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On Codes in $R = \mathbb{Z}_3 + u\mathbb{Z}_3$, $u^2 = 0$ with Covering Radius

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Abstract
In this paper, lower bound and upper bound on the covering radius of codes and block repetition codes over $R = \mathbb{Z}_3 + u\mathbb{Z}_3$, $u^2 = 0$ with various weights are found. In addition, the corresponding upper bounds and lower bounds for same length of block repetition codes and different length of block repetition codes are also given.

Keywords: Codes over finite ring, covering radius, generator matrix, various weight.
MSC (2020): 20G30, 20G35.

1 Introduction

In the last three decade, much researchers have more interest in codes of finite rings and the recent years, especially in the rings $\mathbb{Z}_{2k}$, where $2k$ is the ring of integers modulo $2k$. In several papers have mentioned codes over $\mathbb{Z}_4$ received much attention [1, 4–8]. Good binary linear and non-linear codes can be obtained from codes over $\mathbb{Z}_4$ via the gray map. In[2, 3, 12–14], the covering radius of binary linear codes are studied.

The author gave upper and lower bounds on the covering radius of codes in $\mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^3}$ and $R = F_3 + uF_3$, $u^2 = 0$, has been investigated with assigned to different distance [9–11]. In this paper, I obtain the covering radius of some particular codes over $R = \mathbb{Z}_3 + u\mathbb{Z}_3$, $u^2 = 0$ and consider the finite ring R of integers modulo 3.

2 Preliminaries

Let C subset of R is said to be a *code* and C is *subspace* of R is called a R-*linear code*, where R is an sub module of R^n, where n is an length of the code. The elements of C is call to a *codeword* of C. The number of non-zero components in a codeword c is said to be Hamming weight and it is denoted by $wt_H(c)$ and the smallest weights of all the non-zero codewords in a Code C is call a minimum Hamming weights of a code C and it is denoted by $wt_H(c)$.

Let $x, y \in R^n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, the minimum distance of code C is $d(C) = \min\{d(x, y)|x, y \in C, x-y \neq 0 \in C\} = \min\{wt(x-y)|x-y \neq 0 \in C\} = \min\{wt(c)|c \in C \neq 0\} = wt(C)$ is call to *Hamming distance* amid any distinct vectors $x, y \in R^n$ and is denoted by $d_H(x, y) = wt_H(x-y)$. The minimum Hamming distance amid distinct pairs of codewords of a code C and denoted by $d_H(C) = wt_H(C)$.

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In[17], the homogeneous weights(hw) as given

$$w_{hw}(h) = \begin{cases} 0 & \text{if} \quad h = 0 \\ 3 & \text{if} \quad h = 3, 6 \\ 2 & \text{if} \quad \text{otherwise.} \end{cases}$$

The *linear code* C of length n in R and if $c_1, c_2 \in C \subseteq R^n$ be any two different codewords is defined as $d_{hw}(c_1, c_2) = w_{hw}(c_1 - c_2)$. Thus $d_{hw}(z_1, C) = \min\{d_{hw}(z_1, c) | \forall c \in C\}$, for any $z_1 \in R^n$. Therefore, the parameters of code is an (n, W, d_{hw}), where hw is homogeneous weights.

In euclidean weights(ew), let $z \in R$ is

$$w_{ew}(z) = \begin{cases} 0 & \text{if} \quad z = 0 \\ 1 & \text{if} \quad z = 1, 2 + 2u \\ 4 & \text{if} \quad z = 2, 1 + 2u \\ 9 & \text{if} \quad z = u, 2u \\ 16 & \text{if} \quad z = 1 + u, 2 + u \end{cases}$$

in [16], where z be a codeword of code in R^n.

Defined as $d_{ew}(c_1, c_2) = wt_{ew}(c_1 - c_2)$, here $c_1, c_2 \in C \subseteq R^n$ be any two different codewords of linear code C with length n over R. Then $d_{ew}(z_1, C) = \min\{d_{ew}(z_1, c) | \forall c \in C\}$, for any $z_1 \in R^n$ and the parameters of Euclidean weights code is an (n, W, d_{ew}), where W is the number of codewords in R.

