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A Set of General Theorems for Cyclic Inequalities

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Abstract
This paper introduces a new set of general theorems for cyclic inequalities. A distinguishing feature of these results is the integration of an arbitrary function, which is assumed to be convex and, in specific cases, increasing. In total, we establish sixteen main theorems and two supplementary theorems. Each result is accompanied by a detailed proof and concrete examples to illustrate its practical utility.

Keywords: Cyclic inequalities, Nesbitt inequality, Jensen inequality, Titu lemma.

1 Introduction

Cyclic inequalities form an important class of inequalities characterized by the ordered interaction of variables. Unlike symmetric inequalities, cyclic inequalities are not invariant under arbitrary permutations, which makes their study both challenging and rich in applications.

A fundamental example is the classical Nesbitt inequality, which states that for any $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

or, equivalently,

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}.$$

Here, the notation $\sum_{cyc}$ denotes a cyclic sum over the variables a, b, and c. This means the subsequent terms in the sum are generated by applying the cyclic permutation $a \to b \to c \to a$ to the initial expression.

Since its introduction, the Nesbitt inequality has inspired numerous generalizations incorporating varied cyclic structures, weighted terms, and distinct proof techniques. A representative example is provided by [4, Theorem 2.1], which states that for any $a, b, c, p > 0$,

$$\frac{a}{(b+c)^p} + \frac{b}{(c+a)^p} + \frac{c}{(a+b)^p} \geq \left(\frac{3}{2}\right)^p \frac{1}{(a+b+c)^{p-1}}$$

or, equivalently,

$$\sum_{cyc} \frac{a}{(b+c)^p} \geq \left(\frac{3}{2}\right)^p \frac{1}{(a+b+c)^{p-1}}.$$

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When $p = 1$, this inequality reduces directly to the classical Nesbitt inequality. The formulation of such extensions continues to be an active direction of research in inequality theory (see, e.g., [1–8]).

In this paper, we establish a new set of general theorems for cyclic inequalities characterized by the inclusion of an arbitrary function f, which is assumed to be convex and, in specific cases, increasing. The resulting bounds consistently take the form

$$\alpha(a+b+c)f(\beta),$$

where $\alpha, \beta > 0$ are constants. This formulation renders our results widely applicable across a variety of analytical and algebraic settings. In total, we present sixteen main theorems and two supplementary results. Each theorem is accompanied by a rigorous proof and concrete examples to demonstrate its practical utility.

The remainder of this paper is organized as follows: Section 2 presents the sixteen main theorems, followed by two supplementary theorems in Section 3. Section 4 provides a convenient summary of the core inequalities, and Section 5 offers concluding remarks.

2 Set of sixteen theorems

This section presents our general theorems for cyclic inequalities. These results are sharp, with equality achieved under a specific configuration of a, b, and c. For a concise overview of the core inequalities of these theorems, we refer the reader to Section 4.

2.1 First theorem

Our first theorem, along with its proof and three examples, is presented below.

Theorem 2.1. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$cf\left(\frac{a}{b}\right) + af\left(\frac{b}{c}\right) + bf\left(\frac{c}{a}\right) \geq (a+b+c)f(1).$$

The equality holds for $a = b = c$.

Proof. The (weighted) Jensen inequality applied to the convex function f gives

$$\begin{aligned}
cf\left(\frac{a}{b}\right) + af\left(\frac{b}{c}\right) + bf\left(\frac{c}{a}\right) &= \sum_{cyc} cf\left(\frac{a}{b}\right) \\
&= (a+b+c)\sum_{cyc}\left(\frac{c}{a+b+c}\right) f\left(\frac{a}{b}\right) \\
&\geq (a+b+c) f\left(\frac{\sum_{cyc}\frac{ca}{b}}{a+b+c}\right).
\end{aligned}$$

Using the standard inequality $1/x + x \geq 2$ for any $x > 0$, we have

$$\begin{aligned}
\sum_{cyc}\frac{ca}{b} &= \frac{ca}{b} + \frac{ab}{c} + \frac{bc}{a} \\
&= \frac{a}{2}\left(\frac{c}{b} + \frac{b}{c}\right) + \frac{b}{2}\left(\frac{a}{c} + \frac{c}{a}\right) + \frac{c}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\
&\geq \frac{a}{2}2 + \frac{b}{2}2 + \frac{c}{2}2 = a+b+c.
\end{aligned}$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{ca}{b}}{a+b+c}\right) \geq f\left(\frac{a+b+c}{a+b+c}\right) = f(1).$$

Combining the inequalities above, we obtain

$$cf\left(\frac{a}{b}\right)+af\left(\frac{b}{c}\right)+bf\left(\frac{c}{a}\right)\geq(a+b+c)f(1).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(1)$. This concludes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Three examples of this theorem are presented below.

Applying Theorem 2.1 to the convex and increasing function $f(x)=\sqrt{x^2+1}$ gives

$$c\sqrt{\left(\frac{a}{b}\right)^2+1}+a\sqrt{\left(\frac{b}{c}\right)^2+1}+b\sqrt{\left(\frac{c}{a}\right)^2+1}\geq(a+b+c)\sqrt{2}.$$

Applying Theorem 2.1 to the convex and increasing function $f(x)=q^x$ with $q>1$ gives

$$cq^{a/b}+aq^{b/c}+bq^{c/a}\geq(a+b+c)q.$$

Applying Theorem 2.1 to the convex and increasing function $f(x)=x\arctan(x)$ (with $f''(x)=2/(1+x^2)^2\geq0$) gives

$$\frac{ac}{b}\arctan\left(\frac{a}{b}\right)+\frac{ab}{c}\arctan\left(\frac{b}{c}\right)+\frac{bc}{a}\arctan\left(\frac{c}{a}\right)\geq(a+b+c)\frac{\pi}{4}.$$

2.2 Second theorem

Our second theorem, along with its proof and three examples, is presented below.

Theorem 2.2. Let $f:(0,\infty)\to\mathbb{R}$ be a convex function. For any $a,b,c>0$, the following cyclic inequality holds:

$$bf\left(\frac{a}{b}\right)+cf\left(\frac{b}{c}\right)+af\left(\frac{c}{a}\right)\geq(a+b+c)f(1).$$

The equality holds for $a=b=c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&bf\left(\frac{a}{b}\right)+cf\left(\frac{b}{c}\right)+af\left(\frac{c}{a}\right)=\sum_{cyc}bf\left(\frac{a}{b}\right)\\
&=(a+b+c)\sum_{cyc}\left(\frac{b}{a+b+c}\right)f\left(\frac{a}{b}\right)\\
&\geq(a+b+c)f\left(\frac{\sum_{cyc}\frac{ba}{b}}{a+b+c}\right)=(a+b+c)f\left(\frac{\sum_{cyc}a}{a+b+c}\right)\\
&=(a+b+c)f\left(\frac{a+b+c}{a+b+c}\right)=(a+b+c)f(1).
\end{aligned}$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(1)$. This completes the proof. □

If f is concave instead of convex, the inequality is reversed.

Three examples of this theorem are presented below.

Applying Theorem 2.2 to the convex function $f(x)=1/(1+x)$ gives

$$\frac{b^2}{b+a}+\frac{c^2}{c+b}+\frac{a^2}{a+c}\geq\frac{a+b+c}{2},$$

which can be proved directly with the Titu lemma.

Applying Theorem 2.2 to the convex function $f(x) = q^x$ with $q > 0$ gives

$$bq^{a/b} + cq^{b/c} + aq^{c/a} \geq (a+b+c)q.$$

Applying Theorem 2.2 to the convex function $f(x) = x\ln(x)$ (with $f''(x) = 1/x > 0$) gives

$$a\ln\left(\frac{a}{b}\right) + b\ln\left(\frac{b}{c}\right) + c\ln\left(\frac{c}{a}\right) \geq 0,$$

so that

$$\left(\frac{a}{b}\right)^a \left(\frac{b}{c}\right)^b \left(\frac{c}{a}\right)^c \geq 1.$$

2.3 Third theorem

Our third theorem, along with its proof and three examples, is presented below.

