ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2026) 2026:9

Viscosity Approximation Methods for Monotone α-Nonexpansive Mappings and Variational Inequalities in Ordered Banach Spaces

Seyit Temir[1]

[1]Department of Mathematics, Art and Science Faculty, Adıyaman University, 02040, Adıyaman, Turkey
Email(s): seyittemir@adiyaman.edu.tr

Correspondence should be addressed to Seyit Temir: seyittemir@adiyaman.edu.tr

Abstract
In this paper, we introduce a viscosity approximation method that combines the three-step SP* structure with a projected forward step for an inverse-strongly monotone operator. The convergence analysis is carried out in an ordered real Hilbert space, which is a particular case of an ordered Banach space. For a monotone α-nonexpansive mapping, we prove weak convergence when the viscosity perturbations are summable. When the viscosity coefficients tend to zero and have divergent sum, we prove strong convergence to the unique point in the common solution set that solves the viscosity variational inequality. The order-comparability needed for monotone α-nonexpansive mappings is stated explicitly. An application to a nonlinear Fredholm integral equation and two verified numerical examples are included.

Keywords: Ordered Banach space; fixed point; monotone α-nonexpansive mapping; viscosity approximation; variational inequality; SP*-iteration.
MSC (2020): 47H09, 47H10, 47J20, 47J25.

1 Introduction

Monotone nonexpansive mappings in ordered Banach spaces were studied by Bachar and Khamsi [1] and Bin Dehaish and Khamsi [2]. In 2016, Song et al. [9] introduced the class of monotone α-nonexpansive mappings and investigated the Mann iteration in uniformly convex ordered Banach spaces. This class contains monotone nonexpansive mappings as the case $\alpha = 0$. Related generalized classes and iterative procedures were subsequently considered in [6, 7].

The viscosity approximation method was introduced by Moudafi [5] and developed further by Xu [14]. Its essential feature is that it selects a distinguished point of a fixed-point set, rather than merely producing an unspecified fixed point. Viscosity methods combined with variational inequalities have been studied for nonexpansive mappings in Hilbert spaces; see, for example, [3, 4, 8].

In its standard form, the SP*-iteration consists of three successive applications of the basic step

$$u \longmapsto T\big((1-\theta)u + \theta T u\big),$$

Received: 2026-03-23

with different intermediate points and control parameters at each stage; see [10, 11]. In order to retain this defining structure, the algorithm below includes all three SP* stages and inserts the projected variational-inequality step between the first and second stages. A viscosity convex combination is then placed before the last SP* stage.

There are two points that require particular care. First, the metric projection P_K and the usual inverse-strong monotonicity inequality have the nonexpansive properties used here in a real Hilbert space. Consequently, the main convergence results are stated in that setting. A Banach-space extension would require an appropriate sunny nonexpansive retraction and a duality-mapping formulation; it does not follow merely by replacing the Hilbert identity with a uniform-convexity inequality. Second, a monotone α-nonexpansive mapping is only controlled on comparable pairs. We therefore state an order-comparability hypothesis explicitly. It is automatic when K is a chain, and it is unnecessary when T is globally nonexpansive.

The paper is organized as follows. Section 2 contains the necessary preliminaries. Section 3 introduces the viscosity SP* algorithm and establishes its basic estimates. Weak and strong convergence results are proved in Section 4. Section 5 gives an application to a nonlinear Fredholm integral equation, and Section 6 contains two numerical examples.

2 Preliminaries

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|, \leq)$ be an ordered real Hilbert space. The order is induced by a closed convex cone $\mathcal{P}$:

$$x \leq y \quad \Longleftrightarrow \quad y - x \in \mathcal{P}.$$

The order intervals are closed and convex. In particular, if $x \leq y$ and $t \in [0,1]$, then

$$x \leq (1-t)x + ty \leq y. \tag{1}$$

Let K be a nonempty closed convex subset of H.

Definition 2.1 ([9]). A mapping $T : K \to K$ is called *monotone α-nonexpansive*, where $\alpha \in [0,1)$, if T is monotone and

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x - y\|^2 \tag{2}$$

whenever $x, y \in K$ and $x \leq y$.

We write $\mathrm{Fix}(T) = \{x \in K : Tx = x\}$.