3 Repetition code with covering radius

Let

$$r_d(C) = \max_{s \in R^n} \left\{ \min_{c \in C} \{d(s, c)\} \right\},$$

where C is a code and $r_d(C)$ is a *covering radius* of the code C with different distance(d), such as homogeneous and euclidean.

Block repetition code

A *Block Repetition Code* is a fundamental type of error-correcting channel code in which each individual message bit is repeated multiple times to create a longer codeword.

Example 3.1. Consider a simple $(5, 1)$ repetition binary code, where each single information bit is repeated three times. [Information bit: 0, Codeword: 00000 and Information bit: 1, Codeword: 11111]. Therefore, the binary Block repetition code $C = \{00000, 11111\}$.

Example 3.2. Let C be a block repetition code in R. Thus $C = \{0000, 1111, 2u\,2u\,2u\,2u\}$ be an $(4, 1)$ block repetition code and generated by $G = [uuuu]$.

Let C be a q-ary repetition code C over $\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field consists of the $\{0, 1, \alpha_2, \alpha_3, \cdots, \alpha_{q-1}\}$ and the parameters of the code $C = \{\bar{\alpha} | \alpha \in \mathbb{F}_q\}$, where $\bar{\alpha} = \alpha\alpha \cdots \alpha$ is an $[n, 1, n]$ code. Ref.[15], the covering radius of the code C is $\lfloor \frac{n(q-1)}{q} \rfloor$.

The parameters of the block repetition code C of size n, with $[n(q-1), 1, n(q-1)]$ be a generated by $G = [\overbrace{11 \cdots 1}^{n} \overbrace{\alpha_2\alpha_2 \cdots \alpha_2}^{n} \ldots \overbrace{\alpha_{q-1}\alpha_{q-1} \cdots \alpha_{q-1}}^{n}]$.

In [15], that the covering radius of the code C is $\lfloor \frac{n(q-1)^2}{q} \rfloor$, since it will be equivalent to a repetition code of length $(q-1)n$. The *parameter of repetition code* is (n, k), here n is the codeword length and k is the message length or dimension. There are two type of repetition codes of length n viz.

1. Type A:

   Let $G_A = [\overbrace{11\cdots 1}^{n}]$ be a Generator matrix with unit element in R. This Generator matrix generated by the code C_A of the parameters $[n, 1, 2n, 2n]$.

2. Type B:

   Let $G_B = [\overbrace{u\ u\cdots u}^{n}]$, $[\overbrace{u\ 2u\ u\ 2u\cdots u\ 2u}^{n}]$ or $[\overbrace{2u\ u\ 2u\ u\cdots 2u\ u}^{n}]$ and $[\overbrace{2u\ 2u\cdots 2u}^{n}]$ are the Generator matrix with zero divisor element in R. This Generator matrix generated by the code C_B of the parameters $(n, 3, 3n, 9n)$.

Theorem 3.3.

- $r_{hw}(C_A) = 2n$,
- $\frac{3n}{2} \leq r_{hw}(C_B) \leq 2n$,
- $r_{ew}(C_A) = \frac{20n}{3}$,
- $\frac{9n}{2} \leq r_{ew}(C_B) \leq \frac{20n}{3}$.

Proof. Let $h \in R^n$ with ξ_0 times $0'$s, ξ_1 times $1'$s, ξ_2 times $2'$s, ξ_3 times u's ξ_4 times $1+u'$s, ξ_5 times $2+u'$s, ξ_6 times $2u'$s, ξ_7 times $1+2u'$s ξ_8 times $u+2u'$s, in x and $\xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8 = n$. The code $c_{i,\ i\ =\ 0\ \text{to}\ 8} \in \{\alpha(C_I)|\alpha \in R\}$. Then