Theorem 2.3. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$af\left(\frac{a}{b}\right) + bf\left(\frac{b}{c}\right) + cf\left(\frac{c}{a}\right) \geq (a+b+c)f(1).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned} af\left(\frac{a}{b}\right) + bf\left(\frac{b}{c}\right) + cf\left(\frac{c}{a}\right) &= \sum_{cyc} af\left(\frac{a}{b}\right) \\ &= (a+b+c)\sum_{cyc}\left(\frac{a}{a+b+c}\right) f\left(\frac{a}{b}\right) \\ &\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{a^2}{b}}{a+b+c}\right). \end{aligned}$$

Using the Titu lemma, we have

$$\sum_{cyc}\frac{a^2}{b} = \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{a+b+c} = a+b+c.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{a^2}{b}}{a+b+c}\right) \geq f\left(\frac{a+b+c}{a+b+c}\right) = f(1).$$

Combining the inequalities above, we obtain

$$af\left(\frac{a}{b}\right) + bf\left(\frac{b}{c}\right) + cf\left(\frac{c}{a}\right) \geq (a+b+c)f(1).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(1)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.
Three examples of this theorem are presented below.
Applying Theorem 2.3 to the convex and increasing function $f(x) = \sqrt{x^2+1}$ gives

$$a\sqrt{\left(\frac{a}{b}\right)^2 + 1} + b\sqrt{\left(\frac{b}{c}\right)^2 + 1} + c\sqrt{\left(\frac{c}{a}\right)^2 + 1} \geq (a+b+c)\sqrt{2}.$$

Applying Theorem 2.3 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$aq^{a/b} + bq^{b/c} + cq^{c/a} \geq (a+b+c)q.$$

Applying Theorem 2.3 to the convex and increasing function $f(x) = x\arctan(x)$ gives

$$\frac{a^2}{b}\arctan\left(\frac{a}{b}\right) + \frac{b^2}{c}\arctan\left(\frac{b}{c}\right) + \frac{c^2}{a}\arctan\left(\frac{c}{a}\right) \geq (a+b+c)\frac{\pi}{4}.$$

2.4 Fourth theorem

Our fourth theorem, along with its proof and two examples, is presented below.

Theorem 2.4. Let $f : (0, \infty) \to \mathbb{R}$ be a convex function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$(b+c)f\left(\frac{a}{b+c}\right) + (c+a)f\left(\frac{b}{c+a}\right) + (a+b)f\left(\frac{c}{a+b}\right) \geq 2(a+b+c)f\left(\frac{1}{2}\right).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f and the fact that $\sum_{cyc}(b+c) = 2(a+b+c)$ give

$$\begin{aligned}
&(b+c)f\left(\frac{a}{b+c}\right) + (c+a)f\left(\frac{b}{c+a}\right) + (a+b)f\left(\frac{c}{a+b}\right) \\
&= \sum_{cyc}(b+c)f\left(\frac{a}{b+c}\right) = 2(a+b+c)\sum_{cyc}\left(\frac{b+c}{2(a+b+c)}\right)f\left(\frac{a}{b+c}\right) \\
&\geq 2(a+b+c)f\left(\frac{\sum_{cyc}\frac{a(b+c)}{b+c}}{2(a+b+c)}\right) = 2(a+b+c)f\left(\frac{\sum_{cyc}a}{2(a+b+c)}\right) \\
&= 2(a+b+c)f\left(\frac{a+b+c}{2(a+b+c)}\right) = 2(a+b+c)f\left(\frac{1}{2}\right).
\end{aligned}$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $6af(1/2)$. This concludes the proof. □

If f is concave instead of convex, the inequality is reversed.

An extension of this inequality is that, for any $k > 0$,

$$\begin{aligned}
&(b+kc)f\left(\frac{a}{b+kc}\right) + (c+ka)f\left(\frac{b}{c+ka}\right) + (a+kb)f\left(\frac{c}{a+kb}\right) \\
&\geq (k+1)(a+b+c)f\left(\frac{1}{k+1}\right).
\end{aligned}$$

The proof is identical to that of Theorem 2.4.

Two examples of this theorem are presented below.

Applying Theorem 2.4 to the convex function $f(x) = x^p$ with $p \in \mathbb{R}\backslash(0, 1)$ gives

$$(b+c)\left(\frac{a}{b+c}\right)^p + (c+a)\left(\frac{b}{c+a}\right)^p + (a+b)\left(\frac{c}{a+b}\right)^p \geq (a+b+c)2^{1-p}.$$

Applying Theorem 2.4 to the convex function $f(x) = q^x$ with $q > 0$ gives

$$(b+c)q^{a/(b+c)} + (c+a)q^{b/(c+a)} + (a+b)q^{c/(a+b)} \geq 2(a+b+c)\sqrt{q}.$$

2.5 Fifth theorem

Our fifth theorem, along with its proof and two examples, is presented below.

Theorem 2.5. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$(a+b)f\left(\frac{a}{b+c}\right) + (b+c)f\left(\frac{b}{c+a}\right) + (c+a)f\left(\frac{c}{a+b}\right) \geq 2(a+b+c)f\left(\frac{1}{2}\right).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f and the fact that $\sum_{cyc}(b+c) = 2(a+b+c)$ give

$$\begin{aligned}
&(a+b)f\left(\frac{a}{b+c}\right) + (b+c)f\left(\frac{b}{c+a}\right) + (c+a)f\left(\frac{c}{a+b}\right) \\
&= \sum_{cyc}(a+b)f\left(\frac{a}{b+c}\right) = 2(a+b+c)\sum_{cyc}\left(\frac{a+b}{2(a+b+c)}\right)f\left(\frac{a}{b+c}\right) \\
&\geq 2(a+b+c)f\left(\frac{\sum_{cyc}\frac{a(a+b)}{b+c}}{2(a+b+c)}\right).
\end{aligned}$$

Let us now demonstrate that

$$\sum_{cyc}\frac{a(a+b)}{b+c} \geq a+b+c,$$

which requires some developments.

We have

$$\begin{aligned}
&\sum_{cyc}\frac{a(a+b)}{b+c} - (a+b+c) = \sum_{cyc}\left(\frac{a(a+b)}{b+c} - a\right) \\
&= \sum_{cyc}\frac{a(a-c)}{b+c} = \frac{a(a-c)}{b+c} + \frac{b(b-a)}{c+a} + \frac{c(c-b)}{a+b} \\
&= \frac{a(a-c)(c+a)(a+b) + b(b-a)(b+c)(a+b) + c(c-b)(b+c)(c+a)}{(a+b)(b+c)(c+a)} \\
&= \frac{(a+b+c)(a^3+b^3+c^3-a^2c-b^2a-c^2b)}{(a+b)(b+c)(c+a)}.
\end{aligned}$$

Let us now demonstrate that

$$a^3+b^3+c^3 \geq a^2c+b^2a+c^2b.$$

It follows from the Young inequality that

$$a^2c = (a^3)^{2/3}(c^3)^{1/3} \leq \frac{2a^3+c^3}{3},$$

and, similarly,

$$b^2a \leq \frac{2b^3+a^3}{3}$$

and

$$c^2b \leq \frac{2c^3+b^3}{3}.$$

Adding these three upper bound gives

$$a^2c+b^2a+c^2b \leq \frac{3a^3+3b^3+3c^3}{3} = a^3+b^3+c^3.$$

Thus, we find that

$$\sum_{cyc} \frac{a(a+b)}{b+c} - (a+b+c) = \frac{(a+b+c)(a^3+b^3+c^3-a^2c-b^2a-c^2b)}{(a+b)(b+c)(c+a)} \geq 0,$$

so that

$$\sum_{cyc} \frac{a(a+b)}{b+c} \geq a+b+c.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{a(a+b)}{b+c}}{2(a+b+c)}\right) \geq f\left(\frac{a+b+c}{2(a+b+c)}\right) = f\left(\frac{1}{2}\right).$$

Combining the inequalities above, we have

$$(a+b)f\left(\frac{a}{b+c}\right) + (b+c)f\left(\frac{b}{c+a}\right) + (c+a)f\left(\frac{c}{a+b}\right) \geq 2(a+b+c)f\left(\frac{1}{2}\right).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $6af(1/2)$. This completes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.5 to the convex and increasing function $f(x) = \sqrt{x^2+1}$ gives

$$(a+b)\sqrt{\left(\frac{a}{b+c}\right)^2+1} + (b+c)\sqrt{\left(\frac{b}{c+a}\right)^2+1} + (c+a)\sqrt{\left(\frac{c}{a+b}\right)^2+1}$$
$$\geq (a+b+c)\sqrt{5}.$$

Applying Theorem 2.5 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$(a+b)q^{a/(b+c)} + (b+c)q^{b/(c+a)} + (c+a)q^{c/(a+b)} \geq 2(a+b+c)\sqrt{q}.$$

2.6 Sixth theorem

Our sixth theorem, along with its proof and two examples, is presented below.