Lemma 2.2 ([9, Lemma 2.2]). Let $T : K \to K$ be monotone α-nonexpansive and suppose that $\mathrm{Fix}(T) \neq \emptyset$. Then:

1. T is monotone quasi-nonexpansive on comparable pairs; that is,

$$\|Tx - p\| \leq \|x - p\|$$

whenever $p \in \mathrm{Fix}(T)$ and x is comparable with p;

2. whenever $x \leq y$,

$$\begin{aligned}\|Tx - Ty\|^2 \leq \|x - y\|^2 &+ \frac{2\alpha}{1-\alpha}\|Tx - x\|^2 \\ &+ \frac{2\alpha}{1-\alpha}\|Tx - x\|\big(\|x - y\| + \|Tx - Ty\|\big).\end{aligned} \tag{3}$$

The symmetric form of (3), with $\|Tx - x\|$ replaced by $\|Ty - y\|$, is also valid.

Remark 2.3. Although comparability is written as $x \le y$, inequality (2) is symmetric in the two labelled points. Repeating the argument used in [9, Lemma 2.2], with the algebra based at the other endpoint, yields the symmetric form stated above. Thus, the residual in (3) may be taken at either member of a comparable pair.

The next consequence replaces an unjustified continuity assumption that is sometimes made for this class.

Lemma 2.4. Suppose that $\{\xi_n\}$ and $\{\zeta_n\}$ are bounded and that ξ_n and ζ_n are comparable for every n. If

$$\|\xi_n - \zeta_n\| \to 0 \quad \text{and} \quad \|\xi_n - T\xi_n\| \to 0,$$

then $\|T\xi_n - T\zeta_n\| \to 0$.

Proof. Set $d_n = \|T\xi_n - T\zeta_n\|$, $e_n = \|T\xi_n - \xi_n\|$, and $h_n = \|\xi_n - \zeta_n\|$. Using the form of Lemma 2.2 based at ξ_n gives

$$d_n^2 \le h_n^2 + Ce_n^2 + Ce_n(h_n + d_n), \qquad C = \frac{2\alpha}{1-\alpha}.$$

The associated quadratic inequality first shows that $\{d_n\}$ is bounded; letting $n \to \infty$ then gives $d_n \to 0$. □

Definition 2.5. An operator $A : H \to H$ is called η-*inverse-strongly monotone*, where $\eta > 0$, if

$$\langle Ax - Ay, x - y\rangle \ge \eta\|Ax - Ay\|^2, \qquad x, y \in H. \tag{4}$$

Such an operator is $1/\eta$-Lipschitz continuous. The variational-inequality solution set is

$$\mathrm{VI}(K, A) = \big\{p \in K : \langle Ap, x - p\rangle \ge 0 \text{ for every } x \in K\big\}.$$

The metric projection $P_K : H \to K$ is characterized by

$$z = P_K x \quad \Longleftrightarrow \quad \langle x - z, y - z\rangle \le 0 \quad (y \in K). \tag{5}$$

It is firmly nonexpansive. For $\lambda > 0$, put

$$Q_\lambda = P_K(I - \lambda A). \tag{6}$$

Then

$$\mathrm{Fix}(Q_\lambda) = \mathrm{VI}(K, A). \tag{7}$$

Definition 2.6. A mapping $S : K \to K$ is θ-averaged, where $0 < \theta < 1$, if $S = (1-\theta)I + \theta R$ for some nonexpansive mapping $R : K \to K$.

Lemma 2.7. Let A be η-inverse-strongly monotone and let $0 < \lambda < 2\eta$. Then Q_λ is ϑ_λ-averaged with

$$\vartheta_\lambda = \frac{1}{2} + \frac{\lambda}{4\eta} < 1.$$

Consequently, for $p \in \mathrm{VI}(K, A)$,

$$\|Q_\lambda x - p\|^2 \le \|x - p\|^2 - \frac{2\eta - \lambda}{2\eta + \lambda}\|x - Q_\lambda x\|^2. \tag{8}$$

Proof. Condition (4) implies that $I - 2\eta A$ is nonexpansive. Hence $I - \lambda A$ is $\lambda/(2\eta)$-averaged. Since P_K is $1/2$-averaged, their composition is ϑ_λ-averaged. If S is ϑ-averaged and $p \in \mathrm{Fix}(S)$, then

$$\|Sx - p\|^2 \le \|x - p\|^2 - \frac{1-\vartheta}{\vartheta}\|x - Sx\|^2.$$

Applying this inequality to $S = Q_\lambda$ proves (8). □

We recall the correct sign in Xu's uniform-convexity inequality.

