3.12) and (3.15), the conclusions of the theorem hold for f_1+f_2. Also from (3.3), we can easily obtain that the conclusions of the theorem hold for f_1f_2.

The theorem also holds for $0 \le \psi_p(f_1) < \underline{\psi}(f_2) = \infty$ by the above proof. □

Theorem 3.4. Let $\alpha \in L_2$ and $\beta \in L_1$ with α and β both are increasing functions and f_1, f_2 be meromorphic functions satisfying $\rho^{p,q}(\alpha, \beta, f_1) = \lambda^{p,q}(\alpha, \beta, f_2) = \lambda \ (0 < \lambda < \infty)$, $0 \leq \sigma^{p,q}(\alpha, \beta, f_1) < \mu^{p,q}(\alpha, \beta, f_2) \leq \infty$. Then

$$(i)\ \lambda^{p,q}(\alpha, \beta, f_1 + f_2) = \lambda^{p,q}(\alpha, \beta, f_1 f_2) = \lambda,$$
$$(ii)\ \mu^{p,q}(\alpha, \beta, f_1 + f_2) = \mu^{p,q}(\alpha, \beta, f_1 f_2) = \mu^{p,q}(\alpha, \beta, f_2).$$

Proof. We can obtain the conclusions of this theorem by the similar proof in previous two theorems. □

Theorem 3.5. $f(z)$ be a meromorphic function satisfying $0 < \rho^{p,q}(\alpha, \beta, f) < \infty$ and $\rho^{p,q}(\alpha, \beta, f) = \rho^{p,q}(\alpha, \beta, f')$. Then $\psi^{p,q}(\alpha, \beta, f') = \psi^{p,q}(\alpha, \beta, f)$.

Proof. By the Lemma of logarithmic derivative, we have

$$T(r, f') \leq 3T(r, f) + O\{\log r\}$$

holds outside of an exceptional set E of finite linear measure. So we have $\psi^{p,q}(\alpha, \beta, f') \leq \psi^{p,q}(\alpha, \beta, f)$.
On the other hand, by Lemma 2.2, for all large r,

$$T(r, f) < C(T(2r, f') + \log r),$$

where C is a constant which is only dependent on $f(0)$. By the above inequality, we have $\psi^{p,q}(\alpha, \beta, f') \geq \psi^{p,q}(\alpha, \beta, f)$.
Therefore $\psi^{p,q}(\alpha, \beta, f') = \psi^{p,q}(\alpha, \beta, f)$. □

Theorem 3.6. Let $f(z)$ be an entire function satisfying $0 < \rho^{p,q}(\alpha, \beta, f) < \infty$ and $\rho^{p,q}(\alpha, \beta, f) = \rho^{p,q}(\alpha, \beta, f')$. Then $\sigma^{p,q}(\alpha, \beta, f') = \sigma^{p,q}(\alpha, \beta, f)$.

Proof. For an entire function $f(z)$, we have

$$f(z) = f(0) + \int_0^z f'(\varsigma) d\varsigma, \tag{3.16}$$

where the path is a straight line from 0 to z. By (3.16), we have

$$\begin{aligned} &M(r, f) \leq |f(0)| + rM(r, f') \\ \Rightarrow &M(r, f') \geq \frac{M(r, f) - |f(0)|}{r} \end{aligned} \tag{3.17}$$

By definition 1.9 and (3.17), we have $\sigma^{p,q}(\alpha, \beta, f') \geq \sigma^{p,q}(\alpha, \beta, f)$. On the other hand by Cauchy's integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\varsigma)}{(\varsigma - z)^2} d\varsigma, \tag{3.18}$$

where integral curve is the circle $|\varsigma| = R$. By (3.18), for any $|z| = r < R$,

$$M(r, f') \leq \frac{1}{2\pi} \frac{M(R, f)}{(R - r)^2} \cdot 2\pi R. \tag{3.19}$$

Set $R = 2r$, then by (3.19) for sufficiently large r, we have

$$\begin{aligned} M(r, f') &\leq \frac{2}{r} M(2r, f) \\ &\leq \frac{2}{r} \exp^{[p-1]} \left[\alpha^{-1} \left(\log \left[\left(\sigma^{p,q}(\alpha, \beta, f) + \epsilon \right) \left(\exp \left[\beta \left(\log^{[q-1]} 2r \right) \right] \right)^{\rho^{p,q}(\alpha,\beta,f)} \right] \right) \right]. \end{aligned}$$

Since $\rho^{p,q}(\alpha, \beta, f) = \rho^{p,q}(\alpha, \beta, f')$, we have $\sigma^{p,q}(\alpha, \beta, f') \leq \sigma^{p,q}(\alpha, \beta, f)$.

Therefore $\sigma^{p,q}(\alpha, \beta, f') = \sigma^{p,q}(\alpha, \beta, f)$. □

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