Theorem 2.6. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$(a+c)f\left(\frac{a}{b+c}\right) + (b+a)f\left(\frac{b}{c+a}\right) + (c+b)f\left(\frac{c}{a+b}\right) \geq 2(a+b+c)f\left(\frac{1}{2}\right).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f and the fact that $\sum_{cyc}(b+c) = 2(a+b+c)$ give

$$(a+c)f\left(\frac{a}{b+c}\right) + (b+a)f\left(\frac{b}{c+a}\right) + (c+b)f\left(\frac{c}{a+b}\right)$$
$$= \sum_{cyc}(a+c)f\left(\frac{a}{b+c}\right) = 2(a+b+c)\sum_{cyc}\left(\frac{a+c}{2(a+b+c)}\right)f\left(\frac{a}{b+c}\right)$$
$$\geq 2(a+b+c)f\left(\frac{\sum_{cyc} \frac{a(a+c)}{b+c}}{2(a+b+c)}\right).$$

Let us now demonstrate that

$$\sum_{cyc} \frac{a(a+c)}{b+c} \geq a+b+c,$$

which requires some developments.

We have

$$\begin{aligned}
&\sum_{cyc} \frac{a(a+c)}{b+c} - (a+b+c) = \sum_{cyc} \left(\frac{a(a+c)}{b+c} - a \right) \\
&= \sum_{cyc} \frac{a(a-b)}{b+c} = \frac{a(a-b)}{b+c} + \frac{b(b-c)}{c+a} + \frac{c(c-a)}{a+b} \\
&= \frac{a(a-b)(c+a)(a+b) + b(b-c)(b+c)(a+b) + c(c-a)(b+c)(c+a)}{(a+b)(b+c)(c+a)} \\
&= \frac{(a+b+c)(a^3+b^3+c^3-a^2b-b^2c-c^2a)}{(a+b)(b+c)(c+a)}.
\end{aligned}$$

Let us now demonstrate that

$$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a.$$

It follows from the Young inequality that

$$a^2b = (a^3)^{2/3}(b^3)^{1/3} \leq \frac{2a^3+b^3}{3},$$

and, similarly,

$$b^2c \leq \frac{2b^3+c^3}{3}$$

and

$$c^2a \leq \frac{2c^3+a^3}{3}.$$

Adding these three upper bound gives

$$a^2b+b^2c+c^2a \leq \frac{3a^3+3b^3+3c^3}{3} = a^3+b^3+c^3.$$

Thus, we establish that

$$\sum_{cyc} \frac{a(a+c)}{b+c} - (a+b+c) = \frac{(a+b+c)(a^3+b^3+c^3-a^2b-b^2c-c^2a)}{(a+b)(b+c)(c+a)} \geq 0,$$

so that

$$\sum_{cyc} \frac{a(a+c)}{b+c} \geq a+b+c.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{a(a+c)}{b+c}}{2(a+b+c)} \right) \geq f\left(\frac{a+b+c}{2(a+b+c)} \right) = f\left(\frac{1}{2}\right).$$

Combining the inequalities above, we obtain

$$(a+c)f\left(\frac{a}{b+c}\right) + (b+a)f\left(\frac{b}{c+a}\right) + (c+b)f\left(\frac{c}{a+b}\right) \geq 2(a+b+c)f\left(\frac{1}{2}\right).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $6af(1/2)$. This concludes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.6 to the convex and increasing function $f(x) = x\ln(1+x)$ (with $f''(x) = (x+2)/(1+x)^2 \geq 0$) gives

$$\begin{aligned}&(a+c)\frac{a}{b+c}\ln\left(1+\frac{a}{b+c}\right)+(b+a)\frac{b}{c+a}\ln\left(1+\frac{b}{c+a}\right)\\&+(c+b)\frac{c}{a+b}\ln\left(1+\frac{c}{a+b}\right)\\&\geq (a+b+c)\ln\left(\frac{3}{2}\right),\end{aligned}$$

so that

$$\begin{aligned}&\left(1+\frac{a}{b+c}\right)^{(a+c)a/(b+c)}\left(1+\frac{b}{c+a}\right)^{(b+a)b/(c+a)}\left(1+\frac{c}{a+b}\right)^{(c+b)c/(a+b)}\\&\geq\left(\frac{3}{2}\right)^{(a+b+c)}.\end{aligned}$$

Applying Theorem 2.6 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$(a+c)q^{a/(b+c)}+(b+a)q^{b/(c+a)}+(c+b)q^{c/(a+b)}\geq 2(a+b+c)\sqrt{q}.$$

2.7 Seventh theorem

Our seventh theorem, along with its proof and three examples, is presented below.

Theorem 2.7. Let $f : (0,\infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$af\left(\frac{a}{b+c}\right)+bf\left(\frac{b}{c+a}\right)+cf\left(\frac{c}{a+b}\right)\geq (a+b+c)f\left(\frac{1}{2}\right).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}&af\left(\frac{a}{b+c}\right)+bf\left(\frac{b}{c+a}\right)+cf\left(\frac{c}{a+b}\right)\\&=\sum_{cyc}af\left(\frac{a}{b+c}\right)=(a+b+c)\sum_{cyc}\left(\frac{a}{a+b+c}\right)f\left(\frac{a}{b+c}\right)\\&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{a^2}{b+c}}{a+b+c}\right).\end{aligned}$$

Using the Titu lemma, we have

$$\begin{aligned}&\sum_{cyc}\frac{a^2}{b+c}=\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{(a+b+c)^2}{(b+c)+(c+a)+(a+b)}\\&=\frac{(a+b+c)^2}{2(a+b+c)}=\frac{a+b+c}{2}.\end{aligned}$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{a^2}{b+c}}{a+b+c}\right)\geq f\left(\frac{a+b+c}{2(a+b+c)}\right)=f\left(\frac{1}{2}\right).$$

Combining the inequalities above, we obtain

$$af\left(\frac{a}{b+c}\right)+bf\left(\frac{b}{c+a}\right)+cf\left(\frac{c}{a+b}\right)\geq(a+b+c)f\left(\frac{1}{2}\right).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(1/2)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two variants of this inequality are

$$af\left(\frac{a}{a+b}\right)+bf\left(\frac{b}{b+c}\right)+cf\left(\frac{c}{c+a}\right)\geq(a+b+c)f\left(\frac{1}{2}\right)$$

and

$$af\left(\frac{a}{a+c}\right)+bf\left(\frac{b}{b+a}\right)+cf\left(\frac{c}{c+b}\right)\geq(a+b+c)f\left(\frac{1}{2}\right).$$

Their proofs are identical to that of Theorem 2.7.

Three examples of this theorem are presented below.

Applying Theorem 2.7 to the convex and increasing function $f(x)=x^p$ with $p\geq 1$ gives

$$a\left(\frac{a}{b+c}\right)^p+b\left(\frac{b}{c+a}\right)^p+c\left(\frac{c}{a+b}\right)^p\geq(a+b+c)2^{-p}.$$

Applying Theorem 2.7 to the convex and increasing function $f(x)=q^x$ with $q>1$ gives

$$aq^{a/(b+c)}+bq^{b/(c+a)}+cq^{c/(a+b)}\geq(a+b+c)\sqrt{q}.$$

Applying Theorem 2.7 to the convex and increasing function $f(x)=x\ln(1+x)$ gives

$$\begin{aligned}&\frac{a^2}{b+c}\ln\left(1+\frac{a}{b+c}\right)+\frac{b^2}{c+a}\ln\left(1+\frac{b}{c+a}\right)+\frac{c^2}{a+b}\ln\left(1+\frac{c}{a+b}\right)\\&\geq(a+b+c)\frac{1}{2}\ln\left(\frac{3}{2}\right),\end{aligned}$$

so that

$$\left(1+\frac{a}{b+c}\right)^{a^2/(b+c)}\left(1+\frac{b}{c+a}\right)^{b^2/(c+a)}\left(1+\frac{c}{a+b}\right)^{c^2/(a+b)}\geq\left(\frac{3}{2}\right)^{(a+b+c)/2}.$$

2.8 Eighth theorem

Our eighth theorem, along with its proof and two examples, is presented below.