Lemma 2.8 ([12])**.** Let $q > 1$ and $r > 0$. A Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, convex function $g : [0, \infty) \to [0, \infty)$, with $g(0) = 0$, such that

$$\begin{aligned}\|tx + (1-t)y\|^q &\le t\|x\|^q + (1-t)\|y\|^q \\ &\quad - \big(t^q(1-t) + t(1-t)^q\big) g(\|x-y\|)\end{aligned}$$

for $x, y \in B_r(0)$ and $t \in [0, 1]$.

Definition 2.9. A mapping $f : K \to K$ is a contraction with coefficient $\beta \in [0, 1)$ if

$$\|f(x) - f(y)\| \le \beta\|x - y\|, \qquad x, y \in K.$$

We use the following standard sequence lemma, which is a special case of [13].

Lemma 2.10 ([13])**.** Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\gamma_n\} \subset (0, 1)$, and suppose that

$$\sum_{n=1}^{\infty} \gamma_n = \infty.$$

If

$$s_{n+1} \le (1 - \gamma_n)s_n + \gamma_n \delta_n$$

and

$$\limsup_{n\to\infty} \delta_n \le 0,$$

then

$$\lim_{n\to\infty} s_n = 0.$$

Lemma 2.11. Let $\{s_n\}$ be a nonnegative sequence which is not eventually nonincreasing. For all sufficiently large n, define

$$\tau(n) = \max\{k \le n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ and $s_n \le s_{\tau(n)+1}$ for all sufficiently large n.

Proof. Lemma 2.10 follows by iterating the inequality after replacing δ_n by an arbitrary positive upper bound. For Lemma 2.11, the maximality of $\tau(n)$ implies that the terms following $s_{\tau(n)+1}$ up to s_n form a nonincreasing string. □

Finally, H satisfies Opial's condition. The following form of the demiclosedness principle is adapted to the order setting.

Lemma 2.12. Let $T : K \to K$ be monotone α-nonexpansive. Suppose that $x_n \rightharpoonup x$, that x_n is comparable with x for every n, and that $\|x_n - Tx_n\| \to 0$. Then $x \in \mathrm{Fix}(T)$.

Proof. By (3), $\{Tx_n - Tx\}$ is bounded and

$$\limsup_{n\to\infty} \|Tx_n - Tx\|^2 \le \limsup_{n\to\infty} \|x_n - x\|^2.$$

If $Tx \ne x$, choose a subsequence along which $\|x_n - x\|$ tends to its lower limit. Since $\|x_n - Tx_n\| \to 0$, the preceding inequality contradicts Opial's condition. Thus $Tx = x$. □

3 Viscosity SP*-iteration scheme

Let $T : K \to K$ be monotone α-nonexpansive, let $A : H \to H$ be η-inverse-strongly monotone, and let $f : K \to K$ be a contraction with coefficient $\beta \in [0, 1)$. Set

$$F = \mathrm{Fix}(T) \cap \mathrm{VI}(K, A), \tag{9}$$

and suppose that F is nonempty, closed, and convex.

For the order-dependent estimates used below, we consider the following compatibility setting:

(OC) Either K is a chain with respect to $\leq$, or T is nonexpansive on K.

In the first case, the comparability required in Lemma 2.2 is automatically satisfied. In the second case, the relevant estimates hold for arbitrary pairs of points in K, without any additional comparability requirement.

Starting with $x_1 \in K$, define

$$\begin{aligned} u_n &= T\big((1-c_n)x_n + c_n T x_n\big), & (10)\\ v_n &= Q_{\lambda_n} u_n = P_K(u_n - \lambda_n A u_n), & (11)\\ y_n &= T\big((1-b_n)v_n + b_n T v_n\big), & (12)\\ w_n &= \alpha_n f(y_n) + (1-\alpha_n) T y_n, & (13)\\ x_{n+1} &= T\big((1-a_n)w_n + a_n T w_n\big). & (14) \end{aligned}$$