$$\begin{array}{rcl} d_{hw}(h, c_0) & = & w_{hw}(h - 00\cdots 0) \\ & = & 0\xi_0 + 1\xi_1 + 2\xi_2 + u\xi_3 + (1+u)\xi_4 + (2+u)\xi_5 + 2u\xi_6 \\ & & +(1+2u)\xi_7 + (2+2u)\xi_8 \\ & = & 2\xi_1 + 2\xi_2 + 3\xi_3 + 2\xi_4 + 2\xi_5 + 3\xi_6 + 2\xi_7 + 2\xi_8 \\ d_{hw}(h, c_0) & = & n - \xi_0 + \xi_1 + \xi_2 + 2\xi_3 + \xi_4 + \xi_5 + 2\xi_6 + \xi_7 + \xi_8. \end{array}$$

Similarly,
$d_{hw}(h, c_1) = n - \xi_1 + \xi_2 + \xi_3 + 2\xi_4 + \xi_5 + \xi_6 + 2\xi_7 + \xi_8 + \xi_0$.
$d_{hw}(h, c_2) = n - \xi_2 + \xi_3 + \xi_4 + 2\xi_5 + \xi_6 + \xi_7 + 2\xi_8 + \xi_0 + \xi_1$.
$d_{hw}(h, c_3) = n - \xi_3 + \xi_4 + \xi_5 + 2\xi_6 + \xi_7 + \xi_8 + 2\xi_0 + \xi_1 + \xi_2$.
$d_{hw}(h, c_4) = n - \xi_4 + \xi_5 + \xi_6 + 2\xi_7 + \xi_8 + \xi_0 + 2\xi_1 + \xi_2 + \xi_3$.
$d_{hw}(h, c_5) = n - \xi_5 + \xi_6 + \xi_7 + 2\xi_8 + \xi_0 + \xi_1 + 2\xi_2 + \xi_3 + \xi_4$.
$d_{hw}(h, c_6) = n - \xi_6 + \xi_7 + \xi_8 + 2\xi_0 + \xi_1 + \xi_2 + 2\xi_3 + \xi_4 + \xi_5$.
$d_{hw}(h, c_7) = n - \xi_7 + \xi_8 + \xi_0 + 2\xi_1 + \xi_2 + \xi_3 + 2\xi_4 + \xi_5 + \xi_6$.
$d_{hw}(h, c_8) = n - \xi_8 + \xi_0 + \xi_1 + 2\xi_2 + \xi_3 + \xi_4 + 2\xi_5 + \xi_6 + \xi_7$.

Therefore, $d_{hw}(h, C_A) = \min\{d_{hw}(h, c_0), d_{hw}(h, c_1), d_{hw}(h, c_2), d_{hw}(h, c_3), d_{hw}(h, c_4), d_{hw}(h, c_5), d_{hw}(h, c_6), d_{hw}(h, c_7), d_{hw}(h, c_8)\} \leq n + \frac{9n}{9} = 2n$, where $\xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8 = n$. Therefore, $r_{hw}(C_A) \leq 2n$.

Consider

$$h = \overbrace{00\cdots 0}^{s}\,\overbrace{11\cdots 1}^{s}\,\overbrace{22\cdots 2}^{s}\,\overbrace{uu\cdots u}^{s}\,\overbrace{1+u\ 1+u\cdots 1+u}^{s}$$

$$\overbrace{2+u\ 2+u\cdots 2+u}^{s}\,\overbrace{2u\ 2u\cdots 2u}^{s}\,\overbrace{1+2u\ 1+2u\cdots 1+2u}^{s}\,\overbrace{2+2u\ 2+2u\cdots 2+2u}^{n-8s} \in R^n,$$

where $s = \lfloor \frac{n}{9} \rfloor$, then

$$d_{hw}(h, c_0) = 2n,\ d_{hw}(h, c_1) = 2n,\ d_{hw}(h, c_2) = 2n,$$

$$d_{hw}(h, c_3) = 2n,\ d_{hw}(h, c_4) = 2n,\ d_{hw}(h, c_5) = 2n,$$

$$d_{hw}(h, c_6) = 2n, d_{hw}(h, c_7) = 2n,\ d_{hw}(h, c_8) = 2n.$$

$\therefore\ r_{hw}(C) \geq \min\{2n, 2n, 2n\} = 2n \geq 2n$. Thus $r_{hw}(C_A) = 2n$.