Theorem 2.8. Let $f:(0,\infty)\to\mathbb{R}$ be a convex function. For any $a,b,c>0$, the following cyclic inequality holds:

$$cf\left(\frac{a+b}{c}\right)+af\left(\frac{b+c}{a}\right)+bf\left(\frac{c+a}{b}\right)\geq(a+b+c)f(2).$$

The equality holds for $a=b=c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}&cf\left(\frac{a+b}{c}\right)+af\left(\frac{b+c}{a}\right)+bf\left(\frac{c+a}{b}\right)\\&=\sum_{cyc}cf\left(\frac{a+b}{c}\right)=(a+b+c)\sum_{cyc}\left(\frac{c}{a+b+c}\right)f\left(\frac{a+b}{c}\right)\\&\geq(a+b+c)f\left(\frac{\sum_{cyc}\frac{c(a+b)}{c}}{a+b+c}\right)=(a+b+c)f\left(\frac{\sum_{cyc}(a+b)}{a+b+c}\right)\\&=(a+b+c)f\left(\frac{2(a+b+c)}{a+b+c}\right)=(a+b+c)f(2).\end{aligned}$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(2)$. This concludes the proof. $\square$

If f is concave instead of convex, the inequality is reversed.
An extension of this inequality is that, for any $k > 0$,

$$cf\left(\frac{a+kb}{c}\right) + af\left(\frac{b+kc}{a}\right) + bf\left(\frac{c+ka}{b}\right) \geq (a+b+c)f(k+1).$$

The proof is identical to that of Theorem 2.8.
Two examples of this theorem are presented below.
Applying Theorem 2.8 to the convex function $f(x) = -\ln(x)$ gives

$$-c\ln\left(\frac{a+b}{c}\right) - a\ln\left(\frac{b+c}{a}\right) - b\ln\left(\frac{c+a}{b}\right) \geq -(a+b+c)\ln(2),$$

so that

$$\left(\frac{a+b}{c}\right)^c \left(\frac{b+c}{a}\right)^a \left(\frac{c+a}{b}\right)^b \leq 2^{a+b+c}.$$

Applying Theorem 2.8 to the convex function $f(x) = q^x$ with $q > 0$ gives

$$cq^{(a+b)/c} + aq^{(b+c)/a} + bq^{(c+a)/b} \geq (a+b+c)q^2.$$

2.9 Ninth theorem

Our ninth theorem, along with its proof and two examples, is presented below.

Theorem 2.9. Let $f : (0,\infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$af\left(\frac{a+b}{c}\right) + bf\left(\frac{b+c}{a}\right) + cf\left(\frac{c+a}{b}\right) \geq (a+b+c)f(2).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&af\left(\frac{a+b}{c}\right) + bf\left(\frac{b+c}{a}\right) + cf\left(\frac{c+a}{b}\right)\\
&= \sum_{cyc} af\left(\frac{a+b}{c}\right) = (a+b+c)\sum_{cyc}\left(\frac{a}{a+b+c}\right)f\left(\frac{a+b}{c}\right)\\
&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{a(a+b)}{c}}{a+b+c}\right).
\end{aligned}$$

Let us demonstrate that

$$\sum_{cyc}\frac{a(a+b)}{c} \geq 2(a+b+c).$$

We have

$$\begin{aligned}
\sum_{cyc}\frac{a(a+b)}{c} &= \frac{a^2+ab}{c} + \frac{b^2+bc}{a} + \frac{c^2+ca}{b}\\
&= \underbrace{\left(\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}\right)}_{S_1} + \underbrace{\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)}_{S_2}.
\end{aligned}$$

Using the Titu lemma, we obtain

$$S_1 = \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} \geq \frac{(a+b+c)^2}{c+a+b} = a+b+c.$$

Using the standard inequality $1/x + x \geq 2$ for any $x > 0$, we have

$$\begin{aligned} S_2 &= \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \\ &= \frac{a}{2}\left(\frac{c}{b} + \frac{b}{c}\right) + \frac{b}{2}\left(\frac{a}{c} + \frac{c}{a}\right) + \frac{c}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\ &\geq \frac{a}{2}2 + \frac{b}{2}2 + \frac{c}{2}2 = a+b+c. \end{aligned}$$

Thus, we find that

$$\sum_{cyc} \frac{a(a+b)}{c} \geq 2(a+b+c).$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{a(a+b)}{c}}{a+b+c}\right) \geq f\left(\frac{2(a+b+c)}{a+b+c}\right) = f(2).$$

Combining the inequalities above, we have

$$af\left(\frac{a+b}{c}\right) + bf\left(\frac{b+c}{a}\right) + cf\left(\frac{c+a}{b}\right) \geq (a+b+c)f(2).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(2)$. This concludes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.9 to the convex and increasing function $f(x) = x^p$ with $p \geq 1$ gives

$$a\left(\frac{a+b}{c}\right)^p + b\left(\frac{b+c}{a}\right)^p + c\left(\frac{c+a}{b}\right)^p \geq (a+b+c)2^p.$$

Applying Theorem 2.9 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$aq^{(a+b)/c} + bq^{(b+c)/a} + cq^{(c+a)/b} \geq (a+b+c)q^2.$$

2.10 Tenth theorem

Our tenth theorem, along with its proof and two examples, is presented below.

Theorem 2.10. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$bf\left(\frac{a+b}{c}\right) + cf\left(\frac{b+c}{a}\right) + af\left(\frac{c+a}{b}\right) \geq (a+b+c)f(2).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&bf\left(\frac{a+b}{c}\right)+cf\left(\frac{b+c}{a}\right)+af\left(\frac{c+a}{b}\right)\\
&=\sum_{cyc} bf\left(\frac{a+b}{c}\right)=(a+b+c)\sum_{cyc}\left(\frac{b}{a+b+c}\right)f\left(\frac{a+b}{c}\right)\\
&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{b(a+b)}{c}}{a+b+c}\right).
\end{aligned}$$

Let us demonstrate that

$$\sum_{cyc}\frac{b(a+b)}{c}\geq 2(a+b+c).$$

We have

$$\begin{aligned}
\sum_{cyc}\frac{b(a+b)}{c}&=\frac{ba+b^2}{c}+\frac{cb+c^2}{a}+\frac{ac+a^2}{b}\\
&=\underbrace{\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)}_{S_1}+\underbrace{\left(\frac{ba}{c}+\frac{cb}{a}+\frac{ac}{b}\right)}_{S_2}.
\end{aligned}$$

Using the Titu lemma, we obtain

$$S_1=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)^2}{b+c+a}=a+b+c.$$

Using the standard inequality $1/x+x\geq 2$ for any $x>0$, we have

$$\begin{aligned}
S_2&=\frac{ba}{c}+\frac{cb}{a}+\frac{ac}{b}\\
&=\frac{a}{2}\left(\frac{c}{b}+\frac{b}{c}\right)+\frac{b}{2}\left(\frac{a}{c}+\frac{c}{a}\right)+\frac{c}{2}\left(\frac{b}{a}+\frac{a}{b}\right)\\
&\geq\frac{a}{2}2+\frac{b}{2}2+\frac{c}{2}2=a+b+c.
\end{aligned}$$

Thus, we establish that

$$\sum_{cyc}\frac{b(a+b)}{c}\geq 2(a+b+c).$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{b(a+b)}{c}}{a+b+c}\right)\geq f\left(\frac{2(a+b+c)}{a+b+c}\right)=f(2).$$

Combining the inequalities above, we have

$$bf\left(\frac{a+b}{c}\right)+cf\left(\frac{b+c}{a}\right)+af\left(\frac{c+a}{b}\right)\geq(a+b+c)f(2).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(2)$. This concludes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.10 to the convex and increasing function $f(x)=\sqrt{x^2+1}$ gives

$$b\sqrt{\left(\frac{a+b}{c}\right)^2+1}+c\sqrt{\left(\frac{b+c}{a}\right)^2+1}+a\sqrt{\left(\frac{c+a}{b}\right)^2+1}$$
$$\geq (a+b+c)\sqrt{5}.$$

Applying Theorem 2.10 to the convex and increasing function $f(x)=q^x$ with $q>1$ gives

$$bq^{(a+b)/c}+cq^{(b+c)/a}+aq^{(c+a)/b}\geq (a+b+c)q^2.$$

2.11 Eleventh theorem

Our eleventh theorem, along with its proof and two examples, is presented below.