The control sequences satisfy

$$\begin{aligned} &\text{(C1)} \quad 0 < \underline{a} \leq a_n \leq \overline{a} < 1, \quad 0 < \underline{b} \leq b_n \leq \overline{b} < 1, \\ &\qquad\quad 0 < \underline{c} \leq c_n \leq \overline{c} < 1; \\ &\text{(C2)} \quad 0 < \lambda_0 \leq \lambda_n \leq 2\eta - \lambda_0 < 2\eta; \\ &\text{(C3)} \quad 0 < \alpha_n < 1. \end{aligned}$$

Remark 3.1. Equations (10), (12), and (14) retain the three defining SP* stages from [10, 11]. Omitting the middle stage (12) destroys the uniform-convexity term that controls $\|v_n - Tv_n\|$; this residual cannot in general be recovered from the viscosity step because $\alpha_n \to 0$.

Lemma 3.2 (Boundedness). Under (OC) and (C1)–(C3), the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{w_n\}$ are bounded.

Proof. Fix $p \in F$. Lemma 2.2, or global nonexpansivity in the second case of (OC), gives

$$\|u_n - p\| \leq \|x_n - p\|, \qquad \|y_n - p\| \leq \|v_n - p\|, \qquad \|x_{n+1} - p\| \leq \|w_n - p\|.$$

Since Q_{λ_n} is nonexpansive and fixes p, $\|v_n - p\| \leq \|u_n - p\|$. Moreover,

$$\begin{aligned} \|w_n - p\| &\leq \alpha_n \|f(y_n) - p\| + (1-\alpha_n)\|Ty_n - p\| \\ &\leq \big(1 - \alpha_n(1-\beta)\big)\|y_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

It follows that

$$\|x_{n+1} - p\| \leq \big(1 - \alpha_n(1-\beta)\big)\|x_n - p\| + \alpha_n \|f(p) - p\|.$$

Therefore

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\beta}\right\}$$

for every n. The remaining sequences are bounded by their definitions and the preceding estimates. □

Lemma 3.3. Let $p \in F$. Put

$$\delta_a = \underline{a}(1-\overline{a}), \quad \delta_b = \underline{b}(1-\overline{b}), \quad \delta_c = \underline{c}(1-\overline{c}), \quad \kappa = \frac{\lambda_0}{4\eta - \lambda_0}.$$

Then

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \delta_c \|x_n - Tx_n\|^2, \tag{15}$$
$$\|v_n - p\|^2 \le \|u_n - p\|^2 - \kappa \|u_n - v_n\|^2, \tag{16}$$
$$\|y_n - p\|^2 \le \|v_n - p\|^2 - \delta_b \|v_n - Tv_n\|^2, \tag{17}$$
$$\|x_{n+1} - p\|^2 \le \|w_n - p\|^2 - \delta_a \|w_n - Tw_n\|^2. \tag{18}$$

If $q = P_F f(q)$ and $\gamma_n = \alpha_n(1-\beta)$, then

$$\|w_n - q\|^2 \le (1-\gamma_n)\|y_n - q\|^2 + 2\alpha_n \langle f(q) - q, w_n - q\rangle. \tag{19}$$

Proof. For $z_n = (1-c_n)x_n + c_n T x_n$, quasi-nonexpansivity and the Hilbert identity give

$$\begin{aligned}\|u_n - p\|^2 &\le \|z_n - p\|^2 \\ &= (1-c_n)\|x_n - p\|^2 + c_n \|Tx_n - p\|^2 \\ &\quad - c_n(1-c_n)\|x_n - Tx_n\|^2,\end{aligned}$$

which proves (15). Estimate (16) follows from Lemma 2.7 and (C2). Applying the same Hilbert identity to $(1-b_n)v_n + b_n T v_n$ and to $(1-a_n)w_n + a_n T w_n$ proves (17) and (18).

Since $q = P_F f(q)$,

$$w_n - q = \alpha_n\big(f(y_n) - f(q)\big) + (1-\alpha_n)(Ty_n - q) + \alpha_n(f(q) - q).$$

The norm of the sum of the first two terms is at most $(1-\gamma_n)\|y_n - q\|$. Using $\|a+b\|^2 \le \|a\|^2 + 2\langle b, a+b\rangle$ and $(1-\gamma_n)^2 \le 1 - \gamma_n$ proves (19). □

4 Convergence theorems

The weak and strong results require different conditions on the viscosity coefficients. Combining them into a single condition would be incorrect.