Let $h = \overbrace{u\ u \cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} \in R^n$. The code $C_B = \{\alpha(u\ u\cdots u) \mid \alpha \in R^n\}$, that is $C_{II} = \{00\cdots 0, u\ u\cdots u, 2u\ 2u\cdots 2u\}$, generated by $[u\ u\cdots u]$ is an $(n, 3, 3n)$ code. Then,

$d_{hw}(h, 00\cdots 0) = wt_{hw}(\overbrace{u\ u\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - 00\cdots 0) = \frac{n}{2}wt_{hw}(u) = \frac{3n}{2}$,

$d_{hw}(h, u\ u\cdots u) = wt_{hw}(\overbrace{u\ u\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - u\ u\cdots u) = \frac{n}{2}wt_{hw}(2u) = \frac{3n}{2}$ and

$d_{hw}(h, 2u\ 2u\cdots 2u) = wt_{hw}(\overbrace{u\ u\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} - 2u\ 2u\cdots 2u) = \frac{n}{2}wt_{hw}(2u) + \frac{n}{2}wt_{hw}(u) = 3n$.
Therefore, $d_{hw}(h, C_B) = \min\{\frac{3n}{2}, 3n\} = \frac{3n}{2}$. Thus, $r_{hw}(C_B) \geq \frac{3n}{2}$.

Let $h \in R^n$ be any codeword and take h has ξ_0 links as $0'$s, ξ_1 links as $1'$s, ξ_2 links as $2'$s, ξ_3 links as u's, ξ_4 links as $1 + u'$s, ξ_5 links as $2 + u'$s, ξ_6 links as $2u'$s, ξ_7 links as $1 + 2u'$s and ξ_8 links as $2 + 2u'$s, with $\xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_8 = n$. Since $C_B = \{00\cdots 0, u\ u\cdots u, 2u\ 2u\cdots 2u\}$. So, $r_{hw}(C_{II}) \leq 2n$. Hence, $\frac{3n}{2} \leq r_{hw}(C_B) \leq 2n$. Hence proved for homogeneous weight in C_I and C_B.
using above idea, $r_{ew}(C_A) = \frac{20n}{3}$ and $\frac{9n}{2} \leq r_{ew}(C_B) \leq \frac{20n}{3}$. □

Same size of length n

The covering radius for the same size of block repetition code C of a length n with respect to different weights are given.

Same size of length(n) of the block repetition code C of a parameters $[6n, 1, 12n, 42n]$, that is $BRep^{6n}$: $[6n, 1, 12n, 42n]$ is generated by

$$G_1 = [\overbrace{1\cdots 1}^{n}\overbrace{2\cdots 2}^{n}\overbrace{1+u\cdots 1+u}^{n}\overbrace{2+u\cdots 2+u}^{n}\overbrace{1+2u\cdots 1+2u}^{n}\overbrace{2u\cdots 2u}^{n}] \tag{3.1}$$

Theorem 3.4.

1. $r_{hw}(BRep^{6n}) = 12n$.
2. $r_{ew}(BRep^{6n}) = 40n$.
3. $\frac{7n}{2} \leq r_{hw}(BRep^{2n}) \leq 4n$ in $G = [\overbrace{11\cdots 1}^{n}\overbrace{u\ u\cdots u}^{n}]$.
4. $\frac{67n}{6} \leq r_{ew}(BRep^{2n}) \leq \frac{40n}{3}$ in $G = [\overbrace{11\cdots 1}^{n}\overbrace{u\ u\cdots u}^{n}]$.

where n same size of length in R.