Theorem 2.11. Let $f:(0,\infty)\to\mathbb{R}$ be a convex function. For any $a,b,c>0$, the following cyclic inequality holds:

$$(b+c)f\left(\frac{a+b}{b+c}\right)+(c+a)f\left(\frac{b+c}{c+a}\right)+(a+b)f\left(\frac{c+a}{a+b}\right)\geq 2(a+b+c)f(1).$$

The equality holds for $a=b=c$.

Proof. The Jensen inequality applied to the convex function f and the fact that $\sum_{cyc}(b+c)=2(a+b+c)$ give

$$\begin{aligned}&(b+c)f\left(\frac{a+b}{b+c}\right)+(c+a)f\left(\frac{b+c}{c+a}\right)+(a+b)f\left(\frac{c+a}{a+b}\right)\\ &=\sum_{cyc}(b+c)f\left(\frac{a+b}{b+c}\right)=2(a+b+c)\sum_{cyc}\left(\frac{b+c}{2(a+b+c)}\right)f\left(\frac{a+b}{b+c}\right)\\ &\geq 2(a+b+c)f\left(\frac{\sum_{cyc}\frac{(b+c)(a+b)}{b+c}}{2(a+b+c)}\right)=2(a+b+c)f\left(\frac{\sum_{cyc}(a+b)}{2(a+b+c)}\right)\\ &=2(a+b+c)f\left(\frac{2(a+b+c)}{2(a+b+c)}\right)=2(a+b+c)f(1).\end{aligned}$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $6af(1)$. This completes the proof. □

If f is concave instead of convex, the inequality is reversed.

An extension of this inequality is that, for any $k>0$,

$$(b+kc)f\left(\frac{a+b}{b+kc}\right)+(c+ka)f\left(\frac{b+c}{c+ka}\right)+(a+kb)f\left(\frac{c+a}{a+kb}\right)$$
$$\geq (k+1)(a+b+c)f\left(\frac{2}{k+1}\right).$$

The proof is identical to that of Theorem 2.11.

Two examples of this theorem are presented below.

Applying Theorem 2.11 to the convex function $f(x)=-\ln(x)$ gives

$$-(b+c)\ln\left(\frac{a+b}{b+c}\right)-(c+a)\ln\left(\frac{b+c}{c+a}\right)-(a+b)\ln\left(\frac{c+a}{a+b}\right)\geq 0,$$

so that

$$\left(\frac{a+b}{b+c}\right)^{b+c}\left(\frac{b+c}{c+a}\right)^{c+a}\left(\frac{c+a}{a+b}\right)^{a+b}\leq 1.$$

Applying Theorem 2.11 to the convex function $f(x)=q^x$ with $q>0$ gives

$$(b+c)q^{(a+b)/(b+c)}+(c+a)q^{(b+c)/(c+a)}+(a+b)q^{(c+a)/(a+b)}\geq 2(a+b+c)q.$$

2.12 Twelfth theorem

Our twelfth theorem, along with its proof and three examples, is presented below.

Theorem 2.12. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$af\left(\frac{a+b}{a+c}\right) + bf\left(\frac{b+c}{b+a}\right) + cf\left(\frac{c+a}{c+b}\right) \geq (a+b+c)f(1).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&af\left(\frac{a+b}{a+c}\right) + bf\left(\frac{b+c}{b+a}\right) + cf\left(\frac{c+a}{c+b}\right) \\
&= \sum_{cyc} af\left(\frac{a+b}{a+c}\right) = (a+b+c)\sum_{cyc}\left(\frac{a}{a+b+c}\right)f\left(\frac{a+b}{a+c}\right) \\
&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{a(a+b)}{a+c}}{a+b+c}\right).
\end{aligned}$$

Let us now demonstrate that

$$\sum_{cyc}\frac{a(a+b)}{a+c} \geq a+b+c,$$

which required some developments.

We have

$$\begin{aligned}
&\sum_{cyc}\frac{a(a+b)}{a+c} - (a+b+c) = \sum_{cyc}\left(\frac{a(a+b)}{a+c} - a\right) = \sum_{cyc}\frac{a(b-c)}{a+c} \\
&= \frac{a(b-c)}{a+c} + \frac{b(c-a)}{b+a} + \frac{c(a-b)}{c+b} \\
&= \frac{a(b-c)(b+a)(c+b) + b(c-a)(a+c)(c+b) + c(a-b)(a+c)(b+a)}{(a+c)(b+a)(c+b)} \\
&= \frac{ab^3 + bc^3 + ca^3 - abc(a+b+c)}{(a+c)(b+a)(c+b)}.
\end{aligned}$$

Let us now demonstrate that

$$ab^3 + bc^3 + ca^3 \geq abc(a+b+c).$$

We have

$$ab^3 + bc^3 + ca^3 - abc(a+b+c) = abc\left(\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} - (a+b+c)\right).$$

Using the Titu Lemma, we get

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)^2}{b+c+a} = a+b+c.$$

Therefore, we have

$$\sum_{cyc}\frac{a(a+b)}{a+c} - (a+b+c) \geq \frac{ab^3 + bc^3 + ca^3 - abc(a+b+c)}{(a+c)(b+a)(c+b)} \geq 0,$$

so that

$$\sum_{cyc}\frac{a(a+b)}{a+c} \geq a+b+c.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{a(a+b)}{a+c}}{a+b+c}\right) \geq f\left(\frac{a+b+c}{a+b+c}\right) = f(1).$$

Combining the inequalities above, we obtain

$$af\left(\frac{a+b}{a+c}\right) + bf\left(\frac{b+c}{b+a}\right) + cf\left(\frac{c+a}{c+b}\right) \geq (a+b+c)f(1).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(1)$. This concludes the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Three examples of this theorem are presented below.

Applying Theorem 2.12 to the convex and increasing function $f(x) = x^p$ with $p \geq 1$ gives

$$a\left(\frac{a+b}{a+c}\right)^p + b\left(\frac{b+c}{b+a}\right)^p + c\left(\frac{c+a}{c+b}\right)^p \geq a+b+c.$$

Applying Theorem 2.12 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$aq^{(a+b)/(a+c)} + bq^{(b+c)/(b+a)} + cq^{(c+a)/(c+b)} \geq (a+b+c)q.$$

Applying Theorem 2.12 to the convex and increasing function $f(x) = x\arctan(x)$ gives

$$\frac{a(a+b)}{a+c}\arctan\left(\frac{a+b}{a+c}\right) + \frac{b(b+c)}{b+a}\arctan\left(\frac{b+c}{b+a}\right) + \frac{c(c+a)}{c+b}\arctan\left(\frac{c+a}{c+b}\right)$$
$$\geq (a+b+c)\frac{\pi}{4}.$$

2.13 Thirteenth theorem

Our thirteenth theorem, along with its proof and two examples, is presented below.

Theorem 2.13. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$bf\left(\frac{a+b}{a+c}\right) + cf\left(\frac{b+c}{b+a}\right) + af\left(\frac{c+a}{c+b}\right) \geq (a+b+c)f(1).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&bf\left(\frac{a+b}{a+c}\right) + cf\left(\frac{b+c}{b+a}\right) + af\left(\frac{c+a}{c+b}\right)\\
&= \sum_{cyc} bf\left(\frac{a+b}{a+c}\right) = (a+b+c)\sum_{cyc}\left(\frac{b}{a+b+c}\right)f\left(\frac{a+b}{a+c}\right)\\
&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{b(a+b)}{a+c}}{a+b+c}\right).
\end{aligned}$$

Let us now demonstrate that

$$\sum_{cyc}\frac{b(a+b)}{a+c} \geq a+b+c,$$

which required some developments.