Theorem 4.1. Assume (OC) and (C1)–(C3) and

$$\sum_{n=1}^{\infty} \alpha_n < \infty. \tag{20}$$

Then the sequence $\{x_n\}$ generated by (10)–(14) converges weakly to a point of F.

Proof. Fix $p \in F$. By Lemma 3.2, all sequences are bounded. Hence there is a constant $C_p > 0$ such that

$$\|w_n - p\|^2 \le (1-\alpha_n)\|Ty_n - p\|^2 + \alpha_n \|f(y_n) - p\|^2 \le \|y_n - p\|^2 + C_p \alpha_n.$$

Using Lemma 3.3, we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 &- \delta_c \|x_n - Tx_n\|^2 - \kappa \|u_n - v_n\|^2 \\ &- \delta_b \|v_n - Tv_n\|^2 - \delta_a \|w_n - Tw_n\|^2 + C_p \alpha_n.\end{aligned} \tag{21}$$

Since (20) holds, the sequence $\|x_n - p\|$ has a limit and

$$\|x_n - Tx_n\| + \|u_n - v_n\| + \|v_n - Tv_n\| + \|w_n - Tw_n\| \longrightarrow 0. \tag{22}$$

We next record that all stages have the same asymptotic location. Let $z_n = (1-c_n)x_n + c_nTx_n$. Then $\|z_n - x_n\| \to 0$, and Lemma 2.4 gives $\|u_n - Tx_n\| = \|Tz_n - Tx_n\| \to 0$. Thus $\|u_n - x_n\| \to 0$, and hence $\|v_n - x_n\| \to 0$. Similarly, using $\|v_n - Tv_n\| \to 0$ in the middle SP* stage gives $\|y_n - v_n\| \to 0$ and $\|Ty_n - y_n\| \to 0$. Since $\alpha_n \to 0$, it follows that $\|w_n - y_n\| \to 0$. The last residual in (22) and the final SP* stage then give

$$\|x_n - u_n\| + \|x_n - v_n\| + \|x_n - y_n\| + \|x_n - w_n\| \longrightarrow 0. \tag{23}$$

Let $x_{n_j} \rightharpoonup x$. Lemma 2.12, or the standard demiclosedness principle when T is globally nonexpansive, gives $x \in \mathrm{Fix}(T)$. Passing to a further subsequence, assume that $\lambda_{n_j} \to \bar\lambda \in [\lambda_0, 2\eta - \lambda_0]$. Since $v_{n_j} = Q_{\lambda_{n_j}} u_{n_j}$, the Lipschitz continuity of A and $\|u_{n_j} - v_{n_j}\| \to 0$ yield

$$\begin{aligned}\|v_{n_j} - Q_{\bar\lambda} v_{n_j}\| \le \|u_{n_j} - v_{n_j}\| + \lambda_{n_j}\|Au_{n_j} - Av_{n_j}\| \\ + |\lambda_{n_j} - \bar\lambda|\,\|Av_{n_j}\| \longrightarrow 0.\end{aligned}$$

The mapping $Q_{\bar\lambda}$ is nonexpansive, so its demiclosedness gives $x \in \mathrm{Fix}(Q_{\bar\lambda}) = \mathrm{VI}(K, A)$. Therefore every weak cluster point belongs to F. The existence of $\lim_n \|x_n - p\|$ for every $p \in F$ and Opial's condition show that two distinct weak cluster points are impossible. Hence the whole sequence converges weakly to a point of F. □

Theorem 4.2. Assume (OC) and (C1)–(C3) and

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{24}$$

Then $\{x_n\}$ converges strongly to the unique point $q \in F$ satisfying

$$\langle (I - f)q, p - q\rangle \ge 0, \qquad p \in F. \tag{25}$$

Equivalently, $q = P_F f(q)$.

Proof. Because F is nonempty, closed, and convex, $P_F f : F \to F$ is a contraction. It therefore has a unique fixed point q, and the projection characterization shows that this point is the unique solution of (25).

Put

$$r_n = \|x_n - q\|^2, \qquad h_n = \langle f(q) - q, w_n - q\rangle,$$

and

$$D_n = \delta_c\|x_n - Tx_n\|^2 + \kappa\|u_n - v_n\|^2 + \delta_b\|v_n - Tv_n\|^2.$$

Lemmas 3.2 and 3.3 give

$$r_{n+1} \le (1-\gamma_n)r_n - (1-\gamma_n)D_n - \delta_a\|w_n - Tw_n\|^2 + 2\alpha_n h_n, \tag{26}$$

where $\gamma_n = \alpha_n(1-\beta)$.