Proof. From (3.1), [12] and by using theorem (3.3), thus

$$r_{hw}(BRep^{6n}) \geq 12n \tag{3.2}$$

let $y = (z_1 \mid z_2 \mid z_3 \mid z_4 \mid z_5 \mid z_6) \in R^{6n}$, where $z_i \in R^{6n}, i = 1$ to 6. Let us take in u_1, 0 appears r_0 times, 1 appears r_1 times, 2 appears r_2 times u appears r_3 times $1 + u$ appears r_4 times, $2 + u$ appears $r_5$times, $2u$ appears $r_6$times, $1 + 2u$ appears $r_7$times and $2 + 2u$ appears $r_8$times, in u_2, j appears s_j times, in

u_3, j appears t_j times, in u_4, j appears v_j times, in u_5, j appears u_j times, in u_6, j appears w_j times, with $\sum_j r_j = \sum_j s_j = \sum_j t_j = n = \sum_j v_j = \sum_j u_j = \sum_j w_j$ and $c_j \in \{\alpha(G_1) | \alpha \in R^{6n}\}$, $j = 0$ to 8.

Then, $d_{hw}(y, BRep^{6n}) = \min\{d_{hw}(y, c_j) | j = 0 \text{ to } 8\}$ is less than or equal to $6n + \frac{6n+6n+6n+6n+6n+6n+6n+6n+6n}{9} = 12n$. Thus, $d_{hw}(y, BRep^{6n}) \leq 12n$. Hence,

$$r_{hw}(BRep^{6n}) \leq 12n \tag{3.3}$$

By (3.2) and (3.3). Then,

$$r_{hw}(BRep^{6n}) = 12n.$$

The remaining Proof is follows from the above idea. □

Different size of the length (m_1, m_2)

In this section, only detail that the covering radius are found for different size of block repetition code with assigned to different weight.

There are two different size of length (m_1, m_2) of the block repetition code C in R, $BRep^{m_1+m_2} : [m_1 + m_2, 1, \min\{2m_1, 2m_1 + 3m_2\}, \min\{9m_1, 4m_1 + 9m_2\}]$ generated by $G_3 = [\overbrace{11\cdots1}^{m_1}\overbrace{u\,u\cdots u}^{m_2}]$ and also found the following

Theorem 3.5. In different size of the length (m_1, m_2),

- $2m_1 + m_2 \leq r_{hw}(BRep^{m_1+m_2}) \leq 2m_1 + 2m_2$,
- $\frac{20m_1}{3} + \frac{9m_2}{2} \leq r_{hw}(BRep^{m_1+m_2}) \leq \frac{20}{3}(m_1 + m_2)$.

Proof. Use to Theorem (3.4) and apply the two different size of length(m_1, m_2). □

In general, the Block repetition code for six different size of length $BRep^{m_1+m_2+m_3+m_4+m_5+m_6} : [m_1 + m_2 + m_3 + m_4 + m_5 + m_6, 1, \min\{2(m_1 + m_2 + m_3 + m_4 + m_5 + m_6), 3(m_1 + m_2 + m_3 + m_4 + m_5 + m_6), \min\{(m_1 + 4m_2 + 9m_3 + 16m_4 + 4m_5 + m_6), 6(m_1 + m_2 + m_3 + m_4 + m_5 + m_6)\}]$ generated by $G_4 = [\overbrace{1\cdots1}^{m_1}\overbrace{2\cdots2}^{m_2}\overbrace{1+u\cdots1+u}^{m_3}\overbrace{2+u\cdots2+u}^{m_4}\overbrace{1+2u\cdots1+2u}^{m_5}\overbrace{2u\cdots2u}^{m_6}]$.

Theorem 3.6.

- $r_{hw}(BRep^{m_1+m_2+m_3+m_4+m_5+m_6}) = 2(m_1 + m_2 + m_3 + m_4 + m_5 + m_6)$
- $r_{ew}(BRep^{m_1+m_2+m_3+m_4+m_5+m_6}) = \frac{20}{3}(m_1 + m_2 + m_3 + m_4 + m_5 + m_6)$.

Proof. By theorem (3.5) and apply to six different size of length. □

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