We have

$$\sum_{cyc}\frac{b(a+b)}{a+c}-(a+b+c)=\sum_{cyc}\left(\frac{b(a+b)}{a+c}-b\right)=\sum_{cyc}\frac{b(b-c)}{a+c}$$
$$=\frac{b(b-c)}{a+c}+\frac{c(c-a)}{b+a}+\frac{a(a-b)}{c+b}$$
$$=\frac{b(b-c)(b+a)(c+b)+c(c-a)(a+c)(c+b)+a(a-b)(a+c)(b+a)}{(a+c)(b+a)(c+b)}$$
$$=\frac{(a+b+c)(a^3+b^3+c^3-a^2b-b^2c-c^2a)}{(a+c)(b+a)(c+b)}.$$

Let us now demonstrate that

$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a.$$

It follows from the Young inequality that

$$a^2b=(a^3)^{2/3}(b^3)^{1/3}\leq\frac{2a^3+b^3}{3},$$

and, similarly,

$$b^2c\leq\frac{2b^3+c^3}{3}$$

and

$$c^2a\leq\frac{2c^3+a^3}{3}.$$

Adding these three upper bound gives

$$a^2b+b^2c+c^2a\leq\frac{3a^3+3b^3+3c^3}{3}=a^3+b^3+c^3.$$

Thus, we find that

$$\sum_{cyc}\frac{b(a+b)}{a+c}-(a+b+c)=\frac{(a+b+c)(a^3+b^3+c^3-a^2b-b^2c-c^2a)}{(a+c)(b+a)(c+b)}\geq 0,$$

so that

$$\sum_{cyc}\frac{b(a+b)}{a+c}\geq a+b+c.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{b(a+b)}{a+c}}{a+b+c}\right)\geq f\left(\frac{a+b+c}{a+b+c}\right)=f(1).$$

Combining the inequalities above, we obtain

$$bf\left(\frac{a+b}{a+c}\right)+cf\left(\frac{b+c}{b+a}\right)+af\left(\frac{c+a}{c+b}\right)\geq(a+b+c)f(1).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(1)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed. Two examples of this theorem are presented below.

Applying Theorem 2.13 to the convex and increasing function $f(x)=x^p$ with $p\geq 1$ gives

$$b\left(\frac{a+b}{a+c}\right)^p+c\left(\frac{b+c}{b+a}\right)^p+a\left(\frac{c+a}{c+b}\right)^p\geq a+b+c.$$

Applying Theorem 2.13 to the convex and increasing function $f(x)=q^x$ with $q>1$ gives

$$bq^{(a+b)/(a+c)}+cq^{(b+c)/(b+a)}+aq^{(c+a)/(c+b)}\geq(a+b+c)q.$$

2.14 Fourteenth theorem

Our fourteenth theorem, along with its proof and two examples, is presented below.

Theorem 2.14. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$af\left(\frac{a(b+c)}{bc}\right) + bf\left(\frac{b(c+a)}{ca}\right) + cf\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&af\left(\frac{a(b+c)}{bc}\right) + bf\left(\frac{b(c+a)}{ca}\right) + cf\left(\frac{c(a+b)}{ab}\right) \\
&= \sum_{cyc} af\left(\frac{a(b+c)}{bc}\right) = (a+b+c)\sum_{cyc}\left(\frac{a}{a+b+c}\right)f\left(\frac{a(b+c)}{bc}\right) \\
&\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{a^2(b+c)}{bc}}{a+b+c}\right).
\end{aligned}$$

Let us now demonstrate that

$$\sum_{cyc}\frac{a^2(b+c)}{bc} \geq 2(a+b+c),$$

which required some developments.

We have

$$\begin{aligned}
\sum_{cyc}\frac{a^2(b+c)}{bc} &= \frac{a^2(b+c)}{bc} + \frac{b^2(c+a)}{ca} + \frac{c^2(a+b)}{ab} \\
&= \frac{a^3(b+c) + b^3(c+a) + c^3(a+b)}{abc} \\
&= \frac{(a^3b + ab^3) + (b^3c + bc^3) + (c^3a + ca^3)}{abc}.
\end{aligned}$$

Using the standard inequality $x^2 + y^2 \geq 2xy$ for any $x, y \in \mathbb{R}$, we have

$$a^3b + ab^3 = ab(a^2 + b^2) \geq ab(2ab) = 2a^2b^2,$$

and, similarly,

$$b^3c + bc^3 = bc(b^2 + c^2) \geq 2b^2c^2$$

and

$$c^3a + ca^3 = ca(c^2 + a^2) \geq 2c^2a^2.$$

Therefore, we have

$$\frac{(a^3b + ab^3) + (b^3c + bc^3) + (c^3a + ca^3)}{abc} \geq \frac{2(a^2b^2 + b^2c^2 + c^2a^2)}{abc}.$$

Now, using the standard inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ for any $x, y, z > 0$, we get

$$a^2b^2 + b^2c^2 + c^2a^2 \geq (ab)(bc) + (bc)(ca) + (ca)(ab) = abc(a+b+c).$$

Therefore, we have

$$\frac{2(a^2b^2+b^2c^2+c^2a^2)}{abc} \geq \frac{2abc(a+b+c)}{abc} = 2(a+b+c).$$

Thus, we establish that

$$\sum_{cyc} \frac{a^2(b+c)}{bc} \geq 2(a+b+c).$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{a^2(b+c)}{bc}}{a+b+c}\right) \geq f\left(\frac{2(a+b+c)}{a+b+c}\right) = f(2).$$

Combining the inequalities above, we obtain

$$af\left(\frac{a(b+c)}{bc}\right)+bf\left(\frac{b(c+a)}{ca}\right)+cf\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

Clearly, the equality holds for $a=b=c$, with both sides equal to $3af(2)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.14 to the convex and increasing function $f(x) = x\ln(1+x)$ gives

$$\frac{a^2(b+c)}{bc}\ln\left(1+\frac{a(b+c)}{bc}\right)+\frac{b^2(c+a)}{ca}\ln\left(1+\frac{b(c+a)}{ca}\right)$$
$$+\frac{c^2(a+b)}{ab}\ln\left(1+\frac{c(a+b)}{ab}\right) \geq (a+b+c)2\ln(3),$$

so that

$$\left(1+\frac{a(b+c)}{bc}\right)^{a^2(b+c)/(bc)}\left(1+\frac{b(c+a)}{ca}\right)^{b^2(c+a)/(ca)}\left(1+\frac{c(a+b)}{ab}\right)^{c^2(a+b)/(ab)}$$
$$\geq 9^{a+b+c}.$$

Applying Theorem 2.14 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$aq^{a(b+c)/(bc)}+bq^{b(c+a)/(ca)}+cq^{c(a+b)/(ab)} \geq (a+b+c)q^2.$$

2.15 Fifteenth theorem

Our fifteenth theorem, along with its proof and two examples, is presented below.