Suppose first that $\{r_n\}$ is eventually nonincreasing. Then $r_n - r_{n+1} \to 0$. Since $\{h_n\}$ is bounded and $\alpha_n \to 0$, (26) implies

$$D_n + \|w_n - Tw_n\|^2 \longrightarrow 0. \tag{27}$$

Exactly as in the proof of Theorem 4.1, all stages align as in (23). Any subsequence used to compute $\limsup h_n$ has a further weakly convergent subsequence. The demiclosedness argument for T and the variable projected-forward mappings Q_{λ_n} shows that its weak limit lies in F. Therefore

$$\limsup_{n\to\infty} h_n \le 0$$

by (25). Dropping the negative residual terms from (26) and applying Lemma 2.10 yields $r_n \to 0$.

Suppose now that $\{r_n\}$ is not eventually nonincreasing, and let $\tau(n)$ be given by Lemma 2.11. Since $r_{\tau(n)} < r_{\tau(n)+1}$, inequality (26), the boundedness of h_n, and $\alpha_n \to 0$ imply

$$D_{\tau(n)} + \|w_{\tau(n)} - Tw_{\tau(n)}\|^2 \longrightarrow 0.$$

The preceding cluster-point argument, now along $\{\tau(n)\}$, gives

$$\limsup_{n\to\infty} h_{\tau(n)} \le 0.$$

Moreover,

$$0 < r_{\tau(n)+1} - r_{\tau(n)} \le -\alpha_{\tau(n)}(1-\beta)r_{\tau(n)} + 2\alpha_{\tau(n)}h_{\tau(n)}.$$

Hence

$$(1-\beta)r_{\tau(n)} < 2h_{\tau(n)},$$

and consequently $r_{\tau(n)} \to 0$. The boundedness of h_n and $\alpha_{\tau(n)} \to 0$ also give $r_{\tau(n)+1} \to 0$. Lemma 2.11 now yields $0 \le r_n \le r_{\tau(n)+1} \to 0$. Thus $x_n \to q$ strongly in both cases. □

Remark 4.3. The assumptions on the viscosity parameters in Theorems 4.1 and 4.2 serve different purposes. The divergent-sum condition in Theorem 4.2 ensures that the viscosity term remains sufficiently influential to select the distinguished solution of (25). By contrast, the summability assumption in Theorem 4.1 allows the viscosity term to be treated as a summable perturbation in the quasi-Fej'er estimates and yields weak convergence to a point of F, without in general identifying that point as the solution of (25).

5 Application to a nonlinear Fredholm integral equation

Let $H = L^2([a,b])$ with its usual inner product and the almost-everywhere order. Consider

$$u(t) = g(t) + \lambda_{\mathrm{F}} \int_a^b K(t,s)\Phi(s,u(s))\,ds, \qquad t \in [a,b], \tag{28}$$

where $\lambda_{\mathrm{F}} \ge 0$. Define

$$(Tu)(t) = g(t) + \lambda_{\mathrm{F}} \int_a^b K(t,s)\Phi(s,u(s))\,ds. \tag{29}$$

Fix $M > 0$ and set

$$\mathcal{K} = \{u \in H : 0 \le u(t) \le M \text{ for almost every } t \in [a,b]\}.$$

Assume:

(A1) $K \in L^2([a,b]^2)$ and $K(t,s) \ge 0$ almost everywhere;

(A2) Φ is a Carathéodory function, $\Phi(\cdot,0) \in L^2([a,b])$, and $r \mapsto \Phi(s,r)$ is nondecreasing for almost every s;

(A3) there exists $L_\Phi > 0$ such that

$$|\Phi(s,r) - \Phi(s,\rho)| \le L_\Phi |r-\rho|$$

for almost every s and all $r, \rho \in [0,M]$;

(A4) $T(\mathcal{K}) \subseteq \mathcal{K}$;

(A5) $\lambda_{\mathrm{F}} L_\Phi \|K\|_{L^2([a,b]^2)} \le 1$.