Theorem 2.15. Let $f:(0,\infty)\to\mathbb{R}$ be a convex and increasing function. For any $a,b,c>0$, the following cyclic inequality holds:

$$bf\left(\frac{a(b+c)}{bc}\right)+cf\left(\frac{b(c+a)}{ca}\right)+af\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

The equality holds for $a=b=c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
& bf\left(\frac{a(b+c)}{bc}\right) + cf\left(\frac{b(c+a)}{ca}\right) + af\left(\frac{c(a+b)}{ab}\right) \\
&= \sum_{cyc} bf\left(\frac{a(b+c)}{bc}\right) = (a+b+c)\sum_{cyc}\left(\frac{b}{a+b+c}\right) f\left(\frac{a(b+c)}{bc}\right) \\
&\geq (a+b+c) f\left(\frac{\sum_{cyc} \frac{ba(b+c)}{bc}}{a+b+c}\right) \\
&= (a+b+c) f\left(\frac{\sum_{cyc} \frac{a(b+c)}{c}}{a+b+c}\right).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{cyc} \frac{a(b+c)}{c} &= \frac{a(b+c)}{c} + \frac{b(c+a)}{a} + \frac{c(a+b)}{b} \\
&= \left(\frac{ab}{c} + a\right) + \left(\frac{bc}{a} + b\right) + \left(\frac{ca}{b} + c\right) \\
&= (a+b+c) + \left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right).
\end{aligned}$$

Using the standard inequality $1/x + x \geq 2$ for any $x > 0$, we have

$$\begin{aligned}
& \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{a}{2}\left(\frac{c}{b} + \frac{b}{c}\right) + \frac{b}{2}\left(\frac{a}{c} + \frac{c}{a}\right) + \frac{c}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\
&\geq \frac{a}{2}2 + \frac{b}{2}2 + \frac{c}{2}2 = a+b+c.
\end{aligned}$$

Thus, we find that

$$\sum_{cyc} \frac{a(b+c)}{c} \geq 2(a+b+c).$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{a(b+c)}{c}}{a+b+c}\right) \geq f\left(\frac{2(a+b+c)}{a+b+c}\right) = f(2).$$

Combining the inequalities above, we obtain

$$bf\left(\frac{a(b+c)}{bc}\right) + cf\left(\frac{b(c+a)}{ca}\right) + af\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(2)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.
Two examples of this theorem are presented below.
Applying Theorem 2.15 to the convex and increasing function $f(x) = x^p$ with $p \geq 1$ gives

$$b\left(\frac{a(b+c)}{bc}\right)^p + c\left(\frac{b(c+a)}{ca}\right)^p + a\left(\frac{c(a+b)}{ab}\right)^p \geq (a+b+c)2^p.$$

Applying Theorem 2.15 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$bq^{a(b+c)/(bc)} + cq^{b(c+a)/(ca)} + aq^{c(a+b)/(ab)} \geq (a+b+c)q^2.$$

2.16 Sixteenth theorem

Our sixteenth theorem, along with its proof and two examples, is presented below.

Theorem 2.16. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$cf\left(\frac{a(b+c)}{bc}\right) + af\left(\frac{b(c+a)}{ca}\right) + bf\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

The equality holds for $a = b = c$.

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned}
&cf\left(\frac{a(b+c)}{bc}\right) + af\left(\frac{b(c+a)}{ca}\right) + bf\left(\frac{c(a+b)}{ab}\right) \\
&= \sum_{cyc} cf\left(\frac{a(b+c)}{bc}\right) = (a+b+c)\sum_{cyc}\left(\frac{c}{a+b+c}\right) f\left(\frac{a(b+c)}{bc}\right) \\
&\geq (a+b+c)f\left(\frac{\sum_{cyc} \frac{ca(b+c)}{bc}}{a+b+c}\right) = (a+b+c)f\left(\frac{\sum_{cyc} \frac{a(b+c)}{b}}{a+b+c}\right).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{cyc} \frac{a(b+c)}{b} &= \frac{a(b+c)}{b} + \frac{b(c+a)}{c} + \frac{c(a+b)}{a} \\
&= \left(a + \frac{ca}{b}\right) + \left(b + \frac{ab}{c}\right) + \left(c + \frac{bc}{a}\right) \\
&= (a+b+c) + \left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right).
\end{aligned}$$

Using the standard inequality $1/x + x \geq 2$ for any $x > 0$, we have

$$\begin{aligned}
&\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = \frac{a}{2}\left(\frac{c}{b} + \frac{b}{c}\right) + \frac{b}{2}\left(\frac{a}{c} + \frac{c}{a}\right) + \frac{c}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\
&\geq \frac{a}{2}2 + \frac{b}{2}2 + \frac{c}{2}2 = a + b + c.
\end{aligned}$$

Thus, we establish that

$$\sum_{cyc} \frac{a(b+c)}{b} \geq 2(a+b+c).$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{a(b+c)}{b}}{a+b+c}\right) \geq f\left(\frac{2(a+b+c)}{a+b+c}\right) = f(2).$$

Combining the inequalities above, we obtain

$$cf\left(\frac{a(b+c)}{bc}\right) + af\left(\frac{b(c+a)}{ca}\right) + bf\left(\frac{c(a+b)}{ab}\right) \geq (a+b+c)f(2).$$

Clearly, the equality holds for $a = b = c$, with both sides equal to $3af(2)$. This ends the proof. □

If f is concave and decreasing instead of convex and increasing, the inequality is reversed.

Two examples of this theorem are presented below.

Applying Theorem 2.16 to the convex and increasing function $f(x) = \sqrt{x^2+1}$ gives

$$c\sqrt{\left(\frac{a(b+c)}{bc}\right)^2+1}+a\sqrt{\left(\frac{b(c+a)}{ca}\right)^2+1}+b\sqrt{\left(\frac{c(a+b)}{ab}\right)^2+1}$$
$$\geq (a+b+c)\sqrt{5}.$$

Applying Theorem 2.16 to the convex and increasing function $f(x) = q^x$ with $q > 1$ gives

$$cq^{a(b+c)/(bc)}+aq^{b(c+a)/(ca)}+bq^{c(a+b)/(ab)} \geq (a+b+c)q^2.$$

3 Supplementary theorems

We complete our set of theorems with two results that we believe are not sharp, as the equality does not hold for $a = b = c$. Nevertheless, they may be of interest for both the results themselves and the proof techniques developed.

Our first supplementary theorem, along with its proof, is presented below.

Theorem 3.1. Let $f : (0,\infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$bf\left(\frac{a}{b+c}\right)+cf\left(\frac{b}{c+a}\right)+af\left(\frac{c}{a+b}\right) \geq (a+b+c)f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

Proof. The Jensen inequality applied to the convex function f gives

$$bf\left(\frac{a}{b+c}\right)+cf\left(\frac{b}{c+a}\right)+af\left(\frac{c}{a+b}\right)$$
$$= \sum_{cyc} bf\left(\frac{a}{b+c}\right) = (a+b+c)\sum_{cyc}\left(\frac{b}{a+b+c}\right)f\left(\frac{a}{b+c}\right)$$
$$\geq (a+b+c)f\left(\frac{\sum_{cyc}\frac{ba}{b+c}}{a+b+c}\right).$$

Using the Titu lemma, we have

$$\sum_{cyc}\frac{ba}{b+c} = \sum_{cyc}\frac{(ba)^2}{ba(b+c)} = \frac{(ba)^2}{ba(b+c)}+\frac{(cb)^2}{cb(c+a)}+\frac{(ac)^2}{ac(a+b)}$$
$$\geq \frac{(ba+cb+ac)^2}{ba(b+c)+cb(c+a)+ac(a+b)} = \frac{(ab+bc+ca)^2}{ab^2+bc^2+ca^2+3abc}.$$

Note that

$$(a+b+c)(ab+bc+ca) = a^2b+a^2c+ab^2+b^2c+ac^2+bc^2+3abc$$
$$\geq ab^2+bc^2+ca^2+3abc.$$

Therefore, we obtain

$$\frac{(ab+bc+ca)^2}{ab^2+bc^2+ca^2+3abc} \geq \frac{(ab+bc+ca)^2}{(a+b+c)(ab+bc+ca)} = \frac{ab+bc+ca}{a+b+c}.$$

Combining the inequalities above, we obtain

$$\sum_{cyc} \frac{ba}{b+c} \geq \frac{ab+bc+ca}{a+b+c}.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc} \frac{ba}{b+c}}{a+b+c}\right) \geq f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

Combining the inequalities above, we have

$$bf\left(\frac{a}{b+c}\right) + cf\left(\frac{b}{c+a}\right) + af\left(\frac{c}{a+b}\right) \geq (a+b+c)f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

This concludes the proof. □

By applying the same logic from the first steps of the proof, we can establish a refined version of the theorem. This yields the following more complicated, but strictly sharper, lower bound:

$$\begin{aligned} &bf\left(\frac{a}{b+c}\right) + cf\left(\frac{b}{c+a}\right) + af\left(\frac{c}{a+b}\right) \\ &\geq (a+b+c)f\left(\frac{(ab+bc+ca)^2}{(a+b+c)(ab^2+bc^2+ca^2+3abc)}\right). \end{aligned}$$

Notably, the equality holds for $a = b = c$. On this basis, finding an improved lower bound of the form $\alpha(a+b+c)f(\beta)$, where $\alpha, \beta > 0$ are constants, remains an open problem.