Then $T : \mathcal{K} \to \mathcal{K}$ is monotone by (A1)–(A3). Moreover, the Cauchy–Schwarz inequality and Fubini's theorem give

$$\|Tu - Tv\|_2^2 \le \lambda_{\mathrm{F}}^2 L_{\Phi}^2 \|K\|^2_{L^2([a,b]^2)} \|u - v\|_2^2 \le \|u - v\|_2^2.$$

Thus T is globally nonexpansive, and hence monotone α-nonexpansive with $\alpha = 0$. Notice that no conclusion of α-nonexpansivity follows merely from a Lipschitz constant larger than one.

To incorporate a nontrivial variational constraint, let D be a nonempty closed convex subset of $\mathcal{K}$ and define

$$A = I - P_D.$$

The operator A is 1-inverse-strongly monotone. Furthermore,

$$\mathrm{VI}(\mathcal{K}, A) = D. \tag{30}$$

Indeed, $Au = 0$ for $u \in D$. Conversely, if $u \in \mathrm{VI}(\mathcal{K}, A)$, then taking $v = P_D u \in D \subseteq \mathcal{K}$ gives

$$0 \le \langle u - P_D u, P_D u - u\rangle = -\|u - P_D u\|^2,$$

so $u \in D$.

Assume that

$$F = \mathrm{Fix}(T) \cap D \neq \emptyset.$$

Because T is nonexpansive, F is closed and convex. For any contraction $f : \mathcal{K} \to \mathcal{K}$, Theorem 4.2 applies without an additional comparability requirement and yields strong convergence to the unique $q \in F$ satisfying

$$\langle (I - f)q, u - q\rangle \ge 0, \qquad u \in F.$$

Thus the algorithm approximates a distinguished solution of the Fredholm equation that also lies in the prescribed closed convex constraint set D.

6 Numerical examples

All the numbers below are obtained by direct evaluation of (10)–(14). In both examples, we take

$$a_n = 0.7, \qquad b_n = 0.5, \qquad c_n = 0.6, \qquad \alpha_n = \frac{1}{n+1}, \qquad \lambda_n = 0.05.$$

Example 6.1 (A monotone α-nonexpansive mapping which is not nonexpansive). Let $H = \mathbb{R}$, $K = [0, 3/5]$, and

$$T(x) = \frac{4}{5}x + \frac{1}{5}x^2, \qquad f(x) = \frac{1}{3}x, \qquad A(x) = x.$$

The mapping T is increasing, maps K into itself, and has the unique fixed point 0 in K. It is not nonexpansive because $T'(x) = 4/5 + 2x/5 > 1$ when $x > 1/2$.

We verify that T is monotone 1/2-nonexpansive. Let $0 \le x \le y \le 3/5$, put $d = y - x$, $s = x + y$, and $h(t) = t(1-t)$. Since $Tt = t - h(t)/5$,

$$Ty - Tx = d\left(\frac{4}{5} + \frac{s}{5}\right).$$

For $s \le 1$, direct expansion gives

$$(Tx - y)^2 + (Ty - x)^2 - 2(Ty - Tx)^2 \ge 0.$$

If $1 < s \le 6/5$, then $x, y \in [2/5, 3/5]$, $d \le 1/5$, and $h(x), h(y) \ge 6/25$. Expanding the same difference gives

$$\begin{aligned}&(Tx-y)^2 + (Ty-x)^2 - 2(Ty-Tx)^2 \\ &\quad = \frac{1}{25}\left(h(x)^2 + h(y)^2\right) - \frac{2}{5}\left(\frac{4}{5} + \frac{s}{5}\right)(s-1)d^2 \ge 0,\end{aligned}$$

where the final inequality follows from

$$\frac{1}{5}\left(h(x)^2 + h(y)^2\right) \ge \frac{72}{3125} > \frac{52}{3125} \ge 2\left(\frac{4}{5} + \frac{s}{5}\right)(s-1)d^2.$$

This is exactly (2) with $\alpha = 1/2$.

Here $\eta = 1$, $\mathrm{VI}(K, A) = \{0\}$, and $F = \{0\}$. Starting from $x_1 = 0.55$, we obtain Table 1.

Table 1: Iterates for Example 6.1.

n	x_n
1	$5.5000000000 \times 10^{-1}$
2	$1.6893076205 \times 10^{-1}$
3	$4.1530775970 \times 10^{-2}$
4	$9.7277812434 \times 10^{-3}$
5	$2.2956919030 \times 10^{-3}$
10	$2.0334704800 \times 10^{-6}$
20	$2.3353820439 \times 10^{-12}$

Example 6.2 (Two-dimensional case). Let $H = \mathbb{R}^2$ with the Euclidean norm and componentwise order, let $K = [0, 1]^2$, and define

$$T(x_1, x_2) = (0.9x_1, 0.8x_2), \qquad f(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}\right), \qquad A = I.$$

Then T is monotone and globally nonexpansive, so it is monotone 0-nonexpansive. Again $F = \{(0, 0)\}$. Starting from $x_1 = (0.8, 0.9)$ gives Table 2.

Table 2: Iterates for Example 6.2.

n	first component	second component
1	$8.0000000000 \times 10^{-1}$	$9.0000000000 \times 10^{-1}$
2	$3.2208727572 \times 10^{-1}$	$1.9380860928 \times 10^{-1}$
3	$1.4202529185 \times 10^{-1}$	$4.4945716106 \times 10^{-2}$
4	$6.5349341910 \times 10^{-2}$	$1.0795518599 \times 10^{-2}$
5	$3.0820566133 \times 10^{-2}$	$2.6466251113 \times 10^{-3}$
10	$8.5213153062 \times 10^{-4}$	$2.6999434732 \times 10^{-6}$
20	$8.7029703973 \times 10^{-7}$	$3.5815199440 \times 10^{-12}$

The examples illustrate the two alternatives in (OC). In Example 6.1, K is a chain and T is genuinely α-nonexpansive without being nonexpansive. In Example 6.2, the componentwise order is not total, but T is globally nonexpansive, so the order-comparability issue does not arise.

7 Conclusion

We introduced a projected viscosity SP* scheme for a common fixed-point and variational-inequality problem. The proof distinguishes two mathematically different regimes: summable viscosity perturbations yield weak convergence, whereas vanishing coefficients with divergent sum yield strong convergence to the unique solution of the hierarchical viscosity variational inequality. The full three-stage SP* structure, the Hilbert-space projection framework, and the order-comparability hypothesis are essential to the arguments. The Fredholm application was formulated under an explicit Hilbert–Schmidt bound; no unsupported implication from an expansive Lipschitz estimate to α-nonexpansivity is used.

References

[1] M. Bachar and M. A. Khamsi, Fixed points of monotone mappings and application to integral equations, Fixed Point Theory Appl. 2015 (2015), Article 110.

[2] B. A. Bin Dehaish and M. A. Khamsi, Mann iteration process for monotone nonexpansive mappings, Fixed Point Theory Appl. 2015 (2015), Article 177.

[3] J. Chen, L. Zhang and T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, J. Math. Anal. Appl. 334 (2007), no. 2, 1450–1461.

[4] P. Kumam, U. W. Humphries and C. Jaiboon, Convergence theorems by the viscosity approximation method for equilibrium problems and variational inequality problems, J. Comput. Math. Optim. 5 (2009), no. 1, 29–56.

[5] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000), no. 1, 46–55.

[6] K. Muangchoo-in, D. Thongtha, P. Kumam and Y. J. Cho, Fixed point theorems and convergence theorems for monotone (α, β)-nonexpansive mappings in ordered Banach spaces, Creat. Math. Inform. 26 (2017), no. 2, 163–180.

[7] R. Pant and R. Shukla, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim. 38 (2017), no. 2, 248–266.

[8] S. Plubtieng and R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (2007), no. 1, 455–469.

[9] Y. Song, K. Promluang, P. Kumam and Y. J. Cho, Some convergence theorems of the Mann iteration for monotone α-nonexpansive mappings, Appl. Math. Comput. 287 (2016), 74–82.

[10] S. Temir and O. Korkut, Approximating fixed points of the new SP*-iteration for generalized α-nonexpansive mappings in CAT(0) spaces, J. Nonlinear Anal. Optim. 12 (2021), no. 2, 83–93.

[11] S. Temir, Approximating fixed points of the SP*-iteration for generalized nonexpansive mappings in CAT(0) spaces, Creat. Math. Inform. 34 (2025), no. 1, 113–132.

[12] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991), no. 12, 1127–1138.

[13] H. K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. (2) 66 (2002), no. 1, 240–256.

[14] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004), no. 1, 279–291.