Our second supplementary theorem, along with its proof, is presented below. This is a slight variant of Theorem 3.1.

Theorem 3.2. Let $f : (0, \infty) \to \mathbb{R}$ be a convex and increasing function. For any $a, b, c > 0$, the following cyclic inequality holds:

$$cf\left(\frac{a}{b+c}\right) + af\left(\frac{b}{c+a}\right) + bf\left(\frac{c}{a+b}\right) \geq (a+b+c)f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

Proof. The Jensen inequality applied to the convex function f gives

$$\begin{aligned} &cf\left(\frac{a}{b+c}\right) + af\left(\frac{b}{c+a}\right) + bf\left(\frac{c}{a+b}\right) \\ &= \sum_{cyc} cf\left(\frac{a}{b+c}\right) = (a+b+c)\sum_{cyc}\left(\frac{c}{a+b+c}\right)f\left(\frac{a}{b+c}\right) \\ &\geq (a+b+c)f\left(\frac{\sum_{cyc} \frac{ca}{b+c}}{a+b+c}\right). \end{aligned}$$

Using the Titu lemma, we have

$$\begin{aligned} &\sum_{cyc} \frac{ca}{b+c} = \sum_{cyc} \frac{(ca)^2}{ca(b+c)} = \frac{(ca)^2}{ca(b+c)} + \frac{(ab)^2}{ab(c+a)} + \frac{(bc)^2}{bc(a+b)} \\ &\geq \frac{(ca+ab+bc)^2}{ca(b+c)+ab(c+a)+bc(a+b)} = \frac{(ab+bc+ca)^2}{a^2b+b^2c+c^2a+3abc}. \end{aligned}$$

Note that

$$\begin{aligned} &(a+b+c)(ab+bc+ca) = a^2b+a^2c+ab^2+b^2c+ac^2+bc^2+3abc \\ &\geq a^2b+b^2c+c^2a+3abc. \end{aligned}$$

Therefore, we get

$$\frac{(ab+bc+ca)^2}{a^2b+b^2c+c^2a+3abc} \geq \frac{(ab+bc+ca)^2}{(a+b+c)(ab+bc+ca)} = \frac{ab+bc+ca}{a+b+c}.$$

Combining the inequalities above, we obtain

$$\sum_{cyc} \frac{ca}{b+c} \geq \frac{ab+bc+ca}{a+b+c}.$$

Using this and the fact that f is increasing, we get

$$f\left(\frac{\sum_{cyc}\frac{ca}{b+c}}{a+b+c}\right) \geq f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

Combining the inequalities above, we have

$$cf\left(\frac{a}{b+c}\right) + af\left(\frac{b}{c+a}\right) + bf\left(\frac{c}{a+b}\right) \geq (a+b+c)f\left(\frac{ab+bc+ca}{(a+b+c)^2}\right).$$

This completes the proof. $\square$

By applying the same logic from the first steps of the proof, we can establish a refined version of the theorem. This yields the following more complicated, but strictly sharper, lower bound:

$$\begin{aligned} &cf\left(\frac{a}{b+c}\right) + af\left(\frac{b}{c+a}\right) + bf\left(\frac{c}{a+b}\right) \\ &\geq (a+b+c)f\left(\frac{(ab+bc+ca)^2}{(a+b+c)(a^2b+b^2c+c^2a+3abc)}\right). \end{aligned}$$

Notably, the equality holds for $a = b = c$. On this basis, finding an improved lower bound of the form $\alpha(a+b+c)f(\beta)$, where $\alpha, \beta > 0$ are constants, remains an open problem.

4 Summary

For ease of reference, the sixteen core inequalities established in the main theorems are summarized in Table 1. The specific assumptions regarding the function f are detailed in the respective theorem statements.

Theorem	**Core cyclic inequality**
Theorem 2.1	$cf\left(\frac{a}{b}\right)+af\left(\frac{b}{c}\right)+bf\left(\frac{c}{a}\right)\geq(a+b+c)f(1)$
Theorem 2.2	$bf\left(\frac{a}{b}\right)+cf\left(\frac{b}{c}\right)+af\left(\frac{c}{a}\right)\geq(a+b+c)f(1)$
Theorem 2.3	$af\left(\frac{a}{b}\right)+bf\left(\frac{b}{c}\right)+cf\left(\frac{c}{a}\right)\geq(a+b+c)f(1)$
Theorem 2.4	$(b+c)f\left(\frac{a}{b+c}\right)+(c+a)f\left(\frac{b}{c+a}\right)+(a+b)f\left(\frac{c}{a+b}\right)\geq 2(a+b+c)f\left(\frac{1}{2}\right)$
Theorem 2.5	$(a+b)f\left(\frac{a}{b+c}\right)+(b+c)f\left(\frac{b}{c+a}\right)+(c+a)f\left(\frac{c}{a+b}\right)\geq 2(a+b+c)f\left(\frac{1}{2}\right)$
Theorem 2.6	$(a+c)f\left(\frac{a}{b+c}\right)+(b+a)f\left(\frac{b}{c+a}\right)+(c+b)f\left(\frac{c}{a+b}\right)\geq 2(a+b+c)f\left(\frac{1}{2}\right)$
Theorem 2.7	$af\left(\frac{a}{b+c}\right)+bf\left(\frac{b}{c+a}\right)+cf\left(\frac{c}{a+b}\right)\geq(a+b+c)f\left(\frac{1}{2}\right)$
Theorem 2.8	$cf\left(\frac{a+b}{c}\right)+af\left(\frac{b+c}{a}\right)+bf\left(\frac{c+a}{b}\right)\geq(a+b+c)f(2)$
Theorem 2.9	$af\left(\frac{a+b}{c}\right)+bf\left(\frac{b+c}{a}\right)+cf\left(\frac{c+a}{b}\right)\geq(a+b+c)f(2)$
Theorem 2.10	$bf\left(\frac{a+b}{c}\right)+cf\left(\frac{b+c}{a}\right)+af\left(\frac{c+a}{b}\right)\geq(a+b+c)f(2)$
Theorem 2.11	$(b+c)f\left(\frac{a+b}{b+c}\right)+(c+a)f\left(\frac{b+c}{c+a}\right)+(a+b)f\left(\frac{c+a}{a+b}\right)\geq 2(a+b+c)f(1)$
Theorem 2.12	$af\left(\frac{a+b}{a+c}\right)+bf\left(\frac{b+c}{b+a}\right)+cf\left(\frac{c+a}{c+b}\right)\geq(a+b+c)f(1)$
Theorem 2.13	$bf\left(\frac{a+b}{a+c}\right)+cf\left(\frac{b+c}{b+a}\right)+af\left(\frac{c+a}{c+b}\right)\geq(a+b+c)f(1)$
Theorem 2.14	$af\left(\frac{a(b+c)}{bc}\right)+bf\left(\frac{b(c+a)}{ca}\right)+cf\left(\frac{c(a+b)}{ab}\right)\geq(a+b+c)f(2)$
Theorem 2.15	$bf\left(\frac{a(b+c)}{bc}\right)+cf\left(\frac{b(c+a)}{ca}\right)+af\left(\frac{c(a+b)}{ab}\right)\geq(a+b+c)f(2)$
Theorem 2.16	$cf\left(\frac{a(b+c)}{bc}\right)+af\left(\frac{b(c+a)}{ca}\right)+bf\left(\frac{c(a+b)}{ab}\right)\geq(a+b+c)f(2)$

Table 1: Core inequalities of the sixteen main theorems

5 Conclusion

In this paper, we have established a new set of general theorems for cyclic inequalities involving an arbitrary function that is convex and, in certain cases, increasing. The detailed proofs and concrete examples demonstrate the practical utility of this functional approach. Future research may extend these methodologies to n-variable formulations with $n>3$, non-convex functional settings, or semi-algebraic optimization techniques. Additionally, the open problems arising from the two supplementary theorems warrant further investigation